NAME:____

Final Exam—Winter 99—Math 252H—100 points

Solutions

1. (10 points) Suppose that $y = f(x)$ is a function defined for all $x > 0$ that satisfies the equation

$$\frac{d}{dx}\left(xf'(x)\right) = 12x.$$

Assuming that $f(1) = 3$ and $f'(2) = 0$, find $f(x)$ as an explicit function of x.

- *Integrate both sides:* $xf'(x) = \int 12x\ dx = 6x^2 + C$.
- *Divide by* x*:* $f'(x) = 6x + \frac{C}{x}$.
- *Use* $f'(2) = 0$ *to find* C*:* $0 = f'(2) = 12 + \frac{C}{2}$, *so* $C = -24$ *and hence* $f'(x) = 6x - \frac{24}{x}$.
- *Integrate again:* $f(x) = \int 6x - \frac{24}{x}\ dx = 3x^2 - 24\ln x + D$.
- *Use* $f(1) = 3$ *to find* D*:* $3 = f(1) = 3 - 24\ln 1 = 3$, *so* $D = 0$.
- *Conclude:* $f(x) = 3x^2 - 24\ln x$.

2. A particle moving along the x-axis is at position $x = s(t)$ at time t (measured in seconds). The velocity of the particle at time t is given by $v(t) = t^2 - 1$ (measured in feet per second). Assuming that $s(0) = 1$, determine

(a) (10 points) $s(2)$, the position of the particle at time $t = 2$;
By the Total Change Theorem, since $s'(t) = v(t)$,

$$s(2) - s(0) = \int_0^2 v(t)\ dt = \int_0^2 t^2 - 1\ dt = \left[\frac{t^3}{3} - t\right]_{t=0}^{t=2} = \frac{8}{3} - 2 = \frac{2}{3}.$$

Thus $s(2) - 1 = \frac{2}{3}$, *so* $s(2) = \frac{5}{3}$.

(b) (10 points) the distance travelled by the particle between $t = 0$ and $t = 2$.
The distance travelled is equal to the definite integral of the speed function:

$$\begin{aligned}\int_0^2 |v(t)|\ dt &= \int_0^2 |t^2 - 1|\ dt \\ &= \int_0^1 1 - t^2\ dt + \int_1^2 t^2 - 1\ dt \\ &= \left[t - \frac{t^3}{3}\right]_{t=0}^{t=1} + \left[\frac{t^3}{3} - t\right]_{t=1}^{t=2} \\ &= \left(\frac{8}{3} - 1\right) + \left(\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right) \\ &= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \\ &= 2\end{aligned}$$

The particle travelled two feet between $t = 0$ *and* $t = 2$.

3. The **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\ dt$$

arises in the study of diffraction of light waves.

(a) (5 points) What is $S(0)$?

$$S(0) = \int_0^0 \sin(\pi t^2/2)\ dt = 0.$$

(b) (5 points) Use the Midpoint Rule with $n = 10$ to approximate $S(2)$.

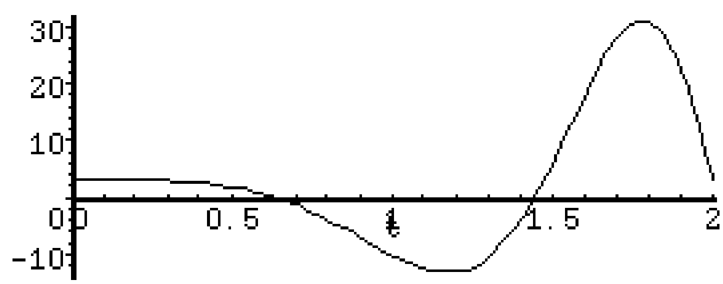
The value $S(2)$ is equal to the definite integral $S(2) = \int_0^2 \sin(\pi t^2/2)\ dt$. We apply the Midpoint Rule with $a = 0$, $b = 2$, $n = 10$, and $f(t) = \sin(\pi t^2/2)$. Thus, setting $\Delta t = \frac{b-a}{n} = \frac{1}{5}$, $t_i = a + i\Delta t = \frac{i}{5}$ and $t_i^* = \frac{t_{i-1}+t_i}{2} = \frac{2i-1}{10}$, we have

$$S(2) \approx M_{10} = \sum_{i=1}^{10} f(t_i^*)\Delta t = .3324429911$$

by using the RSUM program.

(c) (10 points) Use the Midpoint Rule Error Estimate to find an upper bound on the error of your approximation in part (b).

With $f(t) = \sin(\pi t^2/2)$, we find that $f'(t) = \pi t \cos(\pi t^2/2)$ and $f''(t) = \pi\cos(\pi t^2/2) - (\pi t)^2 \sin(\pi t^2/2)$. We can use a graphing calculator to find an upper bound K for the values of $|f''(t)|$ where $0 \le t \le 2$. A good upper bound K is $K = 31.02$. Just to be safe, let's use $K = 32$.



Now the Midpoint Rule Error Estimate shows that

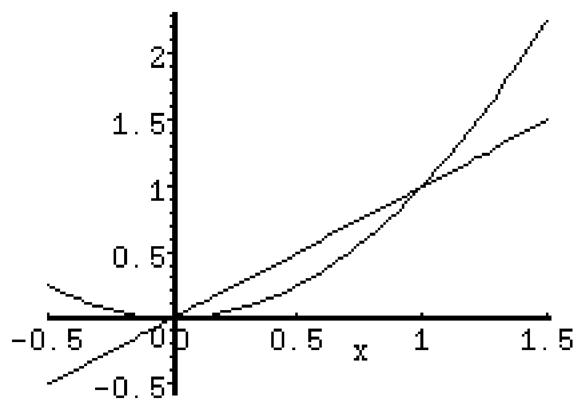
$$|S(2) - M_{10}| \le \frac{K(b-a)^3}{24(10)^2} = \frac{(32)(8)}{(24)(100)} = .106666666667.$$

We can safely conclude that the estimate in part (b) differs from the actual value of $S(2)$ by no more that .11.

(d) (10 points) Find the x-coordinate of the first local minimum on the graph of $y = S(x)$ that is strictly to the right of the y-axis.

We differentiate $S(x)$ to locate the critical points. By the Fundamental Theorem of Calculus (Part I), $S'(x) = \sin(\pi x^2/2)$. Setting $S'(x) = 0$ for $x > 0$, we find that $\pi x^2/2 = \pi, 2\pi, \ldots$ and so $x = \sqrt{2}, 2, \ldots$. To see which of these critical points gives a local minimum, we use the second derivative test: $S''(x) = \pi x \cos(\pi x^2/2)$. Since $S''(\sqrt{2}) = \pi\sqrt{2}\cos(\pi) < 0$, the function S has a local maximum (!) at $x = \sqrt{2}$. Since $S''(2) = 2\pi\cos(2\pi) > 0$, the first local minimum of S to the right of the y-axis occurs at $x = 2$.

4. (10 points) Calculate the volume of the solid that is obtained by revolving the planar region bounded by the curves $y = x$ and $y = x^2$ about the y-axis.



By cylindrical shells,

$$V = \int_0^1 2\pi x(x - x^2)\ dx$$

and by disc/washers,

$$V = \int_0^1 \pi((\sqrt{y})^2 - y^2)\ dy.$$

Either way, $V = \pi/6$.

5. (10 points) Find the centroid of the area under the curve $y = e^x$ for $0 \leq x \leq 1$.

The area of the region in question is

$$A = \int_0^1 e^x\ dx = [e^x]_{x=0}^{x=1} = e - 1.$$

Assuming, as we may, that the uniform density is $\rho = 1$, the moment in the x-direction is calculated via integration by parts:

$$M_x = \int_0^1 xe^x\ dx = [xe^x]_{x=0}^{x=1} - \int_0^1 e^x\ dx$$

(where $u = x$ and $dv = e^x\ dx$ so that $du = dx$ and $v = e^x$). So

$$M_x = e - [e^x]_{x=0}^{x=1} = e - (e-1) = 1.$$

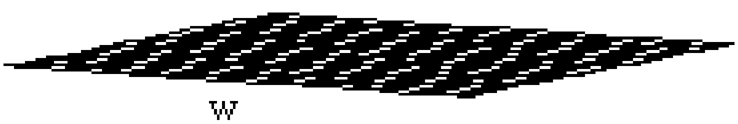
For the moment in the y-direction, we have

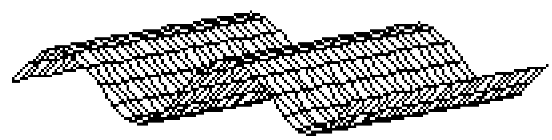
$$M_y = \frac{1}{2}\int_0^1 (e^x)^2\ dx = \frac{1}{4}\left[e^{2x}\right]_{x=0}^{x=1} = \frac{e^2-1}{4}.$$

Thus, the centroid has coordinates

$$(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e^2-1}{4(e-1)}\right) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right).$$

6. (10 points) A manufacturer of corrugated metal roofing wants to produce panels that are 28 inches wide and 2 inches thick by processing flat sheets of metal as shown in the figure below. The profile of the roofing takes the shape of the curve $y = \sin(\pi x/7)$ where $0 \le x \le 28$. Find the width w of a flat metal sheet that is needed to make such a panel.





The length w is equal to the arc length of the graph of $y = \sin(\pi x/7)$ for $0 \le x \le 28$. Since

$$\frac{dy}{dx} = \frac{\pi}{7}\cos(\pi x/7)$$

we have

$$w = \int_0^{28} \sqrt{1 + \left(\frac{\pi}{7}\cos(\pi x/7)\right)^2}\ dx.$$

The only reasonable approach to this definite integral is through numerical methods. For example, using Simpson's Rule with $n = 20$, we find

$$w \approx S_{20} = 29.3607.$$

A flat metal sheet having width 29.36 inches should do the trick.