1 Cayley Graphs

Let $s : x \to G$ be a function from a set $x$ into a group $G$. The Cayley graph $\Gamma = \Gamma(G, x, s)$ is the structure $(V, E^+, \iota, \tau)$ where $V$ is the vertex set, $E^+$ is the set of positive edges, and $\iota, \tau : E^+ \to V$ are the initial and terminal maps, respectively. These items are defined as follows for the Cayley graph $\Gamma = \Gamma(G, x, s)$:

\[
\begin{align*}
V &= G \\
E^+ &= G \times x \\
\iota(g, x) &= g \\
\tau(g, x) &= gs(x)
\end{align*}
\]

This structure should be visualized by taking the vertices $(g \in G)$ to be dots in space and the edges $((g, x) \in G \times x)$ to be arrows joining vertices according to the initial and terminal maps.

Example 1.1 Draw the Cayley graphs corresponding to the following data.

- $G = \mathbb{Z}$, $x = \{x\}$, $s(x) = 1$
- $G = \mathbb{Z}$, $x = \{x\}$, $s(x) = 2$
- $G = \mathbb{Z}$, $x = \{x, y\}$, $s(x) = 2$, $s(y) = 3$
- $G = \mathbb{Z}$, $x = \{x, y\}$, $s(x) = 1$, $s(y) = 0$
• $G = \mathbb{Z}/6\mathbb{Z}, \; x = \{x\}, \; s(x) = 0 + 6\mathbb{Z}$

• $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \; x = \{x, y\}, \; s(x) = (1 + 2\mathbb{Z}, 0 + 3\mathbb{Z}), \; s(y) = (0 + 2\mathbb{Z}, 1 + 3\mathbb{Z})$

• $G = S_3, \; x = \{x, y\}, \; s(x) = (1 2), \; s(y) = (1 2 3)$

• $G = Q_8 \; (\text{quaternion group of order eight}), \; x = \{x, y\}, \; s(x) = i, \; s(y) = j$

Each edge of $\Gamma(G, x, s)$ is a pair $(g, x)$, but in drawing the graph there is no harm in abbreviating this to just $x$ since the $G$-coordinate is already implicit as the name of the initial vertex of $(g, s)$. The $x$-coordinate $x$ of $(g, x)$ is the label of the edge and should be indicated in the picture. One of the main points that we want to understand is that Cayley graphs are highly symmetric. Here is a starting point that addresses the local structure near a vertex.

**Lemma 1.2 (One-In-One-Out)** Let $g \in G$ be a vertex of $\Gamma = \Gamma(G, x, s)$. For each $x \in x$, there is a unique edge $e^+ \in E^+$ with label $x$ such that $\iota(e^+) = g$ and there is a unique edge $f^+ \in E^+$ with label $x$ such that $\tau(f^+) = g$.

**Proof:** $e^+ = (g, x)$ and $f^+ = (gs(x)^{-1}, x)$.

**Words and Paths** A **word** in $x^{\pm 1}$ is a finite sequence $w$ of letters of the form $x^\epsilon$ where $x \in x$ and $\epsilon = \pm 1$:

$$w = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}.$$ 

Using the function $s : x \to G$, any such word uniquely determines a finite product

$$s(w) = s(x_1)^{\epsilon_1} \ldots s(x_n)^{\epsilon_n}$$

in the group $G$. The **linguistic** approach to group theory is to see if we can study local and global properties of groups by organizing words into a language with identifiable properties. This approach is also called combinatorial group theory.

The structure of the Cayley graph is related to the structure of the group according to the connections made by the edges, which can be strung together to make paths in the same way that letters can be strung together...
to make words. Here are the details. The One-In-One-Out lemma shows that if we are given any word and any vertex \( g \in G \) of \( \Gamma \), then we can follow the word to consider an edge-path in \( \Gamma \). Specifically, we start at \( g \) and read the first letter of \( w \), which is \( x_1^{\epsilon_1} \). If \( \epsilon_1 = +1 \) then we look for the unique edge \( e^+ \) of Gamma with label \( x_1 \) and initial vertex \( g \). The first edge in our path is then \((g, x_1)\), which we traverse in the direction indicated by the orientation of that edge: from initial to terminal. If \( \epsilon_1 = -1 \), then we look for the unique edge \( f^+ \) of \( \Gamma \) with label \( x_1 \) and terminal vertex \( g \). In this case the first edge in our path is the reverse edge \( f^- \), which is thought of as the edge \( f \) with reverse orientation. We then repeat the process starting from the new vertex at the other end of the first edge in the path. We stop when the word runs out of letters.

Conversely, an edge path in \( \Gamma \) is a finite sequence of oriented edges (which can be of the form \( e^+ \) or \( e^- \); we allow edge traversal in the negative direction), then we can read the labels to determine a word in \( x^{\pm 1} \).

**Lemma 1.3** The path in \( \Gamma \) that starts at the vertex \( g \in G \) and reads the word \( w = \prod_i x_i^{\epsilon_i} \) has ending vertex \( g \prod_i s(x_i)^{\epsilon_i} \in G \). There are these consequences.

- A word determines a loop in \( \Gamma \) if and only if the corresponding product in \( G \) is the identity. Such a word is called a relation.

- \( \Gamma \) is path-connected (in the sense that any two vertices can be joined by an edge-path) if and only if the group \( G \) is generated by the elements \( s(x) \) (\( x \in x \)).

**Symmetry:** \( G \)-Action on \( \Gamma \) The group \( G \) acts on the graph \( \Gamma \). This means that there are (left) actions by \( G \) on the vertex and edge sets that respect the initial and terminal maps:

\[
\begin{align*}
(0) \quad \iota(g \cdot (h, x)) &= g \cdot \iota(h, x) \\
(1) \quad \tau(g \cdot (h, x)) &= g \cdot \tau(h, x)
\end{align*}
\]

These requirements should be interpreted as a continuity condition for \( G \) acting on the graph structure. One checks that such a \( G \)-action is given on vertices by

\[ g \cdot h = gh \]

and on (positive) edges by
\[ g \cdot (h, x) = (gh, x). \]

As with any action, we investigate stabilizers and orbits. Stabilizers are trivial, literally. Namely, if an element \( g \in G \) fixes either a vertex or an edge must be the identity: \( g = 1 \).

\[
\begin{align*}
  g \cdot h &= h \quad \Rightarrow \quad g = 1 \\
  g \cdot (h, x) &= (h, x) \quad \Rightarrow \quad g = 1
\end{align*}
\]

This is summarized by saying that \( G \) acts freely on \( \Gamma \). As for orbits, there is only one orbit of vertices: given vertices \( h, k \in G \), there is a group element \( g \in G \) such that \( g \cdot h = k \). Namely, \( g = kh^{-1} \). This is summarized by saying that \( G \) acts transitively on the vertices of \( \Gamma \). As for edges, the \( G \)-action preserves edge labels and orientations and any two positive edges with the same label \( x \in \mathbf{x} \) lie in the same orbit. (The argument is the same as for transitivity on vertices.) In other words, the set of orbits of positive edges is in 1-1 correspondence with the set \( \mathbf{x} \).

**The Orbit Graph and the Orbit Map** Group actions determine orbit structures. So we should think about the orbit graph that is determined by the \( G \)-action on the Cayley graph \( \Gamma = \Gamma(G, \mathbf{x}, s) \). The vertex set of the orbit graph \( G \backslash \Gamma \) should be the set of orbits of the \( G \)-action on \( \Gamma \). The set of positive edges of the orbit graph should be the set of orbits of the \( G \)-action on the positive edges of \( \Gamma \). We are led to consider the Cayley graph \( \Gamma(\{1\}, \mathbf{x}, \sigma) \) that has only one vertex and one positive edge for each label \( x \in \mathbf{x} \).

You should draw this object now.

We can even see a natural graph morphism that connects \( \Gamma(G, \mathbf{x}, s) \to \Gamma(\{1\}, \mathbf{x}, \sigma) \), taking vertices to vertices and positive edges to positive edges with the same label. This arises from functions on vertices and edges that respect the graph structures (initial and terminal maps) as in a sense similar to the continuity equations (0) and (1). Moreover, the One-In-One-Out lemma shows that the local structure at any vertex in either graph is the same (aka evenly covered). Surely, it ought to be possible to make enough definitions to make sense of the following statement.

**Theorem 1.4** The orbit map \( \pi : \Gamma \to G \backslash \Gamma \) is a covering projection with automorphism group \( \text{Aut}(\pi) \cong G \).
2 The Free Group

In view of the relation between covering spaces and fundamental groups, the discussion in the previous section suggests that we should build an honest-to-goodness topological space modeled on the combinatorial structure of $\Gamma(\{1\}, x)$. (We need not name the function from $x$ to the trivial group $\{1\}$.)

Then we should ask about the fundamental group of that space.

The One-Point Union of Circles

Let $x$ be a set and let $I = [0, 1]$ be the unit interval. Consider the product space $I \times x$, where $x$ is given the discrete topology. For each $x \in bfx$, we have the slice $I_x = I \times \{x\} \subseteq I \times x$.

Lemma 2.1 The product $I \times x$ is the disjoint union of the slices $I_x$, $x \in x$. As a subspace of the product, each slice $I_x = I \times \{x\}$ is homeomorphic to the unit interval $I$ by the map $(s, x) \mapsto s$. Finally, the product $I \times x$ has the weak topology with respect to the slices in the sense that the following two statements describe the open and closed sets:

- $U \subseteq^op I \times x \iff U \cap I_x \subseteq^op I_x \forall x \in x$
- $F \subseteq^cl I \times x \iff F \cap I_x \subseteq^cl I_x \forall x \in x$

Proof: For the claim about weak topology, note that each slice $I_x = I \times \{x\}$ is an open and closed subset of the product $I \times x$ since $x$ is discrete.

Let $W(x)$ be the quotient space obtained from $I \times x$ by identifying all endpoints to a single point. Put another way, there is an equivalence relation $\sim$ on $I \times x$ where

$$(s, x) \sim (t, y) \iff s, t \in \{0, 1\}.$$ 

The quotient space $W(x)$ is then the set of equivalence classes and is topologized via the quotient map $q$ that assigns each point of $I \times x$ to its equivalence class under $\sim$:

$$q : I \times x \to (I \times x)/\sim = W(x).$$

Some notation and terminology will help us to investigate the topology. We define:

$$c^0 = q(0, x) = q(1, y) \quad \text{(0-cell)}$$
$$c^1_x = q((0, 1) \times \{x\}) \quad \text{(open 1-cell)}$$
$$\bar{c}^1_x = q(I \times \{x\}) = q(I_x) \quad \text{(closed 1-cell)}$$
It is convenient to regard the singleton \( \{ c^0 \} \) as both an open and a closed 0-cell. The usage of the words open and closed in connection with 1-cells is justified within the following lemma. It also explains the sense in which \( W(x) \) is a **one-point union of circles**.

**Lemma 2.2** Let \( W = W(x) \).

1. The open cells form a partition of \( W = \{ c^0 \} \cup \bigcup_x c_x^1 \) and distinct open cells are disjoint.
2. \( \bar{c}_x^1 - c_x^1 = \{ c^0 \} \)
3. \( W = \bigcup_x \bar{c}_x^1 \) and \( \bar{c}_x^1 \cap \bar{c}_y^1 = \{ c^0 \} \) if \( x \neq y \).
4. \( W \) has the weak topology with respect to the closed cells. That is, a subset \( F \) of \( W \) is closed in \( W \) if and only if its intersection with each closed cell is closed in that cell.
5. Given \( A \subseteq I \times x \),

\[
q^{-1}(q(A)) = \begin{cases} 
A & \text{if } A \cap (\{0,1\} \times x) = \emptyset \\
A \cup (\{0,1\} \times x) & \text{if } A \cap (\{0,1\} \times x) \neq \emptyset 
\end{cases}
\]
6. The quotient map \( q \) is a continuous closed surjection.
7. The quotient map \( q \) restricts to a homeomorphism \((0,1) \times \{x\} \xrightarrow{\sim} c_x^1\) from an open interval onto an open 1-cell. Each open 1-cell is an open subset of \( W \).
8. The quotient map \( q \) restricts to a continuous closed surjection \( I_x \to \bar{c}_x^1 \) that simply identifies the endpoints of the interval \( I_x \). Thus each closed 1-cell of \( W \) is homeomorphic to the circle \( S^1 \).
9. The closed 0-cell \( \{ c^0 \} \) is closed in \( W \). The closed 1-cell \( \bar{c}_x^1 \) is the closure in \( W \) of the corresponding open 1-cell \( c_x^1 \). Thus closed 1-cells are closed subsets of \( W \).
10. A neighborhood basis of open sets containing \( c^0 \) consists of sets of the form

\[
q \left( \bigcup_{x \in x} [0, \epsilon_x) \cup (1 - \epsilon_x, 1] \right)
\]

where \( \epsilon_x > 0 \) for all \( x \in x \).
11. $W$ is Hausdorff.

12. $W$ is connected and locally path connected.

**Proof:** Much of this involves only an understanding of the relatively simple identifications that are made by $q$. To prove statement 4, suppose that $F \cap \bar{c}_x^{1} \subseteq^{cl} \bar{c}_x^{1}$ for all $x \in x$. This implies that $q^{-1}(F) \cap I_x \subseteq^{cl} I_x$ for all $x \in x$ by continuity of the restricted map $q : I_x \to \bar{c}_x^{1}$. Thus by Lemma 2.1, it follows that $q^{-1}(F) \subseteq^{cl} I \times x$ and hence $F \subseteq^{cl} W$ since $q : I \times x \to W$ is a quotient map. The analogous statement for open sets is a consequence. Statement 10 involves an understanding of the topology of the unit interval and in turn enables a proof of statement 11.

**Definition 2.3** For a set $x$, the free group with basis $x$ is the fundamental group of the one-point union of circles indexed by $x$:

$$F(x) = \pi_1(W(x), e^0).$$

There is the function $s : x \to F(x)$ that assigns to each $x \in x$ the path homotopy class of the loop in $W(x)$ given by $t \mapsto q(t, x)$ for all $t \in I$. This loop wraps once around the closed 1-cell $\bar{c}_x^{1} \approx S^1$: $s(x) = [q|_{I_x}]$.

See if you can draw the Cayley graph $\Gamma(F(x), x, s)$!

To study the group $F(x)$ we need to study the topology of $W(x)$. Here is an important feature.

**Theorem 2.4 (Compact Supports)** Any compact subspace of $W(x)$ is contained in a finite union of closed cells.

**Proof:** Let $A$ be a compact subspace of $W = W(x)$. We can select a subset $Y$ of $A$ such that $Y$ contains exactly one point from each open cell that has non-empty intersection with $A$. It suffices to show that $Y$ is finite. We argue that $Y$ is finite by showing that $Y$ is a closed and discrete subset of $W$, and hence of the compact space $A$. For any subset $y$ of $Y$ and any $x \in x$, the intersection $y \cap \bar{c}_x^{1}$ contains at most two points since $\bar{c}_x^{1} = c_x^{1} \cup \{e^0\}$ is the union of two open cells. This implies that $y \cap \bar{c}_x^{1}$ is closed in the Hausdorff space $\bar{c}_x^{1}$. Since $W$ has the weak topology with respect to its closed cells, this implies that $y$ is a closed subspace of $W$. Thus every subset of $Y$ is closed in $W$, which implies that $Y$ is closed and discrete in $W$, and hence in $A$. 

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Corollary 2.5 The following are equivalent for $W = W(x)$.

1. $x$ is finite;
2. $W$ is compact;
3. $W$ is metrizable;
4. $W$ is first countable.

Theorem 2.6 The elements $s(x)$ generate $F(x)$.

Proof: Use the easy half of the Seifert-van Kampen Theorem.

This means that every element of $F = F(x)$ can be represented by a word in $x^\pm 1$. Insertion or deletion of a syllable of the form $x^\epsilon x^{-\epsilon}$ from a word does not affect the corresponding element of $F$. A word is freely reduced if no such deletions are possible. Otherwise, the word is freely reducible.

Lemma 2.7 Every element of $F$ can be represented by a freely reduced word. (The empty word is freely reduced and represents the identity).

Proof: Induction on the minimum number of letters in a representative word.

Theorem 2.8 If $w$ is a freely reduced word in $x^\pm 1$, then there is a finite-index subgroup $H$ of $F = F(x)$ such that the corresponding element $s(w)$ of $F$ does not lie in $H$.

Proof: Build a finite-sheeted covering in which a loop of $W$ that reads $w$ does not lift to a loop! That’s it.

Corollary 2.9 (Word Problem Solution for $F$) Any word in $x^\pm 1$ that represents the identity element of $F$ is freely reducible.

Corollary 2.10 Two freely reduced words that represent the same element of $F$ are identical.

Corollary 2.11 $\Gamma(F(x), x, s)$ is a tree.

Corollary 2.12 $F$ is residually finite: If $1 \neq u \in F$, then there is a finite group $G$ and a homomorphism $\varphi : F \rightarrow G$ such that $\varphi(u) \neq 1$ in $G$. Put another way, the intersection of all finite index (normal) subgroups of $F$ is trivial.