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On Shift Dynamics for Cyclically Presented Groups

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Abstract

A group defined by a finite presentation with cyclic symmetry admits a shift automorphism that is periodic and word-length preserving. It is shown that if the presentation is combinatorially aspherical and orientable, in the sense that no relator is a cyclic permutation of the inverse of any of its shifts, then the shift acts freely on the non-identity elements of the group presented. For cyclic presentations defined by positive words of length at most three, the shift defines a free action if and only if the presentation is combinatorially aspherical and the shift itself is fixed point free if and only if the group presented is infinite. **MSC** (2010) 20F05, 20E36. **Keywords:** cyclically presented group, shift, fixed point free, asphericity.

1 Cyclically Presented Groups

A celebrated theorem of J. Thompson [28] on Frobenius kernels states that any finite group that possesses a fixed point free automorphism of prime order must be nilpotent. A combinatorial template for a group with a periodic automorphism is one that admits a **cyclic presentation** of the form

$$\mathcal{P}_n(w) = \big(x_0, \ldots, x_{n-1} \ : w, \theta(w), \ldots, \theta^{n-1}(w)\big)$$

where n is a positive integer, $w = w(x_0, \ldots, x_{n-1})$ is an arbitrary word in the generators $x_0, \ldots, x_{n-1}$, and θ is the **shift** automorphism given by $\theta(x_i) = x_{i+1}$ with subscripts modulo n. The shift θ defines an automorphism of exponent n for the **cyclically presented group** $G_n(w)$ defined by $\mathcal{P}_n(w)$. A fundamental problem in the study of cyclic presentations is to determine whether $G_n(w)$ is finite, and if so, to determine its structure. Thompson's theorem has the following consequence related to the dynamics of the shift action by the cyclic group C_n of order n: *If $G_n(w)$ is finite and C_n acts freely via the shift, then $G_n(w)$ is nilpotent.* This is because freeness of the action is enough to guarantee that $G_n(w)$ has a fixed point free automorphism of prime order, so Thompson's theorem applies. For example, the group $G_5(x_0x_2x_1^{-1})$ is the binary icosahedral group [25, Section 5], which is perfect, and so its shift must have a nonidentity fixed point. Further examples are given in [12, Theorem B], which provides an infinite family of finite metacyclic nonabelian (and therefore non-nilpotent) cyclically presented groups. Section 5 below shows how to identify explicit fixed points for the shift on these groups.

The main general result of this paper provides sufficient conditions under which the shift determines a free action on the nonidentity elements of a cyclically presented group. The principal condition is **combinatorial asphericity**; a working definition from [5] is presented in Proposition 3.1 below. A cyclic presentation $\mathcal{P}_n(w)$ is **orientable** if w is not a cyclic permutation of the inverse of any of its shifts. It turns out that $\mathcal{P}_n(w)$ is not orientable if and only if $n = 2m$ is even and there is a word $u = u(x_0, \dots, x_{n-1})$ such that $w = u\theta^m(u)^{-1}$ (Lemma 3.6). Note that u is fixed by $\theta^m \in \mathrm{Aut}(G_{2m}(u\theta^m(u)^{-1}))$.

Theorem A *If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then C_n acts freely via the shift on the nonidentity elements of $G_n(w)$.*

Remark It can happen that $G_n(w)$ is trivial, in which case $\mathcal{P}_n(w)$ is combinatorially aspherical and orientable. See [10, 11] for the current state of this problem.

Theorem A says that when the cyclic presentation $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, the orbits of the shift partition the nonidentity elements of $G_n(w)$ into pairwise disjoint n-cycles. In particular, θ and all of its powers are fixed point free and θ has order n in the automorphism group of $G_n(w)$. (Compare e. g. [4, Corollary 3.2],[19, Theorem 2].) Another way to view this is that if some power of the shift has a nonidentity fixed point, then the presentation $\mathcal{P}_n(w)$ must support an interesting spherical diagram [5]. Perhaps one could prove Theorem A in this way, but that is not the path followed here. Instead, the proof of Theorem A employs the cohomology theory of aspherical relative presentations [3] applied to the split extension $E_n(w) = G_n(w) \rtimes_\theta C_n$ (Theorem 3.4).

Now $E_n(w)$ admits a presentation of the form $(a, x \;:\; a^n, W)$ and $G_n(w)$ is the kernel of a retraction of $E_n(w)$ onto C_n. Depending on n and w, there can be several different retractions $\nu^f : E_n(w) \to C_n$, each determined by the image $\nu^f(x) = a^f \in C_n$. Section 2 outlines a Reidemeister-Schreier rewriting process $\rho^f(W)$ to show that the kernel of such a retraction is cyclically presented (Theorem 2.3). The original word w is recovered as (a shift of) $\rho^0(W)$. As f ranges through allowable values there arise families of commensurable cyclically presented groups. These commensurable groups need not be isomorphic, but they do have identical dynamics under their respective shifts (Lemma 2.2). Even in cases where $E_n(w)$ is finite, these arguments are useful in identifying the structure of $G_n(w)$ (Lemma 5.5).

It has been known for some time that there is a close connection between asphericity for cyclic presentations $\mathcal{P}_n(w)$ and for relative presentations of $E_n(w)$. The following result articulates this connection in a definitive way and is of independent interest.

Theorem 4.1 *Let M be the cellular model of a relative presentation $(C_n, x : W)$ for a group E where C_n is the cyclic group of order n generated by a. Suppose that $\nu^f : E \to C_n$ is a retraction given by $\nu^f(a) = a$ and $\nu^f(x) = a^f$. Let $w = \rho^f(W)$.*

(a) If $\pi_2 M = 0$, then $\mathcal{P}_n(w)$ is combinatorially aspherical.

(b) If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then $\pi_2 M = 0$.

N. D. Gilbert and J. Howie established a special case of Theorem 4.1(a) in [15, Lemma 3.1]. Their picture-theoretic argument was adapted to other classes of presentations in [2, 4, 27]. Theorem 4.1 subsumes and broadens the scope of these results in several ways, first by eliminating any restriction on the length or shape of the cyclic relators, second by encompassing the more general concept of combinatorial asphericity for cyclic presentations, and third by establishing the converse result Theorem 4.1(b), which in turn underpins Theorem A.

The converse to Theorem A holds for a class of cyclically presented groups that was studied by M. Edjvet and G. Williams [12]. Given integers k and l modulo a positive integer n, let $G_n(k,l)$ be the (nontrivial) group given by the (orientable) cyclic presentation

$$\mathcal{P}_n(k,l) = \mathcal{P}_n(x_0 x_k x_l)$$

that has generators $x_0, \ldots, x_{n-1}$ and relators of the form $x_i x_{i+k} x_{i+l}$ for $i = 0, \ldots, n-1$.

Theorem B *The cyclic group C_n acts freely via the shift on the nonidentity elements of $G_n(k,l)$ if and only if the presentation $\mathcal{P}_n(k,l)$ is combinatorially aspherical.*

Theorem C *The group $G_n(k,l)$ is infinite if and only if the shift θ is fixed point free on its nonidentity elements.*

Theorems B and C deal with cyclic presentations $\mathcal{P}_n(w)$ defined by arbitrary positive relator words w of length three. Both results hold verbatim for cyclic presentations defined by positive relator words w of length two, but subtleties arise for longer positive words of composite length. A positive word $w = w(x_0, \ldots, x_{n-1})$ is **decomposable** if it has a subword u of length greater than one such that w is freely expressible in terms of u and its shifts. Letting $u_k = \theta^k(u)$ and substituting, the word $w(u_0, \ldots, u_{n-1})$ is shorter than $w(x_0, \ldots, x_{n-1})$. Levin's theorem [20] implies that $G_n(w(u_0, \ldots, u_{n-1}))$ embeds in $G_n(w(x_0, \ldots, x_{n-1}))$, thus establishing an inductive framework. For example, $\mathcal{P}_3(x_0^2 x_1^2) = \mathcal{P}_3(u\theta(u))$ is decomposable by $u = x_0^2$. The group $G_3(x_0^2 x_1^2)$ is infinite but the order two fixed point $x_0^2 = x_1^{-2} = x_2^2 = x_0^{-2}$ is a member of the subgroup $G_3(u_0 u_1) \cong C_2$ generated by $u_0 = x_0^2$ and $u_1 = x_1^2$. Efforts to extend Theorems B and C are centered around the hypothesis that both theorems should be true for indecomposable positive relator words.

The converse to Theorem A is not true for more general orientable cyclic presentations. For example, the "original" Fibonacci group [6, 7] was $G_5(x_0x_1x_2^{-1})$, which turned out to be cyclic of order 11. This cyclic presentation is not combinatorially aspherical, but the orbits of the shift partition the nonidentity elements of $G_5(x_0x_1x_2^{-1})$ into two disjoint 5-cycles [14]. Much remains to be learned about the dynamics of the shift on cyclically presented groups.

The outline of the paper is as follows. Section 2 explains how the kernel of any retraction of the extension $E_n(w) = G_n(w) \rtimes_\theta C_n$ onto C_n has cyclically presented kernel and that the shift dynamics are independent of the choice of retraction. Section 3 relates the cohomology theory of relative presentations to fixed points of the shift. Example 3.5 and Lemma 3.6 detail the role of orientability for both cyclic and relative presentations. The main general results, Theorem A and Theorem 4.1, are proved in Section 4. Section 5 tests the strength of these results, where the proofs of Theorems B and C describe the shift dynamics for the cyclically presented groups $G_n(k, l) = G_n(x_0x_kx_l)$ of [12].

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2 Retractions

Given a cyclic presentation $\mathcal{P}_n(w)$, the split extension $E_n(w) = G_n(w) \rtimes_\theta C_n$ has a two-generator two-relator presentation of the form $(x, a : a^n, W(a, x))$ where $W(a, x)$ is obtained from $w(x_0, \ldots, x_{n-1})$ by rewriting $x_0 = x$ and $x_i = a^i x a^{-i}$ for $i = 1, \ldots, n-1$. This is because the rewrites of the shifts $\theta^v(w)$ are pairwise conjugate in the free group on a and x. There is a retraction $\nu = \nu^0 : E_n(w) \to C_n$ with kernel $G_n(w)$ given by $\nu(a) = a$ and $\nu(x) = a^0 = 1$, but there can be other retractions.

Example 2.1 If $m \geq 1$ and $n = 3m$, then the split extension $E = E_{3m}(x_0x_1x_m)$ is presented by $(a, x : a^{3m},\ xaxa^{m-1}xa^{-m})$ and retracts onto the cyclic subgroup C_{3m} via maps $\nu^f : E \to C_{3m}$ satisfying $\nu^f(a) = a$ and $\nu^f(x) = a^f$ where $f = 0, \pm m$. •

Theorem 2.3 below shows that the kernel of any retraction of $E_n(w)$ onto C_n has cyclically presented kernel and that the shift on that kernel is induced by the conjugation action of C_n in $E_n(w)$. Any such kernel has index n in $E_n(w)$ and the following lemma shows that the shift dynamics are independent of the choice of retraction.

Lemma 2.2 *Suppose that $\nu : E \to C$ is a retraction of a group E onto a subgroup C. Thus $\nu(c) = c$ for all $c \in C$. Let G denote the kernel of ν so that $E \cong G \rtimes C$. Then $E/C \cong G$ as left C-sets with $1C \in E/C$ corresponding to $1 \in G$.*

Proof: Now C acts on G by (left) conjugation (that is, $c \cdot g = cgc^{-1}$) and on the coset space E/C by left multiplication (that is, $c \cdot wC = cwC$). The desired isomorphism is given by the C-equivariant function $E/C \to G$ that maps $wC \in E/C$ to $w \cdot \nu(w)^{-1} \in G$ for each $w \in E$. •

Cyclic presentations for retraction kernels arise from the following Reidemeister-Schreier rewriting process [21, Theorem 2.9, page 94] for words of the form

$$W(a,x) = x^{\epsilon_1}a^{p_1}\dots x^{\epsilon_L}a^{p_L} \tag{1}$$

where $L \geq 1$, $\epsilon_i = \pm 1$, and the p_i are integers. Given an integer f, the process involves two integer-valued functions. The first function is defined by $v(1) = 0$ and

$$v(i) = \sum_{j=1}^{i-1} \epsilon_j f + p_j.$$

for $i = 2, \dots, L$. The second function is defined in terms of the first by

$$u(i) = v(i) + \frac{\epsilon_i - 1}{2} f = \begin{cases} v(i) & \text{if } \epsilon_i = 1 \\ v(i) - f & \text{if } \epsilon_i = -1. \end{cases}$$

Given the integer f and a word $W(a,x)$, the rewritten word $\rho^f(W(a,x))$ is defined to be

$$\rho^f(W(a,x)) = x_{u(1)}^{\epsilon_1}\dots x_{u(L)}^{\epsilon_L}.$$

If $W(a,x)$ is cyclically reduced, then so is the rewrite $\rho^f(W(a,x))$. Note that the rewrite is independent of the exponent p_L. Some examples:

$$\begin{aligned} \rho^f(xa^pxa^qxa^r) &= x_0x_{f+p}x_{2f+p+q}, \\ \rho^f(x^2a^pxa^q) &= x_0x_fx_{2f+p}, \\ \rho^f(x^2a^px^{-1}a^q) &= x_0x_fx_{f+p}^{-1}, \\ \rho^0(x^3) &= x_0^3, \\ \rho^1(x^3) &= x_0x_1x_2. \end{aligned}$$

When a positive integer n is fixed and $a^n = 1$, $W(a,x)$ determines $W \in C_n * \langle x \rangle$. By interpreting subscripts modulo n, the rewriting process yields a word $\rho^f(W)$ on the alphabet $\{x_0, \dots, x_{n-1}\}$. In its turn, the cyclic presentation $\mathcal{P}_n(\rho^f(W))$ is defined and depends on n. Thus $\mathcal{P}_6(\rho^2(x^3)) = \mathcal{P}_6(x_0x_2x_4) = \mathcal{P}_6(2,4)$ (see Section 1), whereas $\mathcal{P}_3(\rho^2(x^3)) = \mathcal{P}_3(x_0x_2x_1) = \mathcal{P}_3(2,1)$.

Theorem 2.3 *Let $(a, x : a^n,\ W(a,x))$ be a presentation for a group E. If $\nu^f : E \to C_n$ is a retraction given by $\nu^f(a) = a$ and $\nu^f(x) = a^f$, then the element $x_0 = xa^{-f}$ normally generates $\ker \nu^f$ in E and the conjugates $x_i = a^ix_0a^{-i}$ $(0 \leq i \leq n-1)$ form a generating set in a cyclic presentation $\mathcal{P}_n(\rho^f(W))$ for $\ker \nu^f$. In particular,*

$$E \cong G_n(\rho^f(W)) \rtimes_\theta C_n$$

and $\theta(g) = aga^{-1}$ *in* E *for all* $g \in \ker \nu^f$*.*

Proof: With $W(a,x) = x^{\epsilon_1}a^{p_1} \ldots x^{\epsilon_L}a^{p_L}$, the existence of the retraction ν^f simply means that $\sum_{j=1}^{L} \epsilon_j f + p_j$ is divisible by n. This being the case, the rewriting process ρ^f describes defining relations for the kernel of ν^f as in the Reidemeister-Schreier process. See the proof of Theorem 4.1 for a topological rendition of the argument. •

3 Asphericity

A topological space Y is **aspherical** if each spherical map $S^k \to Y$, $k \geq 2$, is homotopic to a constant map. A connected aspherical CW complex with fundamental group G is a **$K(G,1)$-complex**; its homotopy type is uniquely determined by G. The **cellular model** of a group presentation $(\mathbf{x} : \mathbf{r})$ is a connected two-dimensional CW complex K whose one-skeleton $K^{(1)}$ consists of a single zero-cell $*$ together with a single one-cell c^1_x for each generator $x \in \mathbf{x}$. Selection of an orientation for each one-cell determines an isomorphism $\pi_1 K^{(1)} \cong F(\mathbf{x})$ of the fundamental group of the one-skeleton $K^{(1)}$ with the free group $F = F(\mathbf{x})$ with basis $\mathbf{x}$. The cellular model K has a two-cell c^2_r for each relator $r \in \mathbf{r}$ whose boundary attaching map spells the relator $r \in F$. The fundamental group of the cellular model is isomorphic to the group given by the presentation. See [26] for more details.

Combinatorial Asphericity Identity sequences [5, 24] describe spherical maps into cellular models of group presentations. Given a presentation $(\mathbf{x} : \mathbf{r})$ for a group G, the free group $F = F(\mathbf{x})$ with basis $\mathbf{x}$ acts on the free group $\mathbb{F} = \mathbb{F}(\mathbf{x} : \mathbf{r}) = F(F \times \mathbf{r})$ with basis $F \times \mathbf{r}$ by $v \cdot (u,r) = (vu, r)$. The homomorphism $\partial : \mathbb{F} \to F$ given by $\partial(u,r) = uru^{-1}$ is F-equivariant where F acts on itself by (left) conjugation. The image of ∂ is the normal closure of $\mathbf{r}$ in F so the cokernel of ∂ is isomorphic to G. Elements of the kernel $\mathbb{I} = \mathbb{I}(\mathbf{x} : \mathbf{r}) = \ker \partial$ are called **identity sequences** for the presentation $(\mathbf{x} : \mathbf{r})$ [5]. Given $u, v \in F$, $r, s \in \mathbf{r}$, and $\epsilon, \delta = \pm 1$, there is the **Peiffer identity** $(u,r)^\epsilon (v,s)^\delta (u,r)^{-\epsilon} (ur^\epsilon u^{-1} v, s)^{-\delta} \in \mathbb{I}(\mathbf{x} : \mathbf{r})$. Let $\mathbb{P} = \mathbb{P}(\mathbf{x} : \mathbf{r})$ denote the normal closure in $\mathbb{F}$ of the set of all Peiffer identities. The F-action on $\mathbb{F}$ descends to a G-action on the abelian quotient group $\mathbb{I}/\mathbb{P}$. The following is a working definition of combinatorial asphericity.

Proposition 3.1 ([5, Proposition 1.4]) *A presentation* $(\mathbf{x} : \mathbf{r})$ *for a group* G *is* ***combinatorially aspherical*** *if and only if* $\mathbb{I}(\mathbf{x} : \mathbf{r})/\mathbb{P}(\mathbf{x} : \mathbf{r})$ *is generated as a* $\mathbb{Z}G$*-module by (classes of) length two identity sequences, that is by identity sequences of the form*

$$(u,r)^\epsilon (v,s)^\delta$$

where $u, v \in F$, $r, s \in \mathbf{r}$, and $\epsilon, \delta = \pm 1$.

Given a presentation $(\mathbf{x} : \mathbf{r})$ for a group G and having cellular model K, there is an isomorphism

$$\pi_2 K \cong \mathbb{I}(\mathbf{x} : \mathbf{r})/\mathbb{P}(\mathbf{x} : \mathbf{r})$$

of $\mathbb{Z}G$-modules. (See [24, Section 2] or [26], where this is attributed to K. Reidemeister [23].) The homotopy element $\sigma\mathbb{P} \in \mathbb{I}/\mathbb{P}$ is sometimes denoted $[\sigma]$. Since K is two-dimensional, it follows that K is aspherical (in the topological sense) if and only if $\mathbb{I}(\mathbf{x} : \mathbf{r}) = \mathbb{P}(\mathbf{x} : \mathbf{r})$. This implies that a presentation with aspherical cellular model is combinatorially aspherical. However, the cellular model of a combinatorially aspherical presentation can support essential spheres in two different ways.

A relator $r \in \mathbf{r}$ is a **proper power** if it is freely equal to $r = \dot{r}^e$ where $e > 1$. When e is maximal it is the **exponent** of r and $\dot{r}$ is the **root**. A proper power relator gives rise to an identity sequence $(1, r)(\dot{r}, r)^{-1}$.

Example 3.2 *It is well known that the presentation $(a : a^n)$ for the cyclic group C_n of order n is combinatorially aspherical and that*

$$\mathbb{I}(a : a^n)/\mathbb{P}(a : a^n) = \mathbb{Z}C_n \cdot [\Delta_n] \quad \textit{where} \quad \Delta_n = (1, a^n)(a, a^n)^{-1}.$$

There is a $K(C_n, 1)$-complex D_n having just a single three-cell with attaching map $S^2 \to D_n^{(2)}$ determined by the identity sequence $\Delta_n \in \mathbb{I}(a : a^n)$. •

A relator of $(\mathbf{x} : \mathbf{r})$ is **freely redundant** if it is freely trivial or else is freely conjugate to another relator or its inverse. No relator in a combinatorially aspherical presentation can be freely trivial (as every identity sequence has even length), but the concept does allow for the possibility that one relator r is freely conjugate to another relator s or its inverse: $r = vs^{\delta}v^{-1}$. This leads to identity sequences of the form $(1, r)(v, s)^{-\delta}$. When r and s are cyclically reduced, this simply means that r is a cyclic permutation of s^{δ} (and v is an initial segment of r). Identity sequences arising from proper power or redundant relators are called Peiffer identities of the second kind in [5, 18].

To illustrate the possibilities, the cyclic presentation

$$\mathcal{P}_3(1, 2) = \mathcal{P}_3(x_0x_1x_2) = (x_0, x_1, x_2 : x_0x_1x_2, x_1x_2x_0, x_2x_0x_1)$$

for the free group of rank two has two freely redundant relators. In the cyclic presentation $\mathcal{P}_n(0, 0) = \mathcal{P}_n(x_0^3)$ for the free product of n copies of the cyclic group of order three, each relator has exponent three and root of length one. It is not difficult to show that the presentations $\mathcal{P}_3(x_0x_1x_2)$ and $\mathcal{P}_n(x_0^3)$ are combinatorially aspherical.

Aspherical Relative Presentations Consider a group presentation of the form

$$\mathcal{Q} = (a, x : a^n, W(a, x)).$$

Viewing $W(a, x)$ as an element W in the free product $C_n * \langle x \rangle$ of C_n with the infinite cyclic group generated by x, there is the **relative presentation**

$$\mathcal{R} = (C_n, x : W)$$

with **coefficient group** C_n, in the sense of [3]. Conversely, the ordinary presentation $\mathcal{Q}$ is a **lift** of $\mathcal{R}$. Both $\mathcal{Q}$ and $\mathcal{R}$ define the same group:

$$E = F(a, x)/\langle\langle a^n, W(a, x)\rangle\rangle \cong (C_n * \langle x \rangle)/\langle\langle W \rangle\rangle.$$

A **cellular model** for the relative presentation $\mathcal{R}$ was described in [3, Section 4]. Working in the free product $C_n * \langle x \rangle$, first write $W = \dot{W}^e$ where $\dot{W} \in C_n * \langle x \rangle$ and e is maximal, so again $\dot{W}$ is the **root** and e is the **exponent** of W. Let D_n be a $K(C_n, 1)$-complex with two-skeleton modeled on the presentation $(a : a^n)$ and having just a single three-cell as in Example 3.2. In the same way, let D_e be a $K(C_e, 1)$-complex, unless $e = 1$, in which case D_e is a disc. Form the one point union $D_n \vee S^1_x$ with fundamental group $\pi_1(D_n \vee S^1_x) \cong C_n * \langle x \rangle$. Now the one-skeleton $D_e^{(1)} = S^1_e$ is a circle and there is a loop $\omega : S^1_e \to D_n \vee S^1_x$ that reads the root $\dot{W} \in C_n * \langle x \rangle$. The cellular model of $\mathcal{R}$ is the pushout M obtained from the union of $D_n \vee S^1_x$ and D_e by identifying points of $D_e^{(1)} = S^1_e$ with their images in $D_n \vee S^1_x$ under ω:

$$M = \left(D_n \vee S^1_x\right) \cup_{z \sim \omega(z)} D_e.$$

The model M has fundamental group $\pi_1 M = E$ and contains the $K(C_n, 1)$-complex D_n as a subcomplex. When C_n embeds in E and M is a $K(E, 1)$ complex, cohomology calculations produce the main theoretical consequences of asphericity for relative presentations. See [3, Section 4].

Theorem 3.3 *Let M be the cellular model of a relative presentation $(C_n, x : W)$ for a group E, where C_n is a cyclic group of order n. Assume that M has trivial second homotopy group $\pi_2 M = 0$ and that the natural homomorphism $C_n \to E$ is injective. Then M is (topologically) aspherical and the following conclusions hold:*

(a) The root $\dot{W}$ generates a subgroup C_e of E with order equal to the exponent e of W in $C_n * \langle x \rangle$.

(b) Given any left ZE-module N and any $k \geq 3$, the canonical map $H^k(E; N) \to H^k(C_n; N) \times H^k(C_e; N)$ is an isomorphism.

(c) Each finite subgroup of E is conjugate to a subgroup of either C_n or C_e.

(d) *The conjugates of* C_n *and* C_e *are all algebraically disjoint in* E. *More precisely, if* $g \in E$, *then the following conclusions hold:*

  (i) *If* $gC_ng^{-1} \cap C_n \neq 1$, *then* $g \in C_n$;

  (ii) *If* $gC_eg^{-1} \cap C_e \neq 1$, *then* $g \in C_e$; *and*

  (iii) $gC_eg^{-1} \cap C_n = 1$.

Proof: Given injectivity of $C_n \to E$ and the homotopy condition $\pi_2 M = 0$, standard arguments suffice to prove asphericity of M and the statements (a) and (b). The characterization of finite subgroups (c) follows from the cohomology calculation (b) by a theorem of J.-P. Serre that appeared in [18]. The conjugacy results in (d) can be extracted either from Serre's theorem or from [17, Theorem 5]. A nice alternative approach to (c) and (d) for the case where W is not a proper power is given in [13]. •

Theorem 3.4 *Let* M *be the cellular model of a relative presentation* $(C_n, x : W)$ *for a group* E, *where* C_n *is a cyclic group of order* n. *Assume that* M *has trivial second homotopy group* $\pi_2 M = 0$. *If* $\nu^f : E \to C_n$ *is any retraction onto the coefficient group* C_n, *then* C_n *acts freely via the shift on the nonidentity elements of the cyclically presented group* $G_n(\rho^f(W))$.

Proof: Suppose $g \in G_n(\rho^f(W))$ and $\theta^k(g) = g$ where $1 \leq k < n$. By Theorem 2.3, the shift θ on $G_n(\rho^f(W))$ is realized as conjugation by $a \in C_n$ in $E \cong G_n(\rho^f(W)) \rtimes_\theta C_n$. Thus $a^k g a^{-k} = g$ and hence $1 \neq a^k = g a^k g^{-1} \in C_n \cap gC_ng^{-1}$. By Theorem 3.3(d)(i), $g \in C_n \cap G_n(\rho^f(W)) = 1$. •

Using spherical pictures, the paper [3] introduced a geometric concept of asphericity for relative presentations. A relative presentation of the form $(C_n, x : W)$ is **orientable** (see [3, Sec. 1.1]) if W is not a cyclic permutation of its inverse. For orientable relative presentations, the picture-theoretic notion of asphericity implies the homotopy condition $\pi_2 M = 0$ for the cellular model, as well as injectivity of the coefficient group. (See [3, Theorems 4.1, 4.2].) There are many results on asphericity of (orientable) relative presentations [1, 3, 8, 9, 16, 22, 27] that can potentially be applied to the study of the shift on cyclically presented groups.

For nonorientable relative presentations, the conclusions of Theorem 3.3 are not available and there is a direct connection to fixed points of the shift for nonorientiable cyclic presentations.

Example 3.5 If a is the nontrivial element in a cyclic group C_2 or order two, and $[x, a] = xax^{-1}a^{-1}$ is the commutator, then the relative presentation $\mathcal{R} = (C_2, x : [x, a])$ for the direct product $E = C_2 \times \langle x \rangle$ is aspherical in the sense of [3], but is not orientable because $[x, a]^{-1} = [a, x] = [a^{-1}, x] = a^{-1}xax^{-1}$ is a cylic permutation of $[x, a] = xax^{-1}a^{-1}$. The group $C_2 \times \langle x \rangle$ admits a conjugacy relation $xC_2x^{-1} = C_2$, so the conclusions of Theorem

3.3(b,d) do not extend to nonorientable relative presentations that are aspherical in the sense of [3]. Moreover, it follows from [18, Theorem 3] that the ordinary presentation $\mathcal{Q} = \big(a, x : a^2, [x,a]\big)$ is not (combinatorially) aspherical. Indeed, $\mathcal{Q}$ supports an identity sequence

$$(1, a^2)(x, a^2)^{-1}(1, [x,a])(a, [x,a])$$

that arises from the free commutator identity $[x, a^2] = [x,a]a[x,a]a^{-1}$ and which determines a nontrivial homotopy class in $\pi_2 M$, where M is the cellular model of the relative presentation $\mathcal{R} = (C_2, x : [x,a])$.

Continuing with this same example, there is a retraction $\nu^0 : E = C_2 \times \langle x \rangle \to C_2$ given by $\nu^0(a) = a$ and $\nu^0(x) = a^0 = 1$. As in Theorem 2.3, the rewrite $\rho^0(xax^{-1}a^{-1}) = x_0 x_1^{-1}$ determines a cyclic presentation

$$\mathcal{P}_2(x_0 x_1^{-1}) = (x_0, x_1 : x_0 x_1^{-1}, x_1 x_0^{-1})$$

for the infinite cyclic group $\ker \nu^0 \cong G_2(x_0 x_1^{-1})$. This cyclic presentation is combinatorially aspherical but is not orientable because the relators are mutually inverse to one another. Finally, the shift acts as the identity on the infinite cyclic group $G_2(x_0 x_1^{-1})$. •

The next lemma characterizes nonorientability for cyclic presentations.

Lemma 3.6 *Let $w = w(x_0, \ldots, x_{n-1})$ be a nonempty cyclically reduced word. The cyclic presentation $\mathcal{P}_n(w)$ is nonorientable if and only if $n = 2m$ is even and $w = w(x_0, \ldots, x_{n-1})$ is freely equal to $u\theta^m(u)^{-1}$ for some linearly reduced word $u = u(x_0, \ldots, x_{n-1})$.*

Proof: If $n = 2m$ and $w = u\theta^m(u)^{-1}$ then $w\theta^m(w)$ is freely trivial so $\mathcal{P}_n(w)$ is not orientable. Conversely, suppose that $w\theta^v(w)$ is freely trivial where $1 \leq v < n$. Write $w = x_{p_1}^{\epsilon_1} \ldots x_{p_L}^{\epsilon_L}$, so that

$$x_{p_1}^{\epsilon_1} \ldots x_{p_L}^{\epsilon_L} x_{p_1+v}^{\epsilon_1} \ldots x_{p_L+v}^{\epsilon_L} = 1.$$

Since w is reduced, cancellation implies that $\epsilon_{L-j} = -\epsilon_{j+1}$ and $p_{L-j} \equiv p_{j+1} + v$ modulo n for $j = 0, \ldots, L-1$. The conditions on the ϵ's ensure that $L = 2M$ is even. The conditions on the p's show that $p_L \equiv p_1 + v \equiv p_L + 2v$ modulo n, which means that $n = 2v$ is even. Letting $u = x_{p_1}^{\epsilon_1} \ldots x_{p_M}^{\epsilon_M}$, which is linearly reduced, we have

$$u\theta^v(u)^{-1} = x_{p_1}^{\epsilon_1} \ldots x_{p_M}^{\epsilon_M} \left(x_{p_1+v}^{\epsilon_1} \ldots x_{p_M+v}^{\epsilon_M}\right)^{-1} = x_{p_1}^{\epsilon_1} \ldots x_{p_M}^{\epsilon_M} x_{p_M+v}^{-\epsilon_M} \ldots x_{p_1+v}^{-\epsilon_1} = w,$$

whence the result. •

4 Proof of Theorem A

Consider a cyclic presentation $\mathcal{P}_n(w)$ where $w = w(x_0, \dots, x_{n-1})$ is nonempty and cyclically reduced. Introduce $a^n = 1$ and rewrite $x_j = a^j x a^{-j}$ to obtain $W \in C_n * \langle x \rangle$ and a relative presentation $(C_n, x : W)$ for the split extension $E_n(w) = G_n(w) \rtimes_\theta C_n$. There is a retraction $\nu^0 : E_n(w) \to C_n$ and by Theorem 2.3 the kernel of ν^0 is given by the cyclic presentation $\mathcal{P}_n(\rho^0(W)) = \mathcal{P}_n(w)$. Assuming that $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, the fact that C_n acts freely via the shift on the nonidentity elements of $G_n(\rho^0(W)) = G_n(w)$ then follows from Theorem 3.4 and the following result.

Theorem 4.1 *Let M be the cellular model of a relative presentation $(C_n, x : W)$ for a group E where C_n is the cyclic group of order n generated by a. Suppose that $\nu^f : E \to C_n$ is a retraction given by $\nu^f(a) = a$ and $\nu^f(x) = a^f$. Let $w = \rho^f(W)$.*

(a) If $\pi_2 M = 0$, then $\mathcal{P}_n(w)$ is combinatorially aspherical.

(b) If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then $\pi_2 M = 0$.

Proof: The retraction $\nu^f : E \to C_n$ determines a regular covering projection $p : \widehat{M} \to M$ with automorphism group $\mathrm{Aut}(p) \cong C_n$ that freely permutes the cells lying over any given cell of the cellular model $M = (D_n \vee S^1_x) \cup D_e$. (Here, as before, e is the exponent of the word $W \in C_n * \langle x \rangle$.) The pre-image $p^{-1}(D_n)$ is isomorphic to the universal covering complex $\widetilde{D}_n$ of D_n. The quotient map that collapses the contractible C_n-equivariant subcomplex $\widetilde{D}_n$ of $\widehat{M}$ to a point is a homotopy equivalence onto the quotient complex $K = \widehat{M}/\widetilde{D}_n$. The action of C_n on $\widehat{M}$ descends to a cellular C_n-action on K that is free away from the single zero-cell of K, which means that the two-skeleton K^2 models a cyclic presentation for $\pi_1 \widehat{M} \cong \ker \nu^f$. In fact, the two-skeleton K^2 is the cellular model for the (ordinary) cyclic presentation $\mathcal{P}_n(\rho^f(W))$ as in Theorem 2.3. The covering projection and quotient map induce isomorphisms $\pi_2 M \leftarrow \pi_2 \widehat{M} \to \pi_2 K$. To complete the proof of the theorem requires an understanding of how length two identity sequences for $\mathcal{P}_n(\rho^f(W))$ are related to three-cells of K, which are present only when W is a proper power.

Suppose that $W = \dot{W}^e$ has exponent $e > 1$ in $C_n * \langle x \rangle$, so the complement $M \backslash D_n$ contains a single three-cell. Let $\nu^f(\dot{W}) = a^\sigma \in C_n$ be the image of the root $\dot{W}$ under the retraction ν^f. Let $\hat{w} = \rho^f(\dot{W}) = \hat{w}(x_0, \dots, x_{n-1})$ be the rewrite of the root of W. It is worth noting that if $\sigma \not\equiv 0$ modulo n (that is, if the root $\dot{W}$ is not in the kernel of the retraction ν^f), then the word $\hat{w}$ is *not* the root of w. Indeed, the exponent of $w = \rho^f(W)$ is equal to the order of the subgroup that is generated by the image $\nu^f(\dot{W}) = a^\sigma \in C_n$ in the cyclic group of order n. The attaching map for the three-cell of $M \backslash D_n$, if present, admits n distinct lifts through the covering projection $\widehat{M} \to M$. These descend to the quotient complex K to provide attaching maps for the three-cells of K. These attaching maps correspond to the shifts of the identity sequence

$$\Delta^f = (1, w)(\hat{w}, \theta^\sigma(w))^{-1}$$

for $\mathcal{P}_n(w)$, that is, to

$$\theta^k\Delta^f = (1, \theta^k(w))(\theta^k(\hat{w}), \theta^{k+\sigma}(w))^{-1}$$

where $k = 0, \ldots, n-1$. Each of these identity sequences determines a homotopy class in the kernel of the inclusion-induced homomorphism $\pi_2 K^{(2)} \to \pi_2 K$.

To prove statement (a), if $\pi_2 M = 0$, then $\pi_2 K = 0$, which implies that $\pi_2 K^{(2)}$ is either trivial (if W is not a proper power) or is generated by the classes of the length two identity sequences $\theta^k\Delta^f$ for $\mathcal{P}_n(w)$, whence $\mathcal{P}_n(w)$ is combinatorially aspherical by Proposition 3.1.

To prove statement (b), it is necessary and sufficient to show that for each length two identity sequence of the presentation $\mathcal{P}_n(w)$, the corresponding homotopy class is trivial in $\pi_2 K$. We have $W(a, x) = x^{\epsilon_1}a^{p_1} \cdots x^{\epsilon_L}a^{p_L}$ and $w = \rho^f(W) = x_{u(1)}^{\epsilon_1} \cdots x_{u(L)}^{\epsilon_L}$. There is no loss of generality in assuming that w is cyclically reduced. Because $\mathcal{P}(w)$ is orientable, length two identity sequences for $\mathcal{P}_n(w)$ can arise only if a cyclic permutation of w is identically equal to one of its shifts $\theta^v(w)$. It suffices to consider an identity sequence of the form

$$\Sigma = (1, w)(s, \theta^v(w))^{-1}$$

where s is an initial segment of w. This means that $w = st$ where $s = x_{u(1)}^{\epsilon_1} \ldots x_{u(q)}^{\epsilon_q}$ and $t = x_{u(q+1)}^{\epsilon_{q+1}} \ldots x_{u(L)}^{\epsilon_L}$, and that

$$ts = x_{u(q+1)}^{\epsilon_{q+1}} \ldots x_{u(L)}^{\epsilon_L} x_{u(1)}^{\epsilon_1} \ldots x_{u(q)}^{\epsilon_q} = x_{u(1)+v}^{\epsilon_1} \ldots x_{u(L)+v}^{\epsilon_L}$$

involving cyclically reduced words in the free group with basis $\{x_0, \ldots x_{n-1}\}$. This relation leads to the conditions

$$u(q+i) \equiv u(i) + v \text{ modulo } n \text{ and} \tag{2}$$

$$\epsilon_{q+i} = \epsilon_i \tag{3}$$

for $i = 1, \ldots, L$ modulo L. In turn, these lead to

$$v(q+i) \equiv v(i) + v \text{ modulo } n \text{ and} \tag{4}$$

$$p_{q+i} \equiv p_i \text{ modulo } n \tag{5}$$

for $i = 1, \ldots, L$. The conditions (3) and (5) imply the exponent e of W is at least $e \geq L/\gcd(L, q)$, so the result holds if W is not a proper power in $C_n * \langle x \rangle$. Further, if $S = x^{\epsilon_1}a^{p_1} \ldots x^{\epsilon_q}a^{p_q}$ is the initial segment of $W \in C_n * \langle x \rangle$ corresponding to s, so that $s = \rho^f(S)$, then $S = \dot{W}^\ell$ for suitable $\ell \geq 1$ depending on e, L, and q. (In other words, S commutes with W in $C_n * \langle x \rangle$ and so S is a power of the root $\dot{W}$.) Finally, the condition (4) with $i = 1$ implies that $v(q+1) \equiv \nu^f(S) \equiv \ell\sigma \equiv v$ modulo n.

Form the following product of identity sequences for $\mathcal{P}_n(w) = \mathcal{P}_n(\rho^f(W))$,

$$\Pi = \prod_{j=1}^{\ell} \left(\left(\prod_{k=0}^{j-1} \theta^{k\sigma}(\hat{w}) \right) \cdot \theta^{(j-1)\sigma} \Delta^f \right),$$

and note that $[\Pi] \in \ker(\pi_2 K^{(2)} \to \pi_2 K)$. The strategy is to show that $\Sigma = \Pi$. Expanding terms, the product Π telescopes as follows:

$$\begin{aligned}
\Pi &= \left(\Delta^f\right)\left(\hat{w} \cdot \theta^\sigma \Delta^f\right)\left(\hat{w}\theta^\sigma(\hat{w}) \cdot \theta^{2\sigma}\Delta^f\right) \cdots \left(\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-2)}(\hat{w}) \cdot \theta^{\sigma(\ell-1)}\Delta^f\right) \\
&= (1, w)\,(\hat{w}, \theta^\sigma(w))^{-1} \cdot (\hat{w}, \theta^\sigma(w))\left(\hat{w}\theta^\sigma(\hat{w}), \theta^{2\sigma}(w)\right)^{-1} \\
&\quad \cdot \left(\hat{w}\theta^\sigma(\hat{w}), \theta^{2\sigma}(w)\right)\left(\hat{w}\theta^\sigma(\hat{w})\theta^{2\sigma}(\hat{w}), \theta^{3\sigma}(w)\right)^{-1} \\
&\quad \cdots \\
&\quad \cdot \left(\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-2)}(\hat{w}), \theta^{\sigma(\ell-1)}(w)\right)\left(\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-1)}(\hat{w}), \theta^{\sigma\ell}(w)\right)^{-1} \\
&= (1, w)\left(\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-1)}(\hat{w}), \theta^{\sigma\ell}(w)\right)^{-1} \\
&= (1, w)\left(\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-1)}(\hat{w}), \theta^{v}(w)\right)^{-1}.
\end{aligned}$$

The argument is concluded by noting that since $\rho^f(\dot{W}) = \hat{w}$ and $\nu^f(\dot{W}) = a^\sigma$, it follows that

$$\begin{aligned}
\hat{w}\theta^\sigma(\hat{w}) \cdots \theta^{\sigma(\ell-1)}(\hat{w}) &= \rho^f(\dot{W})\theta^\sigma(\rho^f(\dot{W})) \cdots \theta^{\sigma(\ell-1)}(\rho^f(\dot{W})) \\
&= \rho^f(\dot{W}^\ell) = \rho^f(S) = s,
\end{aligned}$$

and so $\Pi = (1, w)(s, \theta^v(w))^{-1} = \Sigma$, whence $[\Sigma] \in \ker(\pi_2 K^{(2)} \to \pi_2 K)$. •

5 Proofs of Theorems B and C

Now consider the groups $G_n(k, l)$ given by the orientable cyclic presentations

$$\mathcal{P}_n(k, l) = (x_0, \ldots, x_{n-1} \mid x_i x_{i+k} x_{i+l} \ (i = 0, \ldots, n-1)).$$

Each group $G_n(k, l)$ has a homomorphic image of order three and so is nontrivial.

Lemma 5.1 *Let $d = \gcd(n, k, l)$ and let θ be the shift on $G_n(k, l)$.*

(a) $\mathcal{P}_n(k, l)$ is a disjoint union of d subpresentations $\mathcal{Q}_1, \ldots, \mathcal{Q}_d$, where $\mathcal{Q}_j$ consists of those generators and relators of $\mathcal{P}_n(k, l)$ that involve only the generators x_i where $i \equiv j \mod d$ and each Q_j is isomorphic to the cyclic presentation $\mathcal{P}_{n/d}(k/d, l/d)$.

(b) The shift θ on $G_n(k, l)$ carries the free factor determined by the subpresentation $\mathcal{Q}_j$ onto the free factor determined by $\mathcal{Q}_{j+1}$ (subscripts modulo d) and the shift θ' on $G_{n/d}(k/d, l/d)$ is the restriction of θ^d.

(c) A power θ^p of the shift on $G_n(k, l)$ has a non-identity fixed point if and only if $d \,|\, p$ and $\theta'^{p/d}$ has a non-identity fixed point on $G_{n/d}(k/d, l/d)$.

Proof: The statement (a) was noted in [4, 12] and (b) follows immediately. To prove (c), if $p \,|\, d$ and $\theta'^{p/d}$ has a non-identity fixed point, then so does θ^p, by (b). Conversely, suppose that θ^p fixes a nonempty reduced word $1 \neq w = a_{j(1)} \ldots a_{j(\ell)}$ in the free product, where each $a_{j(t)}$ is a nontrivial element in one of the free factors of $\mathcal{P}_n(k, l)$ determined by $\mathcal{Q}_{j(t)}$ $(1 \leq j(t) \leq d)$. Thus consecutive letters $a_{j(t)} \neq 1$ lie in distinct factors. Now (b) implies that $p \,|\, d$ and $\theta^p(a_{j(1)}) = \theta'^{p/d}(a_{j(1)}) = a_{j(1)}$. •

Using [5, Theorem 4.2], Lemma 5.1(a) implies that $\mathcal{P}_n(k, l)$ is combinatorially aspherical if and only if $\mathcal{P}_{n/d}(k/d, l/d)$ is combinatorially aspherical. Lemma 5.1(c) implies that the shift θ acts freely on the nonidentity elements of $G_n(k, l)$ if and only if the nonidentity elements of $G_{n/d}(k/d, l/d)$ are permuted freely by its shift θ'. In addition, if $\gcd(n, k, l) > 1$, then Lemma 5.1(a) implies that $G_n(k, l)$ is infinite and Lemma 5.1(b) implies that θ has no nontrivial fixed points. These remarks show that it suffices to prove Theorems B and C in the case where $\gcd(n, k, l) = 1$.

Edjvet and Williams developed a taxonomy for the parameters (n, k, l) and used it to characterize asphericity for the cellular models of the presentations $\mathcal{P}_n(k, l)$ [12, Theorem A] and finiteness for the groups $G_n(k, l)$ [12, Theorem B]. Their taxonomy involved the following divisibility conditions.

(A) $3 \mid n$ and $3 \mid k + l$.

(B) $n \mid k + l$ or $n \mid 2k - l$ or $n \mid 2l - k$.

(C) $n \mid 3k$ or $n \mid 3l$ or $n \mid 3(k - l)$.

The split extension $E_n(k, l) = G_n(k, l) \rtimes_\theta C_n$ has presentation $(a, x : a^n, xa^k xa^{l-k} xa^{-l})$ and admits the retraction $\nu^0 : E_n(k, l) \to C_n$ with kernel

$$G_n(\rho^0(xa^k xa^{l-k} xa^{-l})) = G_n(x_0 x_k x_l) = G_n(k, l).$$

Lemmas 5.2 and 5.4 below show that if condition (B) or (C) holds, then $E_n(k,l)$ is virtually the fundamental group of a graph of cyclic groups.

Lemma 5.2 *Assume that* $\gcd(n,k,l) = 1$ *and that the condition (B) is satisfied. With a change of variable of the form* $u = xa^t$ *for suitable* t*, the relator* W *is conjugate in* $C_n * \langle x \rangle = C_n * \langle u \rangle$ *to a word of the form* u^3a^{3p}*, where* $\gcd(p,n) = 1$*. Consequently:*

(a) The element a^3 *is central in the semidirect product* $E_n(k,l)$ *and so the shift* θ *on* $G_n(k,l)$ *satisfies* $\theta^3 = 1$*.*

(b) The cyclic presentation $\mathcal{P}_n(k,l)$ *is combinatorially aspherical if and only if* $n = 3$*.*

(c) If $n = 3$*, then* $E_3(k,l) \cong C_3 * C_3$ *and* C_3 *acts freely via the shift on the nonidentity elements of* $G_3(k,l)$*.*

(d) If $3 \mid n$ *and* $n \neq 3$*, then* $E_n(k,l) \cong \langle u \rangle *_{C_{n/3}} C_n$ *is infinite and the shift* θ *has no fixed points.*

(e) If $3 \nmid n$ *then* $E_n(k,l) \cong C_{3n}$ *is finite cyclic, so* $\theta = 1$*.*

Proof: Taking cases in condition (B), if $n \mid k+l$, then let $u = xa^k$ and rewrite $W = u^2a^{l-2k}ua^{-(k+l)}$, which cyclically permutes to u^3a^{-3k} in $C_n * \langle x \rangle = C_n * \langle u \rangle$. Thus $u^3 = a^{3k}$ is central in E. The conditions $\gcd(n,k,l) = 1$ and $n \mid k+l$ imply that $\gcd(n,-k) = 1$, so a^3 is central in E and hence $\theta^3 = 1$ on $G_n(k,l)$. The other cases in condition (B) proceed similarly. The conclusion (a) follows immediately, so if $n \neq 3$, then C_n does not act freely via the shift, whence Theorem A implies that $\mathcal{P}_n(k,l)$ is not combinatorially aspherical. If $n = 3$, then the relative presentation $(C_n, u : u^3a^{3p}) = (C_n, u : u^3)$ is aspherical in the sense of [3], so $\mathcal{P}_3(k,l)$ is combinatorially aspherical by Theorem 4.1(a). If $3 \mid n$ and $n \neq 3$ then $E \cong \langle u \rangle *_{u^3=a^{-3p}} C_n$ splits as a nontrivial free product with amalgamation where the element $a \in C_n$ lies outside the amalgamating subgroup of order $n/3$ generated by a^3. Thus a generates its centralizer in E [21, Theorem 4.5] and so no nonidentity element of $G_n(k,l) \leq E$ is centralized by a. The remaining conclusions are clear. •

Condition (B): The conclusions of Lemma 5.2 show that Theorems B and C both hold true when $\gcd(n,k,l) = 1$ and the condition (B) is satisfied. That is:

- $G_n(k,l)$ is finite $\Leftrightarrow 3 \nmid n \Leftrightarrow \theta = 1 \Leftrightarrow \theta$ has a nonidentity fixed point; and
- $\mathcal{P}_n(k,l)$ is combinatorially aspherical $\Leftrightarrow n = 3 \Leftrightarrow C_n$ acts freely via the shift on the nonidentity elements of $G_n(k,l)$.

Edjvet and Williams [12] showed that if $3 \mid n$ in the situation of Lemma 5.2, then $G_n(k,l)$ is isomorphic to the free group of rank two. Exemplars for this case are $\mathcal{P}_{3k}(1,2) \cong F(x_0, x_1)$ where the shift is given by $\theta(x_0) = x_1$ and $\theta(x_1) = x_2 = (x_0x_1)^{-1}$, under which C_3 acts freely on the nonidentity elements of the free group $F(x_0, x_1)$. The next lemma describes a very different set of exemplars that apply when the condition (C) is satisfied.

Lemma 5.3 *If* $\gcd(p,n) = 1$*, then the cyclic presentation* $\mathcal{P}_n(0,p)$ *defines a cyclic group* $G_n(0,p) \cong C_s$ *of order* $s = 2^n - (-1)^n$*. Moreover the shift* θ *fixes a subgroup of order three.*

Proof: Taking subscripts modulo n, the relations combine sequentially to show:

$$x_i = x_{i-p}^{-2} = x_{i-2p}^{4} = \cdots = x_{i-kp}^{(-2)^k} \quad \text{for all } i \text{ and } k.$$

Since $\gcd(p,n) = 1$, this shows that $G_n(0,p)$ is cyclic of order $s = 2^n - (-1)^n$, generated by x_0. The shift θ satisfies $\theta^p(x_0) = x_p = x_0^{-2}$ and therefore $\theta^p\left(x_0^{s/3}\right) = x_0^{-2s/3} = x_0^{s/3}$. This means that θ^p, and hence θ, fixes a subgroup of order three. •

Lemma 5.4 *Assume that* $\gcd(n,k,l) = 1$ *and that the condition (C) is satisfied.*

- *(a) If* $3 \nmid n$*, then* $\mathcal{P}_n(k,l) = \mathcal{P}_n(0,p)$ *for some* p *satisfying* $\gcd(p,n) = 1$.
- *(b) If* $3 \mid n$ *and* $3 \nmid k+l$*, then for some* f *and for some* p *satisfying* $\gcd(p,n) = 1$ *and* $n \mid 3f$*, the semidirect product* $E = E_n(k,l)$ *admits a retraction* $\nu^f : E \to C_n$ *where* $\mathcal{P}_n(\rho^f(W)) = \mathcal{P}_n(0,p)$*.*
- *(c) If the condition (A) is satisfied, so that* $3 \mid n$ *and* $3 \mid k+l$*, then for some* f *and for some* p *satisfying* $\gcd(p,n) = 3$ *and* $n \mid 3f$*, the semidirect product* $E = E_n(k,l)$ *admits a retraction* $\nu^f : E \to C_n$ *where* $\mathcal{P}_n(\rho^f(W)) = \mathcal{P}_n(0,p)$*.*

Proof: Suppose first that $3 \nmid n$. Taking cases in condition (C), if $n \mid 3k$, then $n \mid k$, so $\mathcal{P}_n(k,l) = \mathcal{P}_n(0,l)$ and $\gcd(n,l) = 1$. The other cases are handled in the same way, using the fact that $\mathcal{P}_n(k,0) = \mathcal{P}_n(0,k)$ and $\mathcal{P}_n(k,k) = \mathcal{P}_n(0,n-k)$.

Now suppose that n is divisible by three. Again taking cases in condition (C), if $n \mid 3k$, then there is the retraction $\nu^{-k} : E \to C_n$ where $\rho^{-k}(W) = x_0^2 x_{l-2k}$. Using the fact that $\gcd(n,k,l) = 1$, if $3 \nmid k+l$, then $\gcd(l-2k,n) = 1$, whereas if $3 \mid k+l$, then $\gcd(l-2k,n) = 3$. The other cases in condition (C) are handled similarly; if $n \mid 3l$, use the retraction ν^l, and if $n \mid 3(k-l)$, then use the retraction ν^{k-l}. •

Condition (C): Lemmas 2.2 and 5.1-5.4 combine to show that Theorem C holds true when $\gcd(n,k,l) = 1$ and the condition (C) is satisfied. That is:

- $G_n(k,l)$ is finite $\Leftrightarrow$ condition (A) is not satisfied $\Leftrightarrow$ θ has a nonidentity fixed point.

Theorem B also holds here because the shift determines a free action on the nonidentity elements of $G_n(k,l)$ only when $n = 3 \mid k+l$, in which case Lemma 5.2 applies. In all other cases, the cyclic presentation $\mathcal{P}_n(k,l)$ is not combinatorially aspherical by Theorems 3.3(c) and 4.1 because $G_n(0,p)$ has nontrivial torsion.

The method of Lemma 5.4 provides quick access to structural results that were obtained through detailed analysis in [12].

Lemma 5.5 ([12, Lemma 3.2]) *Assume that* $\gcd(n,k,l) = 1$ *and that condition (C) is satisfied. If* $3 \mid n$ *and* $3 \nmid k+l$ *(so that condition (A) is not satisfied) then* $G_n(k,l)$ *is metacyclic of order* $s = 2^n - (-1)^n$.

Proof: By Lemma 5.4(b), there is a retraction $\nu^f : E_n(k,l) \to C_n$ where $n \mid 3f$ and $\ker \nu^f \cong G_n(0,p)$ is such that $\gcd(n,p) = 1$. By Lemma 5.3, $\ker \nu^f$ is cyclic of order s, and by Theorem 2.3 this group is generated by xa^{-f}. In particular, the extension $E_n(k,l)$ is metacyclic of order ns. There is also the retraction $\nu^0 : E_n(k,l) \to C_n$ with $\ker \nu^0 = G_n(k,l)$, whence $G_n(k,l)$ has order s. Now the fact that $n \mid 3f$ implies that $\nu^0((xa^{-f})^3) = 1$ so either $\ker \nu^0 = \ker \nu^f$ or else $\ker \nu^0 \cap \ker \nu^f$ is a cyclic normal subgroup of order $s/3$ in $\ker \nu^0 = G_n(k,l)$, so the result follows. •

Example 5.6 Edjvet and Williams classified finite groups of the form $G_n(k,l)$, which are all cyclic or metacyclic [12, Theorem B]. For example, if $m \geq 2$, then $G = G_{3m}(m, 2m+1)$ is nonabelian metacyclic of order $s = 2^{3m} - (-1)^m$. These groups are non-nilpotent because for metacyclic groups, the second and third terms of the lower central series coincide: $[G,G] = [G,[G,G]]$. Thus Thompson's theorem [28] implies that the shift does not act freely on G. To see this directly, consider the extension $E = E_{3m}(m, 2m+1) = G \rtimes_\theta C$ where $C = C_{3m}$, so E has relative presentation $(C_{3m}, x : W)$ where $W = xa^m xa^{m+1} xa^{-(2m+1)}$. There are retractions $\nu^f : E \to C$ for $f = 0, \pm m$. By Theorem 2.3, the group G is recovered from the rewrite $\rho^0(W) = x_0 x_m x_{2m+1}$. However for $f = -m$, the rewrite is $\rho^{-m}(W) = x_0^2 x_1$ and so by Lemma 5.3, the group $G^* = \ker \nu^{-m} \cong G_{3m}(0,1)$ is cyclic of order s. Now G and G^* are normal subgroups of index $3m$ in E, so $|G| = |G^*| = s$. Moreover the shift on G^*, which arises from conjugation by a in E, has a fixed subgroup of order 3. Using Theorem 2.3, the proof of Lemma 5.3 further implies that a fixed point is given by $1 \neq v = (xa^m)^{s/3} \in G^* = \ker \nu^{-m} \trianglelefteq E$. In fact $v \in G = \ker \nu^0$ as well because $9|s$. Working in the non-nilpotent cyclically presented group $G = G_{3m}(m, 2m+1) = G_{3m}(x_0 x_m x_{2m+1})$ of order s, a nonidentity fixed point for the shift is $v = (x_0 x_m x_{2m})^{s/9}$. •

To conclude the proofs of Theorems B and C, it remains to consider the cases where neither (B) nor (C) is satisfied, at which point asphericity takes over.

Theorem 5.7 ([12]) *Assume that* $\gcd(n,k,l)=1$. *If neither the condition (B) nor the condition (C) is satisfied, then the two-dimensional cellular model of* $\mathcal{P}_n(k,l)$ *is aspherical (in the topological sense), unless* $n=18$ *and* $3 \mid k+l$.

As was noted earlier, if the cellular model of $\mathcal{P}_n(k,l)$ is aspherical in the topological sense, then the presentation is combinatorially aspherical and so the shift acts freely on the nonidentity elements by Theorem A. Further, in this case the group $G_n(k,l)$ is torsion-free and so, being nontrivial, is infinite. Therefore it remains to verify Theorems B and C in the case where $n=18$ and $3 \mid k+l$. In this case, Edjvet and Williams showed that $G_{18}(k,l)$ is the free product of a free group of rank two with the cyclic group of order 19 [12, page 771]. Thus $G_{18}(k,l)$ is infinite and $\mathcal{P}_{18}(k,l)$ is not combinatorially aspherical. It remains to show that the shift θ on $G_{18}(k,l)$ has no nonidentity fixed point (as in Theorem C), but that some nonidentity power of θ has a nonidentity fixed point (as in Theorem B).

Lemma 5.8 *Assume that* $n=18$, *that* $\gcd(18,k,l)=1$, *that* $3 \mid k+l$, *and that neither condition (B) nor condition (C) is satisfied. With* $E_{18}(k,l)=G_{18}(k,l)\rtimes_\theta C_{18}$ *defined by the presentation* $(a,x : a^{18}, xa^kxa^{l-k}xa^{-l})$, *the split short exact sequence*

$$1 \to G_{18}(k,l) \to E_{18}(k,l) \stackrel{\nu^0}{\to} C_{18} \to 1$$

is equivalent to one of the form

$$1 \to G \to E \stackrel{\nu}{\to} C_{18} \to 1$$

where E *has presentation* $(a,u : a^{18}, u^2a^9ua^6)$ *and* $\nu : E \to C_{18}$ *is a retraction onto the cyclic group of order* 18 *generated by* a.

Proof: Applying the change of variable $u = xa^{-l}$ to the presentation $(a,x : a^{18}, xa^kxa^{l-k}xa^{-l})$ leads to $(a,u : a^{18}, u^2a^{k+l}u^{2l-k})$. Both $k+l$ and $2l-k$ are defined modulo 18 and are divisible by 3. Working modulo 18, $k+l \not\equiv \pm(2l-k)$ and $k+l, 2l-k \not\equiv 0$ because neither condition (B) nor (C) is satisfied. And because $\gcd(n,k,l)=1$, it follows that $(k+l)+(2l-k)\not\equiv 9$. All of this means that

$$\{k+l, 2l-k\} \equiv \{3,9\}, \{3,12\}, \{6,9\}, \{6,15\}, \{9,12\}, \text{ or } \{9,15\}.$$

All of these choices lead to the same extension. To see this, begin with the parameters $(k,l)=(1,11)$, which satisfy the hypotheses of the lemma and determine the split extension $E_{18}(1,11)$ with presentation $(a,x : a^{18}, xaxa^{10}xa^{-11})$. The following changes of variable do not affect the group or extension class:

$$\begin{aligned}
u = xa &\quad\to\quad (a,u : a^{18}, u^2a^9ua^{-12}) \cong (a,u : a^{18}, u^2a^9ua^6) \\
u = xa^{10} &\quad\to\quad (a,u : a^{18}, u^2a^{-21}ua^{-9}) \cong (a,u : a^{18}, u^2a^{15}ua^9) \\
u = xa^{-11} &\quad\to\quad (a,u : a^{18}, u^2a^{12}ua^{21}) \cong (a,u : a^{18}, u^2a^{12}ua^3)
\end{aligned}$$

Replacing a relator $u^2a^pua^q$ with its inverse and cyclically permuting in $C_{18} * \langle u \rangle$ leads to $\left(u^2a^{18-q}ua^{18-p}\right)^{-1}$. Replacing a and u by their inverses in $u^2a^pua^q$ and cyclically permuting in $C_{18} * \langle u \rangle$ leads to $\left(u^2a^qua^p\right)^{-1}$. With (k,l) given as above and letting $E \cong E_{18}(1,11)$ be the group with presentation $(a,u : a^{18}, u^2a^9ua^6)$, this implies that there is an isomorphism $\psi : E_{18}(k,l) \to E$ that carries a to a^ϵ for $\epsilon = \pm 1$. Now the retraction $\nu : E \to C_{18}$ given by the composite $E \stackrel{\psi^{-1}}{\to} E_{18}(k,l) \stackrel{\nu^0}{\to} C_{18} \stackrel{\epsilon}{\to} C_{18}$ satisfies $\nu(a) = a$. •

Let $\mathcal{P}_{18}(k,l)$ be a cyclic presentation where $\gcd(18,k,l) = 1$, where $3 \mid k+l$, and where neither (B) nor (C) is satisfied. By Lemma 5.8, the cyclically presented group $G_{18}(k,l)$ can be obtained as the kernel of a retraction $\nu : E \to C_{18}$ where E has presentation $(a,u : a^{18}, u^2a^9ua^6)$, with the shift on $G_{18}(k,l)$ (or its inverse) given by conjugation by $a \in E$. This group E admits retractions $\nu = \nu^f : E \to C_{18}$ where $\nu^f(a) = a$ and $\nu^f(u) = a^f$ for $f = 1, 7$, or 13.

Now consider the group K with presentation $(b,u : b^6, u^2b^3ub^2)$, which appeared in [3, page 25]. The element b has order 6 in K [20] and a computer-assisted coset enumeration, for example using GAP [14], shows that K is finite of order 342. The group E decomposes as a free product with amalgamation

$$E = K *_{b=a^3} C_{18} = K *_{C_6} C_{18}.$$

Since $C_{18} \cap \ker \nu = 1$ and a lies outside the amalgamating subgroup C_6 generated by $b = a^3$, the shift on $G_{18}(k,l)$ has no fixed points, as in Theorem C. However, the subgroup $C_6 \leq K$, generated by b, acts by left multiplication on the left coset space K/C_6. Since $|K/C_6| = 57$, the C_6-action on $K/C_6 - \{1C_6\}$ is not free. In fact, a GAP calculation [14] shows that the C_6-action on K/C_6 actually has three fixed points, including $1C_6$ as well as two others. This means that there exists $g \in K \backslash C_6$ with $a^3g = bg \in gC_6 \subseteq gC_{18}$, whence $g \in E \backslash C_{18}$ and $a^3gC_{18} = gC_{18}$. By Lemma 2.2, θ^3 has a nonidentity fixed point on $G_{18}(k,l)$, which concludes the proof of Theorem B.

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