OMEGA-GROUPS

Group Theory For Wild Topology

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The algebraic topology of locally contractible spaces, such as, CW complexes and manifolds, is fully expressible using discrete group theory. But for *wild spaces*, by which we mean metric spaces with arbitrarily small essential features, new group theory terminology is required. For this purpose, this paper offers the theory of *omega-groups*. In essence, these are groups whose elements are assigned non-negative real weights and form countable products of all possible order-type whenever their weights tend to zero as their depth of appearance in the order-type increases without bound. Omega-groups are to discrete groups as wild spaces are to tame spaces, e.g., as totally disconnected spaces are to discrete spaces. Section 1 has the definitions. Section 2 introduces a *weighted combinatorial group theory* to present omega-groups. Section 3 illustrates the theory with motivating examples of the fundamental group of wild metric cell complexes.

1 OMEGA-GROUPS

In this section, we present the axioms for an omega-group. This is designed to be a group in which countable products of all possible forms of the group elements can be evaluated as elements of the group.

1.1 Order-Types

To express the form of a potential countable product of group elements, we use the following order-type concept.

Order-Types An **order-type** is any partition $\omega : 0 < \ldots < p < \ldots < 1$ of the closed unit interval $[0, 1]$ by a closed, nowhere dense, ordered set ω of points p in $[0, 1]$ that includes the endpoints 0 and 1.

Complementary Intervals For any order-type ω, the complementary open dense set $[0, 1] - \omega$ is necessarily the union of a countable collection $\mathcal{I}_\omega$ of disjoint open intervals $i = (a_i, b_i)$, called **complementary intervals** of ω. The set $\mathcal{I}_\omega$ of complementary intervals inherits a linear order from the standard order on I: $i < j$ if $b_i \leq a_j$. The **length** $l(i) = b_i - a_i$ of each complementary interval $i \in \mathcal{I}_\omega$ measures its depth of appearance in the order-type ω.

Order-Type Examples The simplest order-type is the two point partition $\tau : 0 < 1$, called the **trivial order-type**. One countable order-type is the **harmonic order-type**

$$\eta : 0 < \ldots < \tfrac{1}{k} < \ldots < \tfrac{1}{3} < \tfrac{1}{2} < 1.$$

One uncountable order-type is **Cantor's order-type** κ, namely, the closure of the set of the endpoints of the countable collection of middle-third subintervals:

$$0 < \ldots < \tfrac{1}{9} < \tfrac{2}{9} < \ldots < \tfrac{1}{3} < \tfrac{2}{3} < \ldots < \tfrac{7}{9} < \tfrac{8}{9} < \ldots < 1.$$

Reversion The **reverse** of an order-type ω is the order-type $\bar{\omega}$ produced as the image of ω under the homeomorphism $^- : [0,1] \to [0,1], t \ \to \ \bar{t} = 1 - t$, exchanging the endpoints of $[0,1]$.

Concatenation The **concatenation** $\omega'' = \omega \cdot \omega'$ of a pair of order-types ω and ω' is the order-type ω'' constructed by juxtaposition of half-scale versions of ω and ω' on the sub-intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively.

1.2 WEIGHTED SETS; ORDER-TYPE WORDS

Each order-type serves as a model of a proposed product, with the complementary intervals as place holders for the elements to be multiplied. The elements are required to come from a *weighted set*, and the product models, called *order-type words*, have a *weight restriction* on their entries, as follows.

Weighted Sets Let $(X, \ ^{-1})$ denote a set X equipped with an involution $^{-1} : X \to X$, called **inversion**, that fixes an element $1_X \in X$, called the **identity element**. A **weight function** for $(X, \ ^{-1})$ is a real valued function $wt : X \to [0, \infty)$ satisfying the two **weight axioms**: (a) $wt(x) = 0$ if and only if $x = 1_X$, and (b) $wt(x^{-1}) = wt(x)$ for all $x \in X$. Then the pair (X, wt) is called a **weighted set**. If, in addition, the inversion involution fixes just the identity element $1_X \in X$, then (X, wt) is called a **weighted alphabet**.

Order-Type Words An **(order-type) word** for a weighted set (X, wt) is a pair (ω, x) consisting of an order-type ω and a labeling of the complementary intervals $i \in \mathcal{I}_\omega$ by elements $x(i) \in X$ whose weights $wt(x(i))$ tend to zero as the ω intervals' lengths $l(i)$ tend to zero. More precisely, the intervals' labels $x(i) \in X, i \in \mathcal{I}_\omega$, constitute a function $x : \mathcal{I}_\omega \to X$ satisfying the **weight restriction**: $wt(x(i)) \to 0$ as $l(i) \to 0$.

For finite words, the weight restriction is void. As an extreme example, any single element $x \in X$ can be used to label the sole interval for the trivial order-type $\tau : 0 < 1$ to give a **monosyllabic** word $(\tau, x) = (0 < x < 1)$.

A sequence $x : x_1, \ldots, x_k, \ldots$ in X whose weights $wt(x_1), \ldots, wt(x_k), \ldots$ have limit zero determines, for example, a *harmonic word* (η, x):

$$0 < \ldots < \tfrac{1}{k+1} < x_k < \tfrac{1}{k} < \ldots < \tfrac{1}{3} < x_2 < \tfrac{1}{2} < x_1 < 1$$

and a *Cantorian word* (κ, x):

$$0 < \ldots < \tfrac{1}{9} < x_2 < \tfrac{2}{9} < \ldots < \tfrac{1}{3} < x_1 < \tfrac{2}{3} < \ldots < \tfrac{7}{9} < x_3 < \tfrac{8}{9} < \ldots < 1.$$

In these displays, each label appears between its interval's end-points.

Because any specific non-identity element $1_X \neq x' \in X$ has positive weight $wt(x') > 0$, the weight restriction implies that x' can label at most finitely many intervals of an order-type ω in any word (ω, x) for (X, wt). But the weight restriction on the entries of a word (ω, x) means more, namely, that there can be at most finitely many entries $x(i) \in X, i \in \mathcal{I}_\omega$, of a word (ω, x) whose weights $wt(x(i))$ are bounded away from zero by any given $\epsilon > 0$.

Word Inversion The **inverse** $(\omega, x)^{-1}$ of a word (ω, x) is the word $(\bar{\omega}, x^{-1})$ that results when the order-type ω is reversed in the interval $[0, 1]$ and the element labels are reversed in order and are inverted in the set X.

Identity Words An **identity word** $(\omega, 1_X)$ is one that labels all complementary intervals $i \in \mathcal{I}_\omega$ of an order-type ω with the identity element $1_X \in X$.

Subwords Any pair of distinct points $0 \leq p < q \leq 1$ of an order-type ω bound in any word (ω, g) a portion that can be rescaled into a word (ω_{pq}, g_{pq}), called the **subword** of (ω, g) on the interval $[p, q]$.

Word Products The **product** $(\omega, x) \cdot (\omega', x')$ of a pair of words (ω, x) and (ω', x') is the word (ω'', x'') on the concatenated order-type $\omega'' = \omega \cdot \omega'$ constructed by juxtaposition of half-scale versions of (ω, x) and (ω', x') on the sub-intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively.

Word Associativity Two words (ω, x) and (ω', x') for the same weighted set (X, wt) are **associated**, written $(\omega, x) \overset{a}{\sim} (\omega', x')$, if there is an order-preserving and label-preserving pairing of the complementary intervals $i \in \mathcal{I}_\omega$ and $i' \in \mathcal{I}_{\omega'}$ of ω and ω' that are labeled by non-identity elements $x(i) \neq 1_X \neq x'(i')$ of X. When the words (ω, x) and (ω', x') are viewed as decoration on the top and bottom edges of a unit square, the associativity pairing $(\omega, x) \overset{a}{\sim} (\omega', x')$ can be exhibited by a family of disjoint arcs, called **associativity arcs**, each of which joins the midpoint of a non-identity labeled interval of the word (ω, x) to that of an identically labeled interval of the word (ω', x'), say $\cdot x(i) \cdot$ and $\cdot x'(i') \cdot$ where $x(i) = x'(i') \in X$. Such a display A of associativity arcs (Figure

1(a)) is called an **associativity square** for the words (ω, x) and (ω', x') and is denoted $A : (\omega, x) \overset{a}{\sim} (\omega', x')$. The countably many associativity arcs may accumulate in the square; but, because they can be straighted into disjoint line segments and because they join the midpoints of disjoint non-trivial open intervals, they can be made to have disjoint tubular neighborhoods.

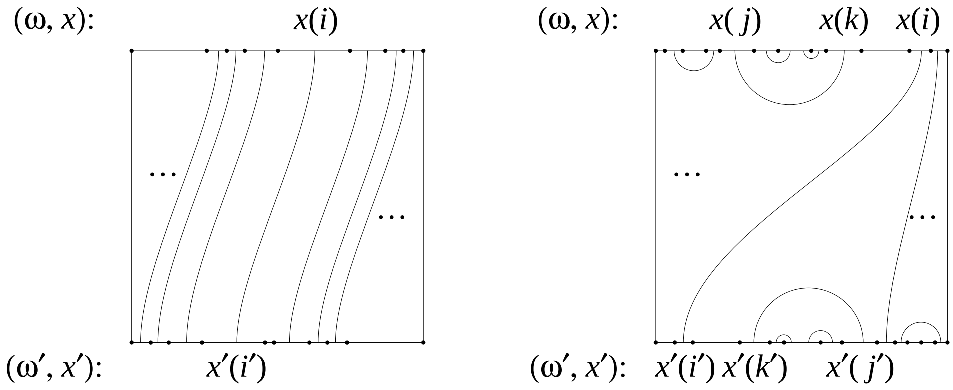

Figure 1: (a) Associativity Square (b) Similarity Square

Word Similarity Two words (ω, x) and (ω', x') for the same weighted set (X, wt) are **similar**, written $(\omega, x) \overset{s}{\sim} (\omega', x')$, if there is a pairing of those complementary intervals for the product word $(\lambda, y) = (\omega, x) \cdot (\omega', x')^{-1}$ having non-identity labels such that: (a) paired intervals $i \leftrightarrow j$ have non-identity inverse labels $y(i) = y(j)^{-1}$, and (b) any two pairs $i \leftrightarrow j$ and $k \leftrightarrow l$ of paired intervals are either nested $(i < k < l < j)$ or disjoint $(i < j < k < l)$. When the two words (ω, x) and (ω', x') are viewed as decoration on the top and bottom edges of a unit square, the similarity pairing $(\omega, x) \overset{s}{\sim} (\omega', x')$ can be exhibited by a family of disjoint arcs joining the midpoints of the intervals of the words having non-identity labels, with each arc ending at identically labeled intervals if in the different words (ω, x) and (ω', x') else inversely labeled intervals if in the same word (ω, x) or (ω', x'). Such a display S of arcs (Figure 1(b)) is called a **similarity square** for the words (ω, x) and (ω', x') and is denoted $S : (\omega, x) \overset{s}{\sim} (\omega', x')$. An arc whose two ends are at identically labeled intervals in the two opposing words, say $\cdot x(i) \cdot$ and $\cdot x'(i') \cdot$ where $x(i) = x'(i')$, is called an **associating arc**; an arc whose two ends are at inversely labeled intervals in the same word, say $\cdot x(j) \cdot$ and $\cdot x(k) \cdot$ where $x(j) = x(k)^{-1}$ or say $\cdot x'(j') \cdot$ and $\cdot x'(k') \cdot$ where $x'(j') = x'(k')^{-1}$, is called a **cancelling arc**.

The disjoint associating and cancelling arcs serve to express the compatibility condition (b) for the similarity pairing $(\omega, x) \overset{s}{\sim} (\omega', x')$, and they can be tightened into disjoint line segments if one conducts business in a circular disc instead of a square. As these arcs join the midpoints of disjoint non-trivial open intervals, they can be made to have disjoint tubular neighborhoods.

The disjointness of the arcs in a similarity square implies that a cancelling arc that joins non-consecutive labeled intervals necessarily bounds a complete family of cancelling arcs for the intervening non-identity labeled intervals. This means, in particular, that a finite word that is similar to an identity word reduces to 1_X by successive applications of *free reduction*, and that two finite words that are similar *freely reduce* to the same word.

Lemma 1.1 *Word similarity* $\overset{s}{\sim}$ *is an equivalence relation on the set of all words* (ω, g) *for a weighted set* (X, wt).

Proof: (*Reflexivity*) Any word (ω, x) is similar to itself, $(\omega, x) \overset{s}{\sim} (\omega, x)$, via a similarity square of parallel linear arcs. (*Symmetry*) Any similarity square $S : (\omega, x) \overset{s}{\sim} (\omega', x')$ can be turned upside down to yield a similarity square $S* : (\omega', x') \overset{s}{\sim} (\omega, x)$. (*Transitivity*) Similarity squares $S : (\omega, x) \overset{s}{\sim} (\omega', x')$ and $S' : (\omega', x') \overset{s}{\sim} (\omega'', x'')$ can be stacked one atop the other (along the edges decorated by (ω', x')) to yield a similarity square $S'' : (\omega, x) \overset{s}{\sim} (\omega'', x'')$. The arcs from the two similarity squares join by their endpoints at (ω', x') to form curves in the resulting stack. These curves can be amazingly complicated, but, once all simple closed curves that result are tossed out, there remains a family of disjoint arcs, each of which either joins identically labeled intervals on the two words (ω, x) and (ω'', x'') or joins inversely labeled intervals for the same word (ω, x) or (ω'', x''). Infinite *chinese dragon* arcs that begin at (ω, x) or (ω'', x''), but end neither at (ω, x) nor at (ω'', x''), do not arise because each non-identity element $x \neq 1_X \in X$ has positive weight $wt(x) > 0$ and so appears at most finitely many times in (ω, x), (ω', x'), and (ω'', x''). □

The word product respects the similarity relation; for an argument, assemble a similarity square for the product by stacking side-by-side similarity squares for the factors. In this way, the set of similarity classes $[(\omega, x)]$ of words (ω, x) for the weighted set (X, wt) is an associative semigroup, denoted by $\Omega(X, wt)$.

Word Groups The definition of word similarity is rigged to insure that: (*a*) any identity word $(\omega, 1_X)$ is similar to the monosyllabic identity word $(\tau, 1_X) = (0 < 1_X < 1)$ via an empty similarity square, (*b*) the product $(\omega, 1_X) \cdot (\omega', x')$ of an identity word $(\omega, 1_X)$ with any word (ω', x') is similar to the word (ω', x') via a similarity square of parallel associating arcs, (*c*) the product $(\omega, x) \cdot (\omega, x)^{-1}$ of any word (ω, x) and its inverse $(\omega, x)^{-1} = (\bar{\omega}, x^{-1})$ is similar to an identity word via a similarity square of concentric cancelling arcs. It follows that the associative semigroup $\Omega(X, wt)$ is a group, called the **word group** on the weighted set (X, wt).

1.3 Omega-Groups

We now present axioms for a *weighted group* (G, wt) in which each order-type word (ω, g) from its own word group $\Omega(G, wt)$ can be evaluated as a countable group product $\langle(\omega, g)\rangle = \omega\langle g\rangle \in G$. We shall call such groups *omega-groups.*

Weighted Groups A **weight function** $wt : G \to [0, \infty)$ for a group $(G, \cdot)$ is required to satisfy three **weight axioms**: (a) $wt(g) = 0$ if and only if $g = 1_G$, (b) $wt(g^{-1}) = wt(g)$ for all $g \in G$, and (c) $wt(g \cdot g') \leq \max\{wt(g), wt(g')\}$ for all $g, g' \in G$. The pair (G, wt) is called a **weighted group**.

Just as for a weighted set, there is the **word group** $\Omega(G, wt)$ for the weighted group (G, wt). One sees that two monosyllabic words $(\tau, g) = (0 < g < 1)$ and $(\tau, g') = (0 < g' < 1)$ are similar if and only if $g = g' \in G$. So G may be identified with the set (of similarity classes) of monosyllabic words in the word group $\Omega(G, wt)$. But G is not a subgroup of $\Omega(G, wt)$ because there is no similarity relation between the product $(\tau, g) \cdot (\tau, g')$ of monosyllabic words and the monosyllabic word $(\tau, g \cdot g')$ for the product element $g \cdot g'$ in G. The word group construction is entirely ignorant of the binary multiplication $\cdot : G \times G \to G$ in the weighted group (G, wt) and utilizes only the existence of a special element $1_G \in G$ invariant under the inversion pairing $^{-1} : G \to G$.

Word Evaluations A **word evaluation** for the weighted group (G, wt) assigns to each word (ω, g) for (G, wt) a group element $\langle(\omega, g)\rangle = \omega\langle g\rangle \in G$, called the **value** of the order-type ω on the group labels $g : \mathcal{I}_\omega \to G$. These group values $\omega\langle g\rangle \in G$ are required to be invariants of similarity classes of words and to determine a group homomorphism $\langle - \rangle : \Omega(G, wt) \to G$ that retracts the word group $\Omega(G, wt)$ to the non-subgroup G: $\tau\langle g\rangle = g, \forall g \in G$.

The value $\omega\langle g\rangle \in G$ of a word (ω, g) is called an **order-type product** with **factors** $g(i) \in G, i \in \mathcal{I}_\omega$, or, specifically, an ω-**product** in G. So *harmonic products* $\eta\langle g\rangle$ and *Cantorian products* $\kappa\langle g\rangle$ are among the possibilities.

Each non-endpoint $0 < p < 1$ of an order-type ω separates any word (ω, g) into two portions, which can be rescaled into subwords (ω_{0p}, g_{0p}) and (ω_{p1}, g_{p1}). These subwords serve as **factors** of the given word (ω, g) in the sense that (ω, g) is similar to their word product $(\omega_{0p}, g_{0p}) \cdot (\omega_{p1}, g_{p1})$. So the word evaluation properties insure that the points of ω separating the group element labels $g(i) \in G, i \in \mathcal{I}_\omega$, generalize multiplication $\cdot$ in the group G. Also the value of any identity word $(\omega, 1_G)$ is the identity element of G: $\omega\langle 1_G\rangle = 1_G$; and inverse words (ω, g) and $(\bar{\omega}, g^{-1})$ have inverse group values in G: $(\omega\langle g\rangle)^{-1} = \bar{\omega}\langle g^{-1}\rangle$.

When the weighted group (G, wt) has a word evaluation, a word (ω, g) is called a **factorization** of the word (ω', g') and (ω', g') is called a **partial evaluation** of (ω, g), written $(\omega', g') \preceq (\omega, g)$, if $\omega' \subset \omega$ and the label $g'(i') \in G$ of each interval $i' \in \mathcal{I}_{\omega'}$ is the value of the subword $(\omega_{p'q'}, g_{p'q'})$ of (ω, g) on $i' = (p', q')$.

Omega-Groups An **omega-group** is a weighted group (G, wt), together with a word evaluation $\langle - \rangle : \Omega(G, wt) \to G$ satisfying the following **weight**, **multiplication**, and **factorization axioms**:

Weight Axiom: *The value* $\omega\langle g\rangle$ *of a word* (ω, g) *for* (G, wt) *has its weight dominated by its factors' weights:* $wt(\omega\langle g\rangle) \leq \max\{wt(g(i)) : i \in \mathcal{I}_\omega\}$.

Multiplication Axiom: *Each word* (ω, g) *for* (G, wt) *and partial evaluation* (ω', g') *of it have the same value in G:* $(\omega', g') \preceq (\omega, g)$ *implies* $\omega\langle g\rangle = \omega'\langle g'\rangle$.

These first two axioms imply that each word (ω, g) is a common factorization of the sequence of finite words

$$(\omega_1, g_1) \preceq (\omega_2, g_2) \preceq \ldots \preceq (\omega_k, g_k) \preceq \ldots \preceq (\omega, g),$$

obtained as follows: ω_k is the finite order-type comprised of $0, 1$, and the endpoints a_i and b_i of those finitely many complementary intervals $i = (a_i, b_i) \in \mathcal{I}_\omega$ having labels $g(i)$ with $wt(g(i) \geq 1/k$; and the function $g_k : \mathcal{I}_\omega \to G$ assigns to each interval $i \in \mathcal{I}_{\omega_k} \cap \mathcal{I}_\omega$ the label $g_k(i) = g(i)$ from the word (ω, g) and assigns to each interval $j \in \mathcal{I}_{\omega_k} \setminus \mathcal{I}_\omega$ the evaluation of the subword of (ω, g) as the label $g_k(j)$. By the Multiplication Axiom, these successive finite factorizations have the same evaluations

$$\omega_1\langle g_1\rangle = \omega_2\langle g_2\rangle = \ldots = \omega_k\langle g_k\rangle = \ldots = \omega\langle g\rangle.$$

These successive finite factorizations in G of the value $\omega\langle g\rangle$ eventually reveal each label $g(i)$ of the given word (ω, g). To express a converse *factorization axiom*, we introduce this terminology:

Consider any word (ω, g) for (G, wt). Successive finite factorizations

$$(\omega_1, h_1) \preceq (\omega_2, h_2) \preceq \ldots \preceq (\omega_k, h_k) \preceq \ldots \tag{1}$$

stabilize to (ω, g) if each labeled interval of (ω, g) appears in (ω_k, h_k) for all sufficiently large $k \geq 1$ and the labels $h_k(j)$ for $j \in \omega_k$ satisfy the **stabilization weight criterion**: $wt(h_k(j)) \to 0$ whenever $k \to \infty$ and $l(j) \to 0$.

This stabilization implies that $\omega_1 \subset \ldots \subset \omega_k \subset \omega_{k+1} \subset \ldots \subset \omega$ and each interval of $\mathcal{I}_{\omega_k}$ is either an interval $i \in \mathcal{I}_{\omega_k} \cap \mathcal{I}_\omega \cap \mathcal{I}_{\omega_{k+1}}$ and has the label $h_k(i) = g(i) = h_{k+1}(i)$ in the words (ω_n, h_n) and (ω_{k+1}, h_{k+1}), or is an interval $j \in \mathcal{I}_{\omega_k} \setminus \mathcal{I}_\omega$ decomposed by finitely many of the intervals of ω_{k+1} and its label in the word (ω_k, g_k) equals the product in G of the finite subword of (ω_{k+1}, g_{k+1}) on the interval j. There is no loss to require that the decomposition of each interval $j \in \mathcal{I}_{\omega_k} \setminus \mathcal{I}_\omega$ by finitely many of the intervals of ω_{k+1} introduces an interval $i \in \mathcal{I}_\omega \cap \mathcal{I}_{\omega_{k+1}}$ of the first type for $\mathcal{I}_{\omega_{k+1}}$, since any (ω_{k+1}, h_{k+1}) not having this property can be discarded from the sequence of successive finite factorizations (1).

Factorization Axiom: *Successive finite factorizations*

$$(\omega_1, h_1) \preceq (\omega_2, h_2) \preceq \ldots \preceq (\omega_k, h_k) \preceq \ldots$$

that stabilize to (ω, g) *have constant evaluation* $\omega\langle g\rangle$*:* $\omega_k\langle h_k\rangle = \omega\langle g\rangle, k \geq 1$.

The Weight and Multiplication Axioms are needed to prove the universal property (Theorem 2.4) for free omega-groups in their category. The Factorization Axiom provides the uniqueness of the word evaluation in an omega group since finite factorizations have the same value under any word evaluation.

Trivial Example Any group $(G, \cdot)$ can be assigned the **trivial weights**: $t\text{-}wt(1_G) = 0$ and $wt(g) = 1$ for all $1_G \neq g \in G$. Any word (ω, g) for the trivially weighted group $(G, t\text{-}wt)$ must label all sufficiently small intervals with the identity element 1_G. Hence, the word (ω, g) is essentially a finite one whose evaluation $\omega\langle g\rangle$ can be taken to be the corresponding finite product in G. In other words, a trivially weighted group $(G, t\text{-}wt)$ has products of all order-type and so is an omega-group in a trivial way.

1.4 Group Theory for Omega-Groups

There are natural conditions to impose on a group homomorphism $\phi : G \to H$ between two weighted groups or omega-groups (G, wt) and (H, wt).

Omega-Group Homomorphisms Let (G, wt) and (H, wt) be weighted groups. A **weighted group homomorphism** $\phi : (G, wt) \to (H, wt)$ is a group homomorphism $\phi : G \to H$ satisfying the **weight restriction**: elements $g \in G$ with weights $wt(g) \to 0$ have images $\phi(g) \in H$ with weights $wt(\phi(g)) \to 0$.

The weight restriction implies that a weighted group homomorphism ϕ converts each word (ω, g) for the weighted group (G, wt) into an **image word** $(\omega, \phi(g))$ for the weighted group (H, wt), using the composite $\phi(g) = \phi \circ g : \mathcal{I}_\omega \to G \to H$ of the labeling function $g : \mathcal{I}_\omega \to G$ with the homomorphism $\phi : G \to H$. Since similarity relations are preserved by the image construction, the weighted group homomorphism ϕ determines a group homomorphism $\phi : \Omega(G, wt) \to \Omega(H, wt)$ of the word groups.

For omega-groups (G, wt) and (H, wt), an **omega-group homomorphism** $\phi : (G, wt) \to (H, wt)$ is a weighted group homomorphism such that the image under ϕ of the value $\omega\langle g\rangle \in G$ of any word (ω, g) for (G, wt) equals the value $\omega\langle\phi(g)\rangle \in H$ of the image word $(\omega, \phi(g))$ for (H, wt): $\phi(\omega\langle g\rangle) = \omega\langle\phi(g)\rangle$.

Omega-Subgroups A **weighted subgroup** (K, wt) of an omega-group (G, wt) consists of a subgroup K of the group G and the restricted weight function $wt : K \subset G \to [0, \infty)$. A weighted subgroup (K, wt) of an omega-group

(G, wt) is an **omega-subgroup** provided that any word (ω, k) for (G, wt) having all its labels $k(i), i \in \mathcal{I}_\omega$, in the subgroup K has its value $\omega\langle k\rangle$ in K.

Lemma 1.2 *Let $\phi : (G, wt) \to (H, wt)$ be an omega-group homomorphism.*

(a) *The kernel* Ker ϕ *of ϕ is an omega-subgroup of (G, wt).*

(b) *The image $\phi(G)$ of ϕ is an omega-subgroup of (H, wt), provided that $wt(\phi(g)) \to 0$ implies $wt(g) \to 0$.*

The lemma is easily checked. The weight condition in (b) is necesary. When an omega-group homomorphism is weight reducing, this weight condition may fail and the image may fail to be an omega-subgroup of the range omega-group. An example in [B-S97(3)] involves an injection of a free omega-group into a coordinate-weighted inverse limit omega-group.

Consequences Let N be a subgroup of a group G. Each finite product $n_1^{g_1} \cdot n_2^{g_2} \cdot \ldots \cdot n_k^{g_k}$ of conjugates $n_i^{g_i} = g_i \cdot n_i \cdot g_i^{-1}$ of subgroup members $n_i \in N$ by group members $g_i \in G$ is called a *consequence* of N in G. Each such consequence of N in G can be written in the form

$$((g_1' \cdot n_1) \cdot (g_2' \cdot n_2) \cdot \ldots \cdot (g_k' \cdot n_k)) \cdot (g_1' \cdot g_2' \cdot \ldots \cdot g_k')^{-1} \quad (2)$$

using the members $g_1' = g_1, g_2' = g_1^{-1} \cdot g_2, \ldots, g_k' = g_{k-1}^{-1} \cdot g_k$ of G. Recall that N is a *normal* subgroup of G provided N contains all consequences (2) of N in G. These concepts of consequences and normality have weighted generalizations for an omega-subgroup (N, wt) of an omega-group (G, wt), as follows.

Weighted Consequences Consider any order-type λ and a pair of words (λ, g) and (λ, n) for the weighted groups (G, wt) and (N, wt), respectively. Then the product labels $g(i) \cdot n(i) \in G$, $i \in \mathcal{I}_\lambda$, determine a word $(\lambda, g \cdot n)$ for (G, wt) as the weight restrictions for the words (λ, g) and (λ, n) carry over by use of the Weight Axiom for (G, wt). Because (G, wt) is an omega-group, the word evaluations $\lambda\langle g \cdot n\rangle$ and $\lambda\langle g\rangle$ exist in G. Emulating (2), we define:

A **weighted consequence** of an omega-subgroup (N, wt) of an omega-group (G, wt) is any product

$$\lambda\langle g \cdot n\rangle \cdot (\lambda\langle g\rangle)^{-1} \in G, \quad (3)$$

associated with an order-type λ and a pair of words (λ, g) and (λ, n) for the two weighted groups (G, wt) and (N, wt), respectively.

An omega-subgroup (N, wt) of an omega-group (G, wt) is an **normal omega-subgroup** provided that N contains each weighted consequence (3) of (N, wt) in (G, wt). (This normality requirement includes the omega-subgroup condition $\lambda\langle n\rangle \in N$ as the special case $\lambda\langle 1_G \cdot n\rangle \cdot (\lambda\langle 1_G\rangle)^{-1} \in N$.)

As expected, one has this

Lemma 1.3 *The kernel subgroup* Ker ϕ *of an omega-group homomorphism* $\phi : (G, wt) \to (H, wt)$ *is a normal omega-subgroup of* (G, wt).

Thus, for a quotient homomorphism from an omega-group (G, wt) to be an omega-group homomorphism, it is necessary that the normal subgroup be a normal omega-subgroup. This necessary condition is also sufficient:

Theorem 1.4 *The quotient of an omega-group* (G, wt) *by a normal omega-subgroup* (N, wt) *is an omega-group* $(G/N, wt)$.

Proof: For convenience, we reparametrize so that the actual weight values of (G, wt) are in the **harmonic set** $\mathcal{H} = \{0, \ldots, \frac{1}{k}, \ldots, \frac{1}{2}, 1\}$, instead of real values in the intervals $\{0\}, \ldots, [\frac{1}{k}, \frac{1}{k-1}), \ldots, [\frac{1}{2}, 1), [1, \infty)$.

Then each coset $g \cdot N = \{g \cdot n : n \in N\}$ has a representative $g \cdot n_g$ of minimum weight: $wt(g \cdot n_g) = \min\{wt(g \cdot n) \in \mathcal{H} : n \in N\}$. To show this, consider a coset $g \cdot N$ that contains a sequence of representatives $g_k \in g \cdot N, k \geq 1$, with strictly decreasing weights in $\mathcal{H}$ that therefore limit at 0. Then their consecutive differences $g_k \cdot g_{k+1}^{-1}, k \geq 1$, belong to the normal omega-subgroup (N, wt) and have weights that limit at 0, so their reverse harmonic product

$$(g_1 \cdot g_2^{-1}) \cdot (g_2 \cdot g_3^{-1}) \cdot \ldots \cdot (g_k \cdot g_{k+1}^{-1}) \cdot \ldots \tag{4}$$

exists and also belongs to this omega-subgroup N. By the Multiplication Axiom and the vanishing of the weights $wt(g_k^{\pm 1})$, this product (4) equals the reverse harmonic product

$$g_1 \cdot g_2^{-1} \cdot g_2 \cdot g_3^{-1} \cdot \ldots \cdot g_k \cdot g_{k+1}^{-1} \cdot \ldots$$

in the omega-group (G, wt), which in turn equals g_1 in G by an obvious word similarity and the similarity invariance of the word evaluation in G. So $g_1 \in N$ and the given coset $g \cdot N = g_1 \cdot N = 1_G \cdot N$ contains a representative of weight zero, i.e., 1_G. This argument shows that any other coset $g \cdot N \neq 1_G \cdot N$ has an element of minimum weight $\frac{1}{k} > 0$.

Thus the quotient group G/N of left cosets $g \cdot N$ acquires a weight function $wt : G/N \to [0, \infty)$ via the definition $wt(g \cdot N) = \min\{wt(g \cdot n) : n \in N\}$. The three weight axioms $(G/N, wt)$ are checked using the subgroup axioms for N, its normality in G, and the analogous weight axioms for (G, wt).

By this weight definition for cosets, minimal weight coset representatives $g(i) \in G, i \in \mathcal{I}_\omega$, of the coset labels $g(i) \cdot N$, $i \in \mathcal{I}_\omega$ of any word $(\omega, g \cdot N)$ for $(G/N, wt)$ determine a word (ω, g) for the omega-group (G, wt), as the weight restrictions carry over. So an evaluation function $\langle - \rangle : \Omega(G/N, wt) \to G/N$ arises from the evaluation function $\langle - \rangle : \Omega(G, wt) \to G$, as follows:

$$\langle(\omega, g \cdot N)\rangle = \omega\langle g \cdot N\rangle := \omega\langle g\rangle \cdot N,$$

for any coset representatives $g(i) \in G, i \in \mathcal{I}_\omega$, that determine a word (ω, g) for (G, wt). Alternative coset representatives $g(i) \cdot n(i) \in G, i \in \mathcal{I}_\omega$, that also determine a word $(\omega, g \cdot n)$ for (G, wt) necessarily involve subgroup elements $n(i) \in N, i \in \mathcal{I}_\omega$, that determine a word (ω, n) for the subgroup (N, wt), as the weight restriction carries over via the weight axioms. Then the weighted consequence $\omega\langle g \cdot n\rangle \cdot (\omega\langle g\rangle)^{-1}$ belongs to the normal omega-subgroup (N, wt), which implies that the evaluation $\omega\langle g \cdot N\rangle = \omega\langle g\rangle \cdot N \in G/N$ is independent of the chosen coset representatives.

The omega-group axioms for the evaluation $\langle - \rangle : \Omega(G, wt) \to G$ can be used to establish the the same axioms for the evaluation $\langle - \rangle : \Omega(G/N, wt) \to G/N$. The checks of the Weight and Multiplication Axioms are straightforward.

To prove the Factorization Axiom, consider successive finite factorizations

$$(\omega_1, h_1 \cdot N) \preceq (\omega_2, h_2 \cdot N) \preceq \ldots \preceq (\omega_k, h_k \cdot N) \preceq \ldots$$

that stabilize to the word $(\omega, g \cdot N)$ for the weighted group $(G/N, wt)$. We must show that their constant evaluations $\omega_k\langle h_k \cdot N\rangle$ equal $\omega\langle g \cdot N\rangle$.

The given words are based on coset labeling function $h_k \cdot N : I_{\omega_k} \to G/N$ and $g \cdot N : I_\omega \to G/N$. Compatible choices of minimal-weight coset representatives provide labeling functions $h_k : \mathcal{I}_{\omega_k} \to G$ that stabilize to the labeling function $g : \mathcal{I}_\omega \to G$. They determine words (ω_k, h_k) and (ω, g) for (G, wt), but the factorization condition $(\omega_k, h_k \cdot N) \preceq (\omega_{k+1}, h_{k+1} \cdot N)$ implies the factorization condition $(\omega_k, h_k) \preceq (\omega_{k+1}, h_{k+1})$ only *modulo* the normal subgroup N: Each interval $j \in \mathcal{I}_{\omega_k} \setminus \mathcal{I}_\omega$ is subdivided into finitely many intervals $j_0, j_1, \ldots j_s \in \mathcal{I}_{\omega_{k+1}} \setminus \mathcal{I}_\omega$ and intervening intervals $i_1, \ldots i_s \in \mathcal{I}_\omega \cap \mathcal{I}_{\omega_{k+1}}$, and it has label

$$h_k(j) \equiv h_{k+1}(j_0)g(i_1)h_{k+1}(j_1)\ldots h_{k+1}(j_{s-1})g(i_s)h_{k+1}(j_s) \bmod N. \tag{5}$$

This congruence relation becomes an equality in the group G, once normality is used to replace one entry $h_{k+1}(i) = g(i)$ for $i \in \{i, \ldots i_s\}$ by a suitable product $g(i) \cdot n(i)$ where $n(i) \in N$. This modification is applied to all the labeling functions $h_k : \mathcal{I}_{\omega_k} \to G$ with indices $k \geq n+1$. By the Weight Axiom and the equality that results from (5), the stabilization weight criterion for the words $(\omega_k, h_k \cdot N)$ carries over to the stabilization weight criterion for the modified words (ω_k, h_k) and the weight restriction for the word $(\omega, g \cdot N)$ implies the weight restriction for the modified word $(\omega, g \cdot n)$. So the modifications of the minimum weight coset representatives yield successive finite factorizations

$$(\omega_1, h_1) \preceq (\omega_2, h_2) \preceq \ldots \preceq (\omega_k, h_k) \preceq \ldots$$

that stabilize to the word $(\omega, g \cdot n)$. We conclude that $\omega_k\langle h_k\rangle = \omega\langle g \cdot n\rangle$ by the Factorization Axiom for the omega-group (G, wt). Because the labeling functions $h_k : I_{\omega_k} \to G$ use coset representatives of the labeling functions $h_k \cdot N : I_{\omega_k} \to G/N$, we have $\omega_k\langle h_k \cdot N\rangle = \omega_k\langle h_k\rangle \cdot N = \omega\langle g \cdot n\rangle \cdot N$. Thus,

$$\omega_k\langle h_k \cdot N\rangle = \omega_k\langle h_k\rangle \cdot N = \omega\langle g \cdot n\rangle \cdot N = \omega\langle g \cdot n \cdot N\rangle = \omega\langle g \cdot N\rangle.$$

This establishes the Factorization Axiom for the quotient group $(G/N, wt)$. □

Whenever (N, wt) is a normal omega-subgroup of the omega-group (G, wt), the omega-group structure on $(G/N, wt)$ in Theorem 1.4 makes the quotient homomorphism an omega-group homomorphism $q : (G, wt) \to (G/N, wt)$.

But we show in Section 3, with Examples 3.4 and 3.5, that not every quotient G/N of an omega-group (G, wt) modulo a normal subgroup N inherits the omega-group property. This makes the omega-group property ephemeral and disguises the omega-group heritage of some groups.

1.5 Weight-Derived Metric

We now explore metric aspects of the theory of omega-groups.

Weight-Derived Metric For any weighted group (G, wt), the **weight-derived metric** $m : G \times G \to [0, \infty)$ is defined by

$$m(g, g') = wt(g^{-1} \cdot g') \text{ for all } g, g' \in G. \tag{6}$$

The arguments for the following lemmas are straightforward:

Lemma 1.5 *The weight-derived metric $m : G \times G \to [0, \infty)$ on a weighted group (G, wt) satisfies the following properties for all $g, g', g'' \in G$:*

- (a) *the non-archimedean condition: $m(g, g'') \leq \max\{m(g, g'), m(g', g'')\}$;*
- (b) *inversion in G preserves distance to the identity: $m(g, 1_G) = m(g^{-1}, 1_G)$;*
- (c) *left multiplication in G preserves the metric: $m(g, g') = m(g'' \cdot g, g'' \cdot g')$.*

Lemma 1.6 *Conversely, any metric m on the group G satisfying $(a) - (c)$ in Lemma 1.5 determines a weight function $wt : G \to [0, \infty)$ via the definition $wt(g) = m(g^{-1}, 1_G)$ for all $g \in G$.* □

A portion of the omega-group definition for a weighted group (G, wt) can be expressed in terms of the completeness of the weight-derived metric on G. But this completeness condition, given below in Theorem 1.7, does not fully capture the omega-group condition, as we shall show in Theorem 2.7.

Theorem 1.7 *For any weighted group (G, wt), the weight-derived metric m is complete if and only if all (reverse) harmonic words can be evaluated in G.*

Proof: Any reverse harmonic word $(\bar{\eta}, g)$ involves a sequence of group elements $g = (g_k)$ with vanishing weights. The finite products $w_k = g_1 \cdot \ldots \cdot g_k, k \geq 1$, form a Cauchy sequence as the metric value $m(w_k, w_l)$, given by

$$wt(w_k^{-1} \cdot w_l) = wt(g_{k+1} \cdot \ \ldots \cdot \ g_l) \leq \max\{wt(g_{k+1}), \ \ldots \ , \ wt(g_l)\},$$

definitely tends to zero as $l \geq k \to \infty$. A limit for the sequence (w_k) serves as the evaluation $\bar{\eta}\langle g \rangle = g_1 \cdot \ldots \cdot g_k \cdot \ldots$ of the given word $(\bar{\eta}, g)$. Conversely, any Cauchy sequence (g_k) for the weight-derived metric gives distances $m(g_k, g_l) = wt(g_k^{-1} \cdot g_l) \to 0$ as $l \geq k \to \infty$. Then the reverse harmonic product

$$g_1 \cdot (g_1^{-1} \cdot g_2) \cdot (g_2^{-1} \cdot g_3) \ \ldots \ \cdot (g_{k-1}^{-1} \cdot g_k) \cdot (g_k^{-1} \cdot g_k) \cdot \ldots$$

serves as a limit of the given Cauchy sequence (g_k). □

From some viewpoints, the weight-derived metric m on a weighted group (G, wt) is perverse. By Lemma 1.5, left multiplication in the weight-derived metric group (G, m) is always continuous and inversion is continuous at the identity 1_G. But (G, m) is not a topological group in general. One way to assure that right multiplication and inversion are continuous in the weight-derived metric is to require that the following equivalent conditions hold.

Lemma 1.8 *For any weighted group (G, wt) and its weight-derived metric $m : G \times G \to [0, \infty)$, the following are equivalent conditions:*

(a) *right multiplication preserves the metric: $m(g, g') = m(g \cdot g'', g' \cdot g'')$;*

(b) *inversion in G preserves the metric: $m(g, g') = m(g^{-1}, g'^{-1})$; and*

(c) *conjugation in G preserves weights: $wt(g') = wt(g \cdot g' \cdot g^{-1})$.* □

Weight-Constrained Filtrations A weighted group (G, wt) has **weight-constrained subgroups** $G^{(k)} = \{g \in G : wt(g) < \frac{1}{k-1}\}, \ k \geq 1$, that constitute a filtration of G

$$\ldots \subseteq G^{(k+1)} \subseteq G^{(k)} \subseteq \ldots \subseteq G^{(2)} \subseteq G^{(1)} = G \tag{7}$$

with intersection $\cap G^{(k)} = \{1_G\}$. By the Weight Axiom, the weight-constrained subgroups $(G^{(k)}, wt)$ of an omega-group (G, wt) are omega-subgroups.

Theorem 1.9 *For a weighted group (G, wt), the weight-derived metric space (G, m) is totally disconnected. It is perfect if and only if all the weight-constrained subgroups $G^{(k)}, k \geq 1$, are non-trivial (hence, infinite).*

Proof: By the definition (6) of the weight-derived metric m, the open m-ball

$$B_m(g, \tfrac{1}{k}) = \{g' \in G : wt(g^{-1} \cdot g') < \tfrac{1}{k})\}$$

equals the coset $g \cdot G^{(k+1)}$ of the weight-constrained subgroup $G^{(k+1)}$. Thus, (G, m) is totally disconnected: the partition of G by the cosets of $G^{(k+1)}$ provide a topological separation in G of any two points $g, g' \in G$ having $m(g, g') = wt(g^{-1} \cdot g') \geq 1/k$ as they have difference $g^{-1} \cdot g' \notin G^{(k+1)}$ and hence disjoint open ball neighborhoods $g \cdot G^{(k+1)}$ and $g \cdot G^{(k+1)}$. Also, (G, m) is perfect precisely when every point $g \in G$ is an accumulation point of (G, m), equivalently, every open m-ball $B_m(g, \frac{1}{k}) = g \cdot G^{(k+1)}$ contains infinitely many members of G. □

In the case of a reparametrized weight function $wt : G \to \mathcal{H}$, the conditions $(a) - (c)$ of Lemma 1.8 are equivalent to the normality of the weight-constrained subgroups $G^{(k)}, k \geq 1$. We shall show in [B-S97(3)] that this normality requirement restricts the theory of omega-groups to those weighted groups (G, wt) that arise as coordinate-weighted inverse limit groups. The omega-group structure of inverse limit groups is interesting but not very subtle, in view of Theorem 1.7 and the following result:

Theorem 1.10 *If the weight-constrained subgroups $G^{(k)}, k \geq 1$, of a weighted group (G, wt) are normal subgroups and all (reverse) harmonic words for (G, wt) can be evaluated in G, then (G, wt) is an omega-group.*

Proof: Consider any word (ω, g) for (G, wt). Let $k \geq 1$. When all values $g(i)$ of labeling function $g : \mathcal{I}_\omega \to G$ that have weight $wt(g(i)) < \frac{1}{k}$ are converted to values 1_G, the result is a labeling function $g_{(k)} : \mathcal{I}_\omega \to G$ all of whose non-identity values have $wt \geq \frac{1}{k}$. Then the words $(\omega, g_{(k)}), k \geq 1$, are finite words, whose evaluations $\omega\langle g_{(k)}\rangle, k \geq 1$, exist in the group G. Because the weight-constrained subgroups $G^{(k)}, k \geq 1$, are normal subgroups and because the product $\omega\langle g_{(k)}\rangle$ arises from the product $\omega\langle g_{(k-1)}\rangle$ by the insertion of finitely many elements having $wt = \frac{1}{k}$, then each difference $(\omega\langle g_{(k-1)}\rangle)^{-1} \cdot (\omega\langle g_{(k)}\rangle)$ has $wt < \frac{1}{k-1}$. Therefore, the reverse harmonic product

$$\omega\langle g_{(1)}\rangle \cdot (\omega\langle g_{(1)}\rangle^{-1} \cdot \omega\langle g_{(2)}\rangle) \cdot \ldots \cdot ((\omega\langle g_{(k-1)}\rangle)^{-1} \cdot \omega\langle g_{(k)}\rangle) \cdot \ldots$$

exists in the group (G, wt) by hypothesis. This product qualifies as the evaluation $\omega\langle g\rangle \in G$. The omega-group axioms for these evaluations carry over from assumed axioms for the harmonic products. □

In the absence of the normality property of the weight-constrained subgroups, the ability to form (reverse) harmonic products in a weighted group does not explain all omega-group structures. In Theorem 2.7, it is shown that there is a subgroup (G, wt) of a free omega-group with this property: all harmonic and reverse harmonic words (η, g) and $(\bar{\eta}, g)$ for (G, wt) can be evaluated in G, but some order-type words (ω, g) for (G, wt) cannot be evaluated in G.

1.6 Weight-Derived Metric and Factorization Axiom

The Factorization Axiom implies that a weighted group (G, wt) has at most one word evaluation that renders it an omega-group. No great use of this point is made in this paper; however, the consequence of the Factorization Axiom expressed in Theorem 1.12 below is crucial for [B-S97(4)].

Let (G, wt) be an omega-group. For each word (ω, g) for (G, wt) and each point $p \in \omega$, let (ω_p, g_p) denote the subword of (ω, g) on the subinterval $[0, p]$. The word (ω, g) has value $\omega\langle g\rangle$ and the subword (ω_p, g_p) has value has value $\omega_p\langle g_p\rangle$. So the labeling function $g : \mathcal{I}_\omega \to G$ for the complementary intervals of ω determines the word (ω, g) and the word, in turn, defines a function $g : \omega \to G$ via the definition: $g(p) = \omega_p\langle g_p\rangle$ for all $p \in \omega$.

Lemma 1.11 *For each word (ω, g) in an omega-group (G, wt), the function $g : \omega \to G$, $g(p) = \omega_p\langle g_p\rangle$, is continuous in the Euclidean metric on $\omega \subset I$ and the weight-derived metric m (6) on G.*

Proof: By the Multiplication Axiom, the function $g : \omega \to G$ satisfies the equation $g(p) \cdot g(p, q) = g(q)$ for each pair $p < q$ in ω, where $g(p, q) \in G$ denotes the evaluation $\omega_{p,q}\langle g_{p,q}\rangle$ of the subword $(\omega_{p,q}, g_{p,q})$ of (ω, g) on the subinterval $[p, q]$. Therefore, by the Weight Axiom,

$$m(g(p), g(q)) = wt(g(p)^{-1} \cdot g(q)) = wt(g(p, q)) \leq \max\{wt(g(i)) : i \subset [p, q]\}.$$

Because $wt(g(i)) \to 0$ as $l(i) \to 0$, then $g(q) \to g(p)$ in G as $q \to p$ in I. □

Notice that the labeling function $g : \mathcal{I}_\omega \to G$ and the continuous function $g : \omega \to G$, $g(p) = \omega_p\langle g_p\rangle$, conspire to have the property: $g(a_i) \cdot g(i) = g(b_i)$ in G for each interval $i = (a_i, b_i) \in \mathcal{I}_\omega$. The Factorization Axiom for the omega-group (G, wt) implies the uniqueness of such a continuous function:

Theorem 1.12 *Let (ω, g) be a word for an omega-group (G, wt). Any continuous function $h : \omega \to G$, such that $h(0) = 1_G$ and $h(a_i) \cdot g(i) = h(b_i)$ in G for each interval $i = (a_i, b_i) \in \mathcal{I}_\omega$, necessarily has the values $h(p) = \omega_p\langle g_p\rangle$ for all $p \in \omega$. In particular, $h(1) = \omega\langle g\rangle$.*

Proof: Given any word (ω, g), consider the successive finite factorizations

$$(\omega_1, h_1) \preceq (\omega_2, h_2) \preceq \ldots \preceq (\omega_k, h_k) \preceq \ldots \preceq (\omega, g),$$

obtained as follows: ω_k is the finite order-type comprised of $0, 1$, and the endpoints a_i and b_i of those finitely many complementary intervals $i = (a_i, b_i) \in \mathcal{I}_\omega$ having labels $g(i)$ with $wt(g(i)) \geq 1/k$; and the labeling function $h_k : \mathcal{I}_\omega \to G$ is defined on each interval $j = (a_j, b_j) \in \mathcal{I}_{\omega_k}$ by $h_k(j) = h(a_j)^{-1} \cdot h(b_j)$. Notice that, by hypothesis, h_k uses the label $g(i) = h(a_i)^{-1} \cdot h(b_i)$ for the intervals $i \in \mathcal{I}_{\omega_k} \cap \mathcal{I}_\omega$. By construction, the evaluations of these successive finite factorizations are finite products of differences $h(a_j)^{-1} \cdot h(b_j)$ in G that collapse to the same difference:

$$h(1) = h(0)^{-1} \cdot h(1) = \omega_1\langle h_1 \rangle = \omega_2\langle h_2 \rangle = \ldots = \omega_k\langle h_k \rangle = \ldots.$$

The intervals $j = (a_j, b_j) \in \mathcal{I}_{\omega_k}$ have label weights

$$wt(h_k(j)) = wt(h(a_j)^{-1} \cdot h(b_j)) = m(h(a_j), h(b_j)) \to 0$$

as $k \to \infty$ and $l(j) \to 0$, because $h : \omega \to G$ is continuous.

This proves that the successive finite factorizations of $h(1)$ stabilize to the word (ω, g). By the Factorization Axiom for the omega-group (G, wt), we conclude that $h(1) = \omega\langle g \rangle$. The same argument applies to the initial subinterval $[0, p]$ for each $p \in \mathcal{I}_\omega$ to prove $h(p) = \omega_p\langle g_p \rangle$. □

Theorem 1.12 is used in [B-S97(4)] to show that, in the metric Cayley graph $\Gamma(G, wt, S)$ with respect to a subset S of an omega-group (G, wt), the vertices visited by a continuous edge path are determined by its initial vertex and the labels from S on the preceding edges it traversed. This is a trivial property in the usual Cayley graph $\Gamma(G, S)$ with respect to a subset S of an ordinary group G, because one considers only finite edge paths corresponding to finite words of the members of $S \subset G$. But for an omega-group (G, wt), one has to consider continuous edge paths corresponding to order-type words (ω, g) with labels from the weighted subset (S, wt) of (G, wt).

2 WEIGHTED COMBINATORICS

In this section, we consider omega-groups as they arise in the combinatorial terms of generators and relators. We give the fundamental construction of a free omega-group with a weighted basis. Then we consider groups presented by a weighted basis and a weighted relator set, namely, the quotients of the free omega-group with weighted basis modulo the weighted normal closure of the weighted relator set. These presented groups do not always inherit the omega-group status of the free omega-group, and so the omega-group origin of some of these groups is obscured.

2.1 Free Omega-Groups

A free group is an algebraic structure that satisfies the group axioms but has no further relations among its members beyond those axioms. We now show that, for any weighted alphabet (X, wt), the word group $\Omega(X, wt)$, introduced in Section 2, of similarity classes of order-type words in (X, wt) is a free example of the omega-group axioms.

Pre-Reduced Words A word (ω, x) for the weighted alphabet (X, wt) is **pre-reduced** if it any subword (ω^*, x^*) of (ω, x) that is similar to an identity word is actually an identity word: $(\omega^*, x^*) \overset{s}{\sim} (\omega^*, 1_X)$ implies $(\omega^*, x^*) = (\omega^*, 1_X)$.

Lemma 2.1 *Any word (ω, x) for a weighted alphabet (X, wt) is similar to a pre-reduced word.*

Proof: For each word (ω, x), there exist maximal non-identity subwords similar to identity words. To prove this, we apply Zorn's Lemma to the set $\mathcal{I}$ of all non-identity subwords of (ω, x) similar to identity words, ordered by inclusion $\subset$ of their supporting intervals in $[0, 1]$. To show that any chain in $\mathcal{I}$ has an upper bound it suffices to consider a countable chain in $\mathcal{I}$:

$$(\gamma_1, y_1) \subset (\gamma_2, y_2) \subset \ldots \subset \gamma_k, y_k) \subset \ldots,$$

since any chain in $\mathcal{I}$ has the same union as a countable chain. The *union word*

$$(\gamma, y) = (\gamma = \text{Closure}(\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k \cup \ldots), y = y_1 \cup y_2 \cup \ldots \cup y_k \cup \ldots)$$

of the countable chain is similar to an identity word, as we can build nested similarity squares $S_1 \subset S_2 \subset \ldots \subset S_k \subset \ldots$ for the similarities

$$(\gamma_1, y_1) \overset{s}{\sim} (\gamma_1, 1_X), (\gamma_2, y_2) \overset{s}{\sim} (\gamma_2, 1_X), \ldots, (\gamma_k, y_k) \overset{s}{\sim} (\gamma_k, 1_X), \ldots,$$

inductively, as follows: Once nested similarity squares $S_1 \subset S_2 \subset \ldots \subset S_{k-1}$ have been constructed and $T_k : (\gamma_k, y_k) \overset{s}{\sim} (\gamma_k, 1_X)$ is any given similarity square, modify T_k by pushing into the subword (γ_{k-1}, y_{k-1}) in its base edge word (γ_k, y_k) an upside-down copy C_{k-1} of the similarity square S_{k-1} (drawn to look like a half-disc with the word (γ_{k-1}, y_{k-1}) on its half-circumference and $(\gamma_{k-1}, 1_X)$ on its diameter.) Then $T_k \cup C_{k-1}$ is a similarity square for the complementary word $(\gamma_k, y_k) - (\gamma_{k-1}, y_{k-1})$ that results when all the labels on the intervals of γ_{k-1} are replaced by 1_X. Now modify $T_k \cup C_{k-1}$ by pushing into the subword $(\gamma_{k-1}, 1_X)$ on its base edge a right-side-up copy of the similarity square S_{k-1} (drawn to look like a half-disc with the word $(\gamma_{k-1}, 1_X)$ on its half-circumference and (γ_{k-1}, y_{k-1}) on its diameter.) Then $S_k = T_k \cup C_{k-1} \cup S_{k-1}$ is a similarity square $S_k : (\gamma_k, y_k) \overset{s}{\sim} (\gamma_k, 1_X)$ that contains the similarity square

$S_{k-1} : (\gamma_{k-1}, y_{k-1}) \overset{s}{\sim} (\gamma_k, 1_X)$. The nested union $S_1 \cup S_2 \cup \ldots \cup S_k \cup \ldots$ is a similarity square proving that the union word (γ, y) is similar to the identity word $(\gamma, 1_X)$.

By Zorn's Lemma, then, any labeled interval of (ω, x) that belongs of a subword (γ, y) with $(\gamma, 1_X) \neq (\gamma, y) \overset{s}{\sim} (\gamma, 1_X)$ belongs to a maximal such subword (though not a unique such maximal subword as the middle label of the word (x, x^{-1}, x) shows). Then a maximal family $\mathcal{F}$ of non-overlapping maximal subwords $(\gamma, 1_X) \neq (\gamma, y) \overset{s}{\sim} (\gamma, 1_X)$ of (ω, x) exist. When all subwords in such a maximal family $\mathcal{F}$ for (ω, x) are converted into identity words the result (ω, z) is similar to the given word (ω, x). The result (ω, z) is pre-reduced, because any non-identity subword of (ω, z) similar to an identity word would either contradict the maximality of the family $\mathcal{F}$ or the maximality of one of the member words (γ, y) of the family $\mathcal{F}$. □

Reduced Words A word (ω, x) for the weighted alphabet (X, wt) is **reduced** if either it is the monosyllabic identity word $0 < 1_X < 1$ or no subword (ω^*, x^*) of (ω, x) is similar to an identity word. So a reduced word not similar to an identity word contains no identity element labels.

Lemma 2.2 *Any pre-reduced word (ω, x) for a weighted alphabet (X, wt) is similar to a reduced word.*

Proof: Let r denote the sum of the lengths $l(i) = b_i - a_i$ of the complementary intervals $i \in \mathcal{I}_\omega$ of the order-type ω that have label $x(i) \neq 1_X$. If $r = 0$, then (ω, x) is an identity word, and so is associated to the monosyllabic identity word $0 < 1_X < 1$. If $0 < r \leq 1$, then (ω, x) is not an identity word, and is associated to the reduced word (ω', x') obtained from (ω, x) by collapsing the intervals $i \in \mathcal{I}_\omega$ that have label $x(i) = 1_X$. To see how the other intervals expand and the points of ω coalesce into points of a new order-type, define ω' as the image of the function $\omega \to [0, 1]$ that sends $p \in \omega$ to $\frac{1}{r}$ times the sum of the lengths $l(i)$ of the complementary intervals $i \in \mathcal{I}_\omega$ that are contained in $[0, p]$ and have label $x(i) \neq 1_X$. The labeling function $x' : \mathcal{I}_{\omega'} \to X$ simply adopts the non-trivial values of the labeling function $x : \mathcal{I}_\omega \to X$. □

Free Reductions A reduced word (ω', x') similar to a given word (ω, x) is called a **free reduction** of (ω, x) in $\Omega(X, wt)$. In two steps, Lemmas 2.1 and 2.2 yield a free reduction (ω', x') of each word (ω, x). When (ω, x) is similar to an identity word, it has the unique free reduction $0 < 1_X < 1$. Similarly, the free reduction of any word is unique, at least up to word associativity:

Lemma 2.3 *Any two free reductions (ω', x') and (ω'', x'') of a word (ω, x) for the weighted alphabet (X, wt) are associated: $(\omega', x') \overset{a}{\sim} (\omega'', x'')$.*

Proof: The two free reductions (ω', x') and (ω'', x'') of (ω, x) come with similarity squares $S' : (\omega', x') \overset{s}{\sim} (\omega, x)$ and $S'' : (\omega, x) \overset{s}{\sim} (\omega'', x'')$, which glue along (ω, x) into a similarity square $A : (\omega', x') \overset{s}{\sim} (\omega'', x'')$ (once simple closed curves that arise are discarded). Any cancellation arc in the similarity square A would cut off both a subword and a similarity square proving the subword is similar to an identity word. Because (ω', x') and (ω'', x'') are reduced words, this similarity square A must be free of such cancellation arcs and so is an associativity square $A : (\omega', x') \overset{a}{\sim} (\omega'', x'')$, as desired. □

Word-Weights The word group $\Omega(X, wt)$ inherits its own weight function. The **word-weight function** $w\text{-}wt : \Omega(X, wt) \to [0, \infty)$ for the word group $\Omega(X, wt)$ assigns to each similarity class $[(\omega, x)]$ the maximal weight $wt(x(i))$ achieved by a label $x(i)$ of the reduced representative (ω, x) of the class. This makes the word group $\Omega(X, wt)$ into a weighted group $(\Omega(X, wt), w\text{-}wt)$.

We now show that this word-weighted word group $(\Omega(X, wt), w\text{-}wt)$ is an omega-group. Basically, the point is that a *word of words* for (X, wt) can be evaluated as a *composite word* for (X, wt), provided that the word-weight is used on the internal words when forming the external word. The story is slightly more complicated since we deal with a word of similarity classes of words and its evaluation is a similarity class of a composite word for (X, wt).

Here are the details of the construction of a composite word: When suitably re-scaled copies of any family $\lambda(i), i \in \mathcal{I}_\omega$, of order-types are placed on the complementary intervals $i \in \mathcal{I}_\omega$ of any order-type ω, the result is an order-type, denoted by $\omega \circ \lambda$ and called the **composite** of ω and the family $\lambda(i), i \in \mathcal{I}_\omega$. The complementary intervals $j \in \mathcal{I}_{\omega\circ\lambda}$ of the composite order-type $\omega \circ \lambda$ are rescaled versions $i(j)$ of the complementary intervals $j \in \mathcal{I}_{\lambda(i)}$, where $i \in \mathcal{I}_\omega$. Given a family of words $g(i) = (\lambda(i), x(i))$ for (X, wt), indexed by $i \in \mathcal{I}_\omega$, having word-weights that tend to zero as the ω intervals' lengths $l(i)$ tend to zero, there is the **composite word** $\omega\langle g\rangle = (\omega \circ \lambda, x)$ whose labeling function $x : \mathcal{I}_{\omega\circ\lambda} \to X$ is given by $x(i(j)) = x(i)(j)$ for $j \in \mathcal{I}_{\lambda(i)}$ and $i \in \mathcal{I}_\omega$. The composite word $\omega\langle g\rangle = (\omega \circ \lambda, x)$ is characterized by the fact that its subwords on the intervals $i \in \mathcal{I}_\omega$ are the given words $g(i) = (\lambda(i), x(i)), i \in \mathcal{I}_\omega$.

Theorem 2.4 *Consider any weighted alphabet (X, wt).*

(*a*) *The word-weighted word group $(\Omega(X, wt), w\text{-}wt)$ is an omega-group.*

(*b*) *Universal Property: A weighted function $\phi : (X, wt) \to (G, wt)$ to an omega-group (G, wt) (i.e., $\phi(1_X) = 1_G$, $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in X$, and $wt(\phi(x)) \to 0$ whenever $wt(x) \to 0$) determines a unique omega-group homomorphism $\phi : (\Omega(X, wt), w\text{-}wt) \to (G, wt)$ such that $\phi([(\tau, x)]) = \phi(x)$ for all the monosyllabic words $(\tau, x) = (0 < x < 1)$.*

Proof: (a) Consider any word $(\omega, [g])$ for the word-weighted word group $(\Omega(X, wt), w\text{-}wt)$. By the word-weight restriction, we can evaluate the word of reduced representative words (ω, g) as the composite word $\omega\langle g\rangle$ having subwords $g(i)$ on the intervals $i \in \mathcal{I}_\omega$ of the order-type ω. We take the similarity class in the word group $\Omega(X, wt)$ of the composite word $\omega\langle g\rangle$ as the evaluation of the given word $(\omega, [g])$: $\omega\langle [g]\rangle := [\omega\langle g\rangle]$. This is an invariant of the similarity class of the word $(\omega, [g])$. The Weight and Multiplication Axiom for easily checked for this evaluation $\langle - \rangle : \Omega(\Omega(X, wt), w\text{-}wt) \to \Omega(X, wt)$.

To establish the Factorization Axiom, consider successive finite factorizations

$$(\omega_1, [h_1]) \preceq (\omega_2, [h_2]) \preceq \ldots \preceq (\omega_k, [h_k]) \preceq \ldots$$

that stabilize to a word $(\omega, [g])$. Because $(\omega_k, [h_k]) \preceq (\omega_{k+1}, [h_{k+1}])$, a similarity square $S_k : S_k : \omega_k\langle h_k\rangle \overset{s}{\sim} \omega_{k+1}\langle h_{k+1}\rangle$ can be assembled by stacking side-by-side similarity squares between each reduced word $h_k(j)$ for $j \in \mathcal{I}_{\omega_k} \setminus \mathcal{I}_\omega$ and the finite product into which it factors in the word (ω_{k+1}, h_{k+1}), as well as (identity-like) similarity squares of parallel arcs between each reduced word $h_k(i) = g(i)$ for $i \in \mathcal{I}_{\omega_k} \cap \mathcal{I}_\omega \cap \mathcal{I}_{\omega_{k+1}}$ and its appearance as $h_{k+1}(i) = g(i)$ in the word (ω_{k+1}, h_{k+1}). The stabilization weight criterion guarantees that the stack of the similarity squares $S_1, \ldots, S_k, \ldots$ rests on the composite word $\omega\langle g\rangle$ to yield a similarity square $S : \omega_1\langle h_1\rangle \overset{s}{\sim} \omega\langle g\rangle$; any arc in the stack with one end on $\omega_1\langle h_1\rangle$ or $\omega\langle g\rangle)$ cannot wander endlessly and so has its other end on $\omega_1\langle h_1\rangle$ or $\omega\langle g\rangle)$. Thus the constant evaluations $\omega_k\langle [h_k]\rangle$ equal $[\omega\langle g\rangle] = \omega\langle [g]\rangle$.

(b) By its weight restriction, the assignment $\phi : X \to G$ converts any word (ω, x) for (X, wt) into a word $(\omega, \phi(x))$ for (G, wt), using the composite labeling function $\phi(x) := \phi \circ x : \mathcal{I}_\omega \to X \to G$. Since any similarity square $S : (\omega, x) \overset{s}{\sim} (\omega', x')$ converts to a similarity square $S : (\omega, \phi(x)) \overset{s}{\sim} (\omega', \phi(x'))$ via an application of the assignment ϕ and since the word evaluation for the omega-group (G, wt) is constant on similarity classes, then the assignment $[(\omega, x)] \to \omega\langle\phi(x)\rangle \in G$ defines a function $\phi : (\Omega(X, wt), w\text{-}wt) \to (G, wt)$. By this definition of ϕ and retraction property of the word evaluation of the omega-group (G, wt), we have $\phi([(\tau, x)]) = \tau\langle\phi(x)\rangle = \phi(x)$.

This function $\phi : (\Omega(X, wt), w\text{-}wt) \to (G, wt)$ is an omega-group homomorphism: First of all, if members $[(\omega, x)] \in \Omega(X, wt)$ have $w\text{-}wt([(\omega, x)]) \to 0$, then their images $\omega\langle\phi(x)\rangle \in G$ have $wt(\omega\langle\phi(x)\rangle) \to 0$. This holds since ($i$) for a reduced word (ω, x), $wt(x(i)) \leq w\text{-}wt([(\omega, x)])$, for all $i \in \mathcal{I}_\omega$, (ii) $wt(\phi(x)) \to 0$ whenever $wt(x) \to 0$ by the hypothesis on ϕ, and (iii) $wt(\omega\langle\phi(x)\rangle) \leq \max\{wt(\phi(x(i)) : i \in \mathcal{I}_\omega\}$ by the Weight Axiom for the omega-group (G, wt). Secondly, the image under ϕ of the value $[(\omega \circ \lambda, x)]$ in $\Omega(X, wt)$ of any word $(\omega, [(\lambda, x)])$ for $\Omega(X, wt)$ equals the value $\omega\langle\lambda\langle\phi(x)\rangle\rangle$ in (G, wt) of the image $(\omega, \phi([(\lambda, x)]))) = (\omega, \lambda\langle\phi(x)\rangle)$ of that word, by the definition of

ϕ and Multiplication Axiom for the omega-group (G, wt):

$$\phi([(\omega \circ \lambda, x)]) = \omega \circ \lambda\langle\phi(x)\rangle = \omega\langle\lambda\langle\phi(x)\rangle\rangle.$$

Now let $\phi' : (\Omega(X, wt), w\text{-}wt) \to (G, wt)$ be a second omega-group homomorphism with the assigned values $\phi'([(\tau, x)]) = \phi(x)$ for all the monosyllabic words (τ, x) for $x \in X$. Because any word (ω, x) can be viewed as a composite word $(\omega \circ \tau, x)$ with monosyllabic subwords $(\tau, x(i))$ on the intervals $i \in \mathcal{I}_\omega$, its similarity class in $\Omega(X, wt)$ is an ω-product: $[(\omega, x)] = [(\omega \circ \tau, x)] = \omega\langle[(\tau, x)]\rangle$ by Multiplication Axiom for $(\Omega(X, wt), w\text{-}wt)$. Then the omega-group homomorphism properties of ϕ' give the calculation:

$$\phi'([(\omega, x)]) = \phi'(\omega\langle[(\tau, x)]\rangle) = \omega\langle\phi'([(\tau, x)])\rangle = \omega\langle\phi(x)\rangle,$$

which proves that the omega-group homomorphisms ϕ' and ϕ coincide. □

Free Omega-Groups In view of Theorem 2.4(b), we call the word-weighted word group $(\Omega(X, wt), w\text{-}wt)$ the **free omega-group with weighted basis** (X, wt). By Lemmas 2.2 and 2.3, we may represent the members of the free omega-group as reduced words (ω, x) for (X, wt). As in the proof of Theorem 2.4(a), order-type products in the free omega-group can be expressed in terms of reduced words via composition followed by free reduction.

In [B-S97(4)], we show that the metric Cayley graph of the weighted-group $(\Omega(X, wt), w\text{-}wt)$ with respect to its weighted basis (X, wt) is a contractible metric 1-complex whose vertex set is a totally disconnected perfect copy of the group $\Omega(X, wt)$, in short, a *wild metric tree.* This is another justification for calling $\Omega(X, wt)$ a free omega-group with weighted basis (X, wt).

The universal property (b) in Theorem 2.4 is not available for unweighted assignments $X \to G$ to an omega-group (G, wt). Infinite words in $\Omega(X, wt)$ preclude arbitrary basis assignment to trivially-weighted omega-groups:

Theorem 2.5 *A group homomorphism $\phi : \Omega(X, wt) \to F(Y)$ to a free group $F(Y)$ trivializes the free omega-subgroup $\Omega(X^{(k)}, wt)$ having restricted basis $X^{(k)} = \{x \in X : wt(x) < \frac{1}{k-1}\}$ for all sufficiently large $k \geq 1$.*

Proof: Let $\phi : \Omega(X, wt) \to F(Y)$ be a group homomorphism to a free group $F(Y)$ with basis Y. Then all words for (X, wt) with sufficiently small word-weight must have trivial image; otherwise, we could construct words $w \in \Omega(X, wt)$ with non-trivial images $W = \phi(w) \in F(Y)$ having impossibly large word length $||W||$ with respect to the basis Y, as follows. For any sequence p_k of positive integers and any sequence u_k of words for (X, wt) having word-weights $w\text{-}wt(u_k) \to 0$ as $k \to \infty$, the system of recursive equations

$$w_0 = u_1 \cdot w_1^{p_1},\ w_1 = u_2 \cdot w_2^{p_2},\ \ldots\ ,\ w_{k-1} = u_k \cdot w_k^{p_k},\ \ldots \tag{8}$$

uniquely defines words $w_0, w_1, \ldots, w_k, \ldots$ for (X, wt) via infinite application of word composition. The group homomorphism ϕ carries the order-type words $w_k \in \Omega(X, wt)$ defined by the system (8) in $\Omega(X, wt)$ to finite words $W_k = \phi(w_k) \in F(Y)$ satisfying the system of equations

$$W_0 = U_1 \cdot W_1^{p_1},\ W_1 = U_2 \cdot W_2^{p_2},\ \ldots\ ,\ W_{k-1} = U_k \cdot W_k^{p_k},\ \ldots \tag{9}$$

in the free group $F(Y)$, where $U_k = \phi(u_k)$. Were there a sequence of group elements $u_k \in \Omega(X, wt)$, as above, with non-trival images $U_k = \phi(u_k) \neq 1_Y$, the latter would have finite word lengths $||U_k|| \geq 1$ with respect to the free basis Y. Using the exponents $p_k = ||U_k|| + 2$ in the defining system of equations (8), the words $W_{k-1} = U_k \cdot W_k^{p_k}$ would have word lengths $||W_{k-1}|| \geq ||W_k|| + 1$. In case $W_k = 1_Y$, this would follow from

$$||W_{k-1}|| = ||U_k \cdot W_k^{p_k}|| = ||U_k|| \geq 1 = ||W_k|| + 1.$$

In case $W_k \neq 1_Y$, this would follow from

$$||W_{k-1}|| = ||U_k \cdot W_k^{p_k}|| \geq ||W_k^{p_k}|| - ||U_k|| \geq ||W_k|| + (p_k - 1) - ||U_k|| = ||W_k|| + 1.$$

So we would have $||W_0|| \geq ||W_k|| + k$ for all $k \geq 1$, which would make W_0 an impossibly long word in the free group $F(Y)$. $\square$

The positive weights in the weighted basis (X, wt) are *bounded away from zero* if there is some bound $0 < \epsilon < wt(x)$ valid for all $1_X \neq x \in X$. In that case, each order-type word (ω, x) for (X, wt) has just finitely many non-identity entries $x(i) \neq 1_X, i \in \mathcal{I}_\omega$ and so is associated to a finite word in the alphabet X. The similarity classes of these finite words constitute the usual free group $F(X)$ with (un-weighted) basis X. In this situation, the free-omega group $\Omega(X, wt)$ is simply the ordinary free group $F(X)$. This is the only way a free-omega group $\Omega(X, wt)$ can be isomorphic to a free group according to the following corollary of Theorem 2.5:

Corollary 2.6 *A free omega-group $\Omega(X, wt)$ is isomorphic to a free group if and only if the positive weights in (X, wt) are bounded away from zero, in which case, $\Omega(X, wt) = F(X)$.*

Proof: By Theorem 2.5, a group isomorphism $\phi : \Omega(X, wt) \to F(Y)$ trivializes the free omega-subgroup $\Omega(X^{(k)}, wt)$ for sufficiently large $k \geq 1$. So the weight restricted basis $X^{(k)}$ is contained in Ker $\phi = \{1_X\}$ for sufficiently large $k \geq 1$. In other words, the weights in (X, wt) are bounded away from zero. $\square$

In [G52], G. Higman gave a similar argument to prove that the inverse limit of a certain sequence of finitely generated free groups is not free.

By Theorem 2.4, every group homomorphism

$$\phi : \Omega(X, wt) \to F(Y) = F(Y, t\text{-}wt)$$

arises from a weighted function $(X, wt) \to (F(Y, t\text{-}wt), w\text{-}wt)$ to the word-weighted free (omega-) group and is actually an omega-group homomorphism

$$\phi : (\Omega(X, wt), w\text{-}wt) \to (F(Y, t\text{-}wt), w\text{-}wt).$$

But not every group homomorphism $\phi : \Omega(X, wt) \to F(X', wt')$ between free omega-groups is an omega-group homomorphism. When the weighted basis (X, wt) has members $x \in X$ with weights $wt(x) \to 0$, conjugation

$$x^* \cdot (-) \cdot x^{*-1} : \Omega(X, wt) \to \Omega(X, wt)$$

by a non-identity basis member $1_X \neq x^* \in X$ is not an omega-group homomorphism $(\Omega(X, wt), w\text{-}wt) \to (\Omega(X, wt), w\text{-}wt)$. This group isomorphism is not even a weighted group homomorphism since $w\text{-}wt(x^* \cdot x \cdot x^{*-1}) \geq wt(x^*)$ for all $x \in X$. The conjugate of an infinite word (ω, x) is not the word of conjugates $(\omega, x^* \cdot x \cdot x^{*-1})$, since the latter is not even defined.

Freeness by Order-Type Class When the weighted alphabet (X, wt) has non-trivial weights that limit on zero, the free group $F(X)$ and the free omega-group $\Omega(X, wt)$ are distinct extremes of the spectrum of *freeness*. Intermediate groups $F(X) \leq G \leq \Omega(X, wt)$ demonstrate other possibilities.

Theorem 2.7 *Consider the word-weighted subgroup $(\mathcal{C}(X, wt), w\text{-}wt)$ of all reduced words (ω, x) for (X, wt) based on countable order-types ω. Words for $(\mathcal{C}(X, wt), w\text{-}wt)$ based on countable order-types can be evaluated in $\mathcal{C}(X, wt)$, but not so for all those words based upon uncountable order-types.*

Proof: The composite of countable order-types within a countable order-type is again a countable order-type. So the weighted group $(\mathcal{C}(X, wt), w\text{-}wt)$ is closed under the evaluation of words based on countable order-types. Each reduced word in $\mathcal{C}(X, wt)$ has only countably many initial subwords, but a reduced word in $\Omega(X, wt)$ based upon an uncountable order-type has uncountably many initial subwords. So a typical word for $(\mathcal{C}(X, wt), w\text{-}wt)$ based upon an uncountable order-type cannot be evaluated in $\mathcal{C}(X, wt)$. $\square$

So the classes $\mathcal{F} \subset \mathcal{C} \subset \Omega$ of finite, countable, and arbitrary order-types, determine their own classes of groups: the $\mathcal{F}$-groups (i.e., ordinary groups), $\mathcal{C}$-groups, and Ω-groups (the omega-groups), satisfying the weakened omega-group evaluation requirement that words of the class can be evaluated in the group. For each weighted alphabet (X, wt), the subgroups

$$F(X) = \mathcal{F}(X, wt) \subset \mathcal{C}(X, wt) \subset \Omega(X, wt)$$

are free groups of their respective classes, in the sense that each one satisfies the universal property for freeness among all weighted groups in its class.

2.2 Weighted Presentations

In a presentation theory for weighted groups, one has the option of weighting the relators in excess of the weights of the generators that they involve. This enables weighted presentations to express the fundamental groups of metric cell-complexes whose 2-cells' diameter may exceed that of their boundary. This possibility is indicated below and is more fully explored in [B-S97(1)].

Weighted Presentations A **weighted presentation**

$$\mathcal{P} \ = < (X, wt) : (R, r\text{-}wt) > \tag{10}$$

consists of a **weighted generator alphabet** (X, wt) and a **weighted relator set** $(R, r\text{-}wt)$. The latter is comprised of reduced words $r = (\gamma_r, y_r)$ and their inverses from the free omega-group $(\Omega(X, wt), w\text{-}wt)$ with weighted basis (X, wt) that are assigned **relator-weights** $r\text{-}wt(r) \geq w\text{-}wt(r)$ independently of their word-weights $w\text{-}wt(r)$ derived from their constituent generators in X. For convenience, the identity element $1_X \in X$ is included in R where it is denoted by 1_R and is assigned relator-weight $r\text{-}wt(1_R) = 0$.

For a weighted presentation $\mathcal{P}$ (10), the *normal closure* in the group $\Omega(X, wt)$ of the set R of relators is the group $N(R) = N_{\Omega(X,wt)}(R)$ of all finite products $r_1^{w_1} \cdot r_2^{w_2} \ldots \cdot r_k^{w_k}$ of conjugates $r_i^{w_i} = w_i \cdot r_i \cdot w_i^{-1}$ in $\Omega(X, wt)$ of the relators $r_i \in R$ by reduced words $w_i \in \Omega(X, wt)$. Each such finite product of conjugates is called a *consequence* of R in $\Omega(X, wt)$ and can be written

$$((w'_1 \cdot r_1) \cdot (w'_2 \cdot r_2) \cdot \ldots \cdot (w'_k \cdot r_k)) \cdot (w'_1 \cdot w'_2 \cdot \ldots \cdot w'_k)^{-1},$$

using the words $w'_1 = w_1, w'_2 = w_1^{-1} \cdot w_2, \ldots, w'_k = w_{k-1}^{-1} \cdot w_k$. This concept of consequences of R in $\Omega(X, wt)$ and this construction of the normal closure $N_{\Omega(X,wt)}(R)$ of R in $\Omega(X, wt)$ have weighted generalizations, as follows:

Weighted Normal Closure Consider any order-type λ and a pair of words (λ, w) and (λ, r) for the two weighted sets $(\Omega(X, wt), w\text{-}wt)$ and $(R, r\text{-}wt)$, respectively. These λ words involve labels $w(i) \in \Omega(X, wt)$ and $r(i) \in R$, for the intervals $i \in \mathcal{I}_\lambda$, having word-weights $w\text{-}wt(w(i))$ and relator-weights $w\text{-}wt(r(i)) \leq r\text{-}wt(r(i))$ that tend to zero as the λ intervals' lengths $l(i) \to 0$. By the product weight axiom for $w\text{-}wt$, the word-weights of the product labels $w(i) \cdot r(i) \in \Omega(X, wt)$, $i \in \mathcal{I}_\lambda$, also tend to zero as $l(i) \to 0$. So $(\lambda, w \cdot r)$ is a second word for the weighted set $(\Omega(X, wt), w\text{-}wt)$. Because $(\Omega(X, wt), w\text{-}wt)$ is an omega-group, the word evaluations $\lambda\langle w \cdot r\rangle$ and $\lambda\langle w\rangle$ exist in $\Omega(X, wt)$.

A **weighted consequence** of the weighted relator set $(R, r\text{-}wt)$ in the free omega-group $(\Omega(X, wt), w\text{-}wt)$ is any product of the form

$$\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1} \in \Omega(X, wt), \tag{11}$$

associated with an order-type λ and a pair of words (λ, w) and (λ, r) for the two weighted sets $(\Omega(X, wt), w\text{-}wt)$ and $(R, r\text{-}wt)$, respectively. The **weighted normal closure** of the weighted relator set $(R, r\text{-}wt)$ is the group $N(R, r\text{-}wt)$ of all weighted consequences (11) of $(R, r\text{-}wt)$ in $\Omega(X, wt)$.

The weighted normal closure $N(R, r\text{-}wt)$ really is a subgroup of $\Omega(X, wt)$ since the product of two weighted consequences

$$(\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}) \cdot (\lambda'\langle w' \cdot r'\rangle \cdot (\lambda\langle w'\rangle)^{-1})$$

can be expressed as a weighted consequence $\lambda''\langle w'' \cdot r''\rangle \cdot (\lambda''\langle w''\rangle)^{-1}$, using the concatenated order-type $\lambda'' = \lambda \cdot \bar{\lambda} \cdot \lambda' \cdot \bar{\lambda}'$ and the word $(\lambda'', w'' \cdot r'')$ comprised of the four subwords $(\lambda, w \cdot r)$, $(\lambda, w \cdot 1_R)^{-1}$, $(\lambda', w' \cdot r')$, and $(\lambda', w' \cdot 1_R)^{-1}$. Indeed, then $\lambda''\langle w'' \cdot r''\rangle$ equals the given product of the two weighted consequences, and $\lambda''\langle w''\rangle$ equals the trivial word. Also $N(R, r\text{-}wt)$ is a normal subgroup of $(\Omega(X, wt)$ since any conjugate $v \cdot (\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}) \cdot v^{-1}$ of a weighted consequence can be expressed as a weighted consequence $\lambda'\langle w' \cdot r'\rangle \cdot (\lambda'\langle w'\rangle)^{-1}$, using the concatenated order-type $\lambda' = \tau \cdot \lambda$ (τ being the trivial order-type $0 < 1$) and the word $(\lambda', w' \cdot r')$ comprised of subwords $(\tau, v \cdot 1_R)$ and $(\lambda, w \cdot r)$.

Presented Groups The weighted presentation $\mathcal{P}$ (10) **presents** the quotient group

$$\Pi(\mathcal{P}) = \Omega(X, wt)/N(R, r\text{-}wt)$$

of the free omega-group $(\Omega(X, wt), w\text{-}wt)$ with weighted basis (X, wt), modulo the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$.

According to Theorem 1.4, the group $\Pi(\mathcal{P})$ presented by $\mathcal{P}$ is an omega-group provided that the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set is a normal omega-subgroup of the free omega-group $\Omega(X, wt)$. Examples 3.4 and 3.5 below show that this is not always the case and the presented group can fail to be an omega-group. For conditions sufficient to guarantee that $\Pi(\mathcal{P})$ is an omega-group, see Theorem 2.11.

Similarity Modulo $(R, r\text{-}wt)$ Consider two words (ω, x) and (ω', x') as decorations on the top and bottom edges of a unit square. An oriented disc in that unit square with its boundary circuit decorated by a cyclic version of a relator $r = (\lambda_r, y_r) \in R$ is called a **relator disc**. The two words (ω, x) and (ω', x') for the weighted basis (X, wt) are **similar modulo** $(R, r\text{-}wt)$, written $(\omega, g) \overset{s}{\sim} (\omega', x')$ (mod R), if there is a family of disjoint relator discs in the unit square whose diameters tend to zero only as the relator-weights tend to zero, such that the intervals of the opposing edges of the square and the boundaries of the relator dics that are labeled by non-identity elements of (X, wt) can be simultaneously paired (connected) in the complement of the relator discs in the unit square by a family of disjoint arcs, preserving labels and orientation. We call such a display S of arcs a **similarity square modulo** $(R, r\text{-}wt)$ for the words (ω, x) and (ω', x'). See Figure 2.

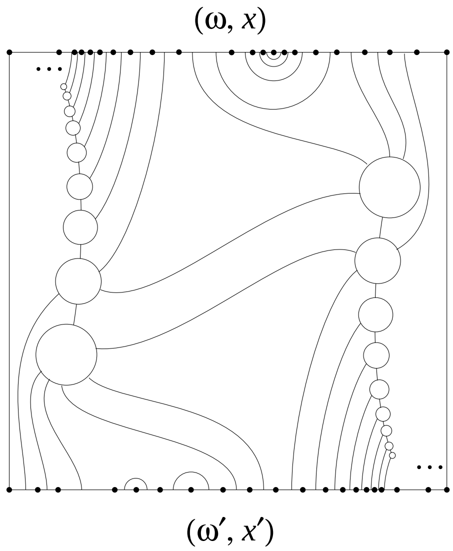

Figure 2: Similarity Square Modulo $(R, r\text{-}wt)$

The similarity relation, modulo the weighted relator set $(R, r\text{-}wt)$, respects the product of words; so the quotient set of similarity classes is a group. As expected, the following weighted generalization of the van Kampen theory of group diagrams is available:

Theorem 2.8 ([B-S97(1), Theorem 2.3]) *The group $\Pi(\mathcal{P})$ presented by a weighted presentation $\mathcal{P}$ is isomorphic to the group of similarity classes, modulo the weighted relator set $(R, r\text{-}wt)$, of words for the weighted basis (X, wt).*

Just as for ordinary presentations and their realization by CW cell complexes, the van Kampen theory of group diagrams is one step in proving that each weighted presentation $\mathcal{P}$ is realized by a metric cell-complex $Z(\mathcal{P})$. The realization $Z(\mathcal{P})$ is constructed in [B-S97(1)] using metric 1-cells and 2-cells, in correspondence with the generator pairs $x^{\pm 1}$ and relators pairs $r^{\pm 1}$ of $\mathcal{P}$, whose diameters are regulated by the generator-weights $wt(x)$ and relator-weights $r\text{-}wt(r)$. This realizes the presentation $\mathcal{P}$ in the obvious sense:

Theorem 2.9 ([B-S97(1), Theorem 2.5]) *A weighted presentation $\mathcal{P}$ presents the fundamental group $\pi_1(Z)$ of its metric realization $Z = Z(\mathcal{P})$.*

Examples of weighted presentations and their metric realizations are used in Section 3 to illustrate the theory of omega-groups. It is shown there that not every presented group $\Pi(\mathcal{P})$ associated with a weighted presentation $\Pi(\mathcal{P})$ acquires the structure of an omega-group.

2.3 Presented Omega-Groups

There are, however, conditions under which a weighted presentation does present an omega-group. By Theorem 1.4, it is sufficient to have the weighted normal closure of the weighted relator set be a normal omega-subgroup of the free omega-group on the weighted basis.

To analyze that possibility for a weighted presentation $\mathcal{P}$ (10), we form, for each $k \geq 1$, the **weight-restricted generator alphabet** and **relator set**

$$X^{(k)} = \{x \in X : wt(x) < \tfrac{1}{k-1}\} \text{ and } R^{(k)} = \{r \in R : r\text{-}wt(r) < \tfrac{1}{k-1}\}.$$

Since we always require that $w\text{-}wt(r) \leq r\text{-}wt(r)$, then $R^{(k)} \subset \Omega(X^{(k)}, wt)$ and $R^{(k)}$ has its own weighted normal closure $N(R^{(k)}, r\text{-}wt)$ in $(\Omega(X^{(k)}, wt), w\text{-}wt)$ comprised of all weighted consequences

$$\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1} \in \Omega(X^{(k)}, wt),$$

associated with an order-type λ and a pair of words (λ, w) and (λ, r) for the two weighted sets $(\Omega(X^{(k)}, wt), w\text{-}wt)$ and $(R^{(k)}, r\text{-}wt)$, respectively. Then, for each $k \geq 1$, there corresponds the **weight-restricted subpresentation**

$$\mathcal{P}^{(k)} \;\; = < (X^{(k)}, wt) : (R^{(k)}, r\text{-}wt) > .$$

Lemma 2.10 *The weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$ is a normal omega-subgroup of $(\Omega(X, wt), w\text{-}wt)$, if, for all $k \geq 1$,*

$$\Omega(X^{(k)}, wt) \cap N(R, r\text{-}wt) \subset N(R^{(k)}, r\text{-}wt). \tag{12}$$

Proof: To establish using the hypotheses (12) that the weighted normal closure $N(R, r\text{-}wt)$ is an omega-subgroup of $(\Omega(X, wt), w\text{-}wt)$, we consider any omega-product $\omega\langle\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}\rangle$ of weighted consequences

$$\lambda(i)\langle w(i) \cdot r(i)\rangle \cdot (\lambda(i)\langle w(i)\rangle)^{-1}\rangle \in \Omega(X, wt), i \in \mathcal{I}_\omega,$$

of $(R, r\text{-}wt)$ in $(\Omega(X, wt), w\text{-}wt)$. We express this omega-product as the difference $\lambda'\langle w' \cdot r'\rangle \cdot (\lambda'\langle w'\rangle)^{-1}$, using the composite order-type $\lambda' = \omega \circ (\lambda \cdot \bar{\lambda})$ and the composite word $(\lambda', w' \cdot r')$ having subwords $(\lambda(i), w(i) \cdot r(i)) \cdot (\lambda, w(i) \cdot 1_R)^{-1}$, $i \in \mathcal{I}_\omega$. Indeed, $\lambda'\langle w' \cdot r'\rangle$ equals the given omega-product $\omega\langle\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}\rangle$ of weighted consequences, and $\lambda'\langle w'\rangle$ equals the trivial word. But the word-weight restriction on the omega-product $\omega\langle\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}\rangle$ does not assure the relator weight restrictions on (λ', r') necessary for $\lambda'\langle w' \cdot r'\rangle \cdot (\lambda'\langle w'\rangle)^{-1}$ to be a weighted consequence for $(R, r\text{-}wt)$ in $(\Omega(X, wt), w\text{-}wt)$. But all is well if we use the containment relations (12) to preliminarily replace each weighted consequence in the given omega-product $\omega\langle\lambda\langle w\cdot r\rangle\cdot(\lambda\langle w\rangle)^{-1}\rangle$ having word-weight $\frac{1}{k}$ with an equal weighted consequence in $N(R^{(k)}, r\text{-}wt)$. Then (λ', w') is a word for $(R, r\text{-}wt)$ and (λ', w') is a word for $(\Omega(X, wt), w\text{-}wt)$. So $N(R, r\text{-}wt)$ is an omega-subgroup of $(\Omega(X, wt), w\text{-}wt)$.

To prove that $N(R, r\text{-}wt)$ is a normal omega-subgroup of $(\Omega(X, wt), w\text{-}wt)$, we consider any words (ω, v) for $(\Omega(X, wt), w\text{-}wt)$ and $(\omega, \lambda\langle w\cdot r\rangle\cdot(\lambda\langle w\rangle)^{-1})$ for $N(R, r\text{-}wt)$ modeled on the same order-type ω. We express the the difference

$$\omega\langle v\cdot(\lambda\langle w\cdot r\rangle\cdot(\lambda\langle w\rangle)^{-1})\rangle\cdot(\omega\langle v\rangle)^{-1}$$

as the product $\lambda'\langle w'\cdot r'\rangle\cdot(\lambda'\langle w'\rangle)^{-1}\in N(R, r\text{-}wt)$, using the composite order-type $\lambda' = \omega\circ(\tau\cdot\lambda\cdot\bar{\lambda})$ and the composite word $(\lambda', w'\cdot r')$ with subwords $(\tau, v(i)\cdot 1_R)\cdot(\lambda(i), w(i)\cdot r(i))\cdot(\lambda(i), w(i)\cdot 1_R)^{-1}, i\in\mathcal{I}_\omega$. Indeed, $\lambda'\langle w'\cdot r'\rangle$ equals the product $\omega\langle v\cdot(\lambda\langle w\cdot r\rangle\cdot(\lambda\langle w\rangle)^{-1})\rangle$, and $\lambda'\langle w'\rangle$ equals the product $\omega\langle v\cdot\lambda\langle w\rangle\cdot(\lambda\langle w\rangle)^{-1}\rangle = \omega\langle v\rangle$. As above, $\lambda'\langle w'\cdot r'\rangle\cdot(\lambda'\langle w'\rangle)^{-1}\in N(R, r\text{-}wt)$, provided each weighted consequence in the product $\omega\langle\lambda\langle w\cdot r\rangle\cdot(\lambda\langle w\rangle)^{-1}\rangle$ having word-weight $\frac{1}{k}$ is preliminarily replaced with an equal weighted consequence in $N(R^{(k)}, r\text{-}wt)$. □

Incidently, for the proof of the lemma, it would be sufficient that there exists an unbounded monotonic sequence k_k such that

$$\Omega(X^{(k)}, wt)\cap N(R, r\text{-}wt)\subset N(R^{(k_k)}, r\text{-}wt)$$

for all $k\geq 1$. The topological analog of this condition is the requirement that sufficiently tiny loops that are homotopic in a metric space are homotopic by a tiny enough homotopy. Even when the relator-weights are equal to the word-weights of the relators, these conditions for the lemma can fail.

One application of Lemma 2.10 is this:

Theorem 2.11 *A weighted presentation $\mathcal{P}$ presents an omega-group if the inclusions among the weight restricted subpresentations $\mathcal{P}^{(k)}$ induce a sequence of injective homomorphisms of the presented groups:*

$$\ldots\to\Pi(\mathcal{P}^{(k+1)})\to\Pi(\mathcal{P}^{(k)})\to\ldots\to\Pi(\mathcal{P}^{(2)})\to\Pi(\mathcal{P}^{(1)}) = \Pi(\mathcal{P}).$$

Proof: By hypothesis, there are composite injections $\Pi(\mathcal{P}^{(k)})\to\Pi(\mathcal{P})$; equivalently, the containment relations (12) of Lemma 2.10 hold. □

Applications of Theorem 2.11 include the omega-groups appearing in Examples 3.6 and 3.7 below. Such applications of this theorem invoke the theory of equations over groups, as is explained in [B-S97(2)].

3 Examples of Weighted Presentations

The following seven examples illustrate the concepts of weighted presentations and their presented groups. Some are omega-groups, others not.

3.1 Hawaiian Earring Group

The Hawaiian earring group $\Pi(\mathcal{P}_{HE})$ *is a group that is neither countably generated nor free, and yet is countably generated and free as an omega-group.*

The *Hawaiian earring* HE is a planar 1-point union of a sequence of circles of radii $\frac{1}{k}$, for $k \geq 1$, as indicated in Figure 3 (a). The Hawaiian earring group $\pi_1(HE)$ is neither countably generated nor free as a group [Sm92]. Yet $\pi_1(HE)$ is the free omega-group $\Omega(X, wt)$ with countable weighted basis

$$(X, wt) = \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\}$$

comprised of the path classes of the circuits on the individual rings, weighted by their metric diameters. For one proof, see [B-S97(1),]. (For notational consistency, let the notation include $(1_X, wt\ 0)$ as the case $k = \infty$.) In short, every path class can be expressed uniquely as a reduced order-type word in this countable weighted basis. In other words, the Hawaiian earring is the realization of weighted presentation $\mathcal{P}_{HE} = < (X, wt) : >$ with the countable weighted alphabet (X, wt) given above and empty weighted relator set.

3.2 Hawaiian Discs Group

The Hawaiian discs group $\Pi(\mathcal{P}_{HD})$ *is the trivial group presented trivially.*

The weighted presentation $\mathcal{P}_{HD}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := x_k^{\pm 1}, r\text{-}wt\ \tfrac{1}{k}) : k \geq 1\} >,$$

whose generator and relator sequences are the same and have generator-weights equal to the relator-weights, presents the trivial group. The assigned weights assure that the generator labels in any word $(\omega, x) \in \Omega(X, wt)$ can be relaced by the corresponding relators $r_{k(i)}^{\pm 1} = x_k^{\pm 1}, i \in \mathcal{I}_\omega$ to yield an equal word $(\omega, x) = (\omega, r)$ that belongs to the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$. This algebraic fact corresponds to the similarity square $S : (\omega, x) \sim (\omega, 1_X)\ mod(R, r\text{-}wt)$ comprised of one relator disc $r_k^{\pm 1}$ arc-connected to each label $x_k^{\pm 1}$ of the boundary word (ω, x). So the weighted normal closure $N(R, r\text{-}wt)$ equals the free omega-group $\Omega(X, wt)$ and the presented group $\Pi(\mathcal{P}_{HD}) = \Omega(X, wt)/N(R, r\text{-}wt)$ is trivial.

The metric realization of the weighted presentation $\mathcal{P}_{HD}$ is the **Hawaiian disc**, i.e., the 1-point union HD in 3-space of discs of vanishing diameter, bounded by a fanned copy of the Hawaiian earring HE. The triviality of the fundamental group $\pi_1(HD)$ is easily visualized as any loop deforms simultaneously over all of the discs in HD bounded by the rings that it traverses.

3.3 Disc of Crescents Group

The disc of crescents group $\Pi(\mathcal{P}_{DC})$ is the trivial group presented non-trivially.

The weighted presentation $\mathcal{P}_{DC}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := (x_k \cdot x_{k+1}^{-1})^{\pm 1}, r\text{-}wt\ \tfrac{1}{k}) : k \geq 1\} >,$$

whose consecutive generators x_k and x_{k+1} are equated by a sequence of relators r_k of vanishing weight, also presents the trivial group. Because of the vanishing relator-weights $r\text{-}wt(r_k) = \frac{1}{k}$, the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$ contains the reverse-harmonic product

$$\bar{\eta}\langle\ r_k,\ r_{k+1},\ \ldots\ \rangle = r_k \cdot r_{k+1} \cdot \ldots \cdot r_m \cdot \ldots,$$

which in $\Omega(X, wt)$ equals the reverse-harmonic product

$$x_k \cdot x_{k+1}^{-1} \cdot x_{k+1} \cdot x_{k+2}^{-1} \cdot \ldots,$$

which, in turn, freely reduces (i.e., is similar) to the monosyllabic word x_k. So each weighted generator of $\mathcal{P}_{DC}$ becomes trivial in the presented group $\Pi(\mathcal{P}_{DC}) = \Omega(X, wt)/N(R, r\text{-}wt)$. The next example shows that this alone does not imply that the presented group $\Pi(\mathcal{P}_{DC})$ is trivial.

The preceding algebraic argument for the triviality of x_k in $\Pi(\mathcal{P}_{DC})$ corresponds to the similarity square $S : x_k \sim 1_X\ mod(R, r\text{-}wt)$ comprised of a vanishing sequence of relator disc $r_k, r_{k+1}, \ldots$ arc-connected together to the boundary label x_k. There is likewise an algebraic argument for the triviality in $\Pi(\mathcal{P}_{DC})$ of any word (ω, x) for (X, wt); the corresponding similarity square $S : (\omega, x) \sim (\omega, 1_X)\ mod(R, r\text{-}wt)$ contains such a vanishing sequence of relator discs strung below each occurrence of the label $x_{k(i)}^{\pm 1}, i \in \mathcal{I}_\omega$, in the boundary word (ω, x). This proves that the weighted normal closure $N(R, r\text{-}wt)$ equals the free omega-group $\Omega(X, wt)$ and so the presented group $\Pi(\mathcal{P}_{DC})$ is trivial. (Notice that the unweighted analog of $\mathcal{P}_{DC}$ presents $\mathbf{Z}$.)

The realization of the weighted presentation $\mathcal{P}_{DC}$ is the disc DC assembled from the lunar crescents of vanishing diameter bounded by the consecutive rings of a planar copy of the Hawaiian earring HE, as in Figure 3 (a). The triviality of the fundamental group $\pi_1(DC)$ is easily visualized since any loop deforms deforms simultaneously over all of the nested subdiscs in DC bounded by the individual rings that it traverses.

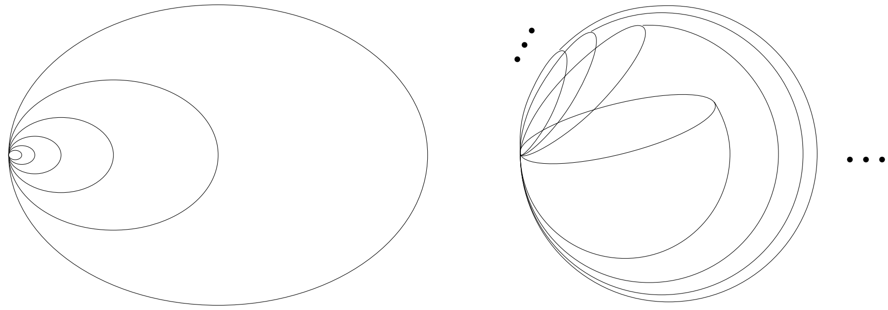

Figure 3: (a) Disc of Crescents (b) Harmonic Onion

3.4 Harmonic Onion Group

The harmonic onion group $\Pi(\mathcal{P}_{HO})$ *is a non-trivial group that paradoxically can be presented entirely by generators that become trivial in* $\Pi(\mathcal{P}_{HO})$.

The weighted presentation $\mathcal{P}_{HO}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := x_k^{\pm 1}, r\text{-}wt\ 1) : k \geq 1\} >,$$

whose generator and relator sequences are the same and are assigned vanishing generator-weights but constant non-trivial relator-weights, does not present the trivial group. Because each generators $x_k^{\pm 1}$ is also a relator $r_k^{\pm 1}$, it belongs to the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$ and so is trivial in the presented group $\Pi(\mathcal{P}_{HO})$.

But $\Pi(\mathcal{P}_{HO})$ is not trivial, in fact, it is uncountable. The constant non-trivial relator-weights assure that any word (λ, r) for $(R, r\text{-}wt)$ is essentially a finite word, i.e., $r(i) = 1_R$ for all but finitely many intervals $i \in \mathcal{I}_\omega$. Thus, an arbitrary weighted consequence $\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}$ of $(R, r\text{-}wt)$ in $(\Omega(X, wt), w\text{-}wt)$ equals a finite consequence

$$((w_1 \cdot r_{k(1)}^{\epsilon(1)}) \cdot (w_2 \cdot r_{k(2)}^{\epsilon(2)}) \cdot \ldots \cdot (w_k \cdot r_{k(m)}^{\epsilon(m)})) \cdot (w_1 \cdot w_2 \cdot \ldots \cdot w_m)^{-1}$$

in $\Omega(X, wt)$ of relators $r_k^{\pm 1} := x_k^{\pm 1} \in R$. Therefore the weighted normal closure $N(R, r\text{-}wt)$ is the (unweighted) ordinary normal closure $N_{\Omega(X,wt)}(F(X))$ in the group $\Omega(X, wt)$ of the ordinary free group $F(X)$ with basis X. Thus the presented group $\Pi(\mathcal{P}_{HO})$ is the quotient group $\Omega(X, wt)/N_{\Omega(X,wt)}(F(X))$ of cosets $(\omega, x) \cdot N_{\Omega(X,wt)}(F(X))$, called **germs**, of reduced order-type words (ω, x) for the weighted alphabet (X, wt); two reduced words (ω, x) and (ω', x') represent the same *germ* if and only if they become similar when finitely many of the generators $x_1, x_2, \ldots, x_k, \ldots$ are replaced by the trivial element 1_X.

The presented group $\Pi(\mathcal{P}_{HO})$ of germs of words for the weighted alphabet (X, wt) is not an omega-group. Order-type products, even harmonic products, of these germs are not well-defined. For example, because the generators x_k have vanishing weights $wt(x_k) = \frac{1}{k}$, the free omega-group $(\Omega(X, wt), w\text{-}wt)$ contains the harmonic product $w_k = \ldots \cdot x_{k+1} \cdot x_k$ with word-weight $\frac{1}{k}$ for each $k \geq 1$. These harmonic products w_k determine the same non-trivial germ $w \cdot N$ of the weighted normal closure $N = N_{\Omega(X,wt)}(F(X))$ because in $\Omega(X, wt)$ the difference $w_{k+1}^{-1} \cdot w_k$ equals $x_k \in F(X)$. But the reverse harmonic products

$$(w_1 \cdot N) \cdot (w_2 \cdot N) \cdot \ldots \cdot (w_k \cdot N) \cdot \ldots = (w_2 \cdot N) \cdot \ldots \cdot (w_k \cdot N) \cdot \ldots$$

cannot be well-defined in $\Pi(\mathcal{P}_{HO})$, else cancellation in $\Pi(\mathcal{P}_{HO})$ would imply the triviality of the germ $w \cdot N \neq N$.

The realization of the weighted presentation $\mathcal{P}_{HO}$ is the *harmonic onion*, the metric cell-complex HO constructed from Hawaiian earring HE sequence of rings of decreasing diameter $\frac{1}{k}$ (corresponding to the relators of vanishing weight) by adjoining hemispherical discs H_k of non-decreasing diameter (corresponding to the relators of constant weight) that form the expanding layers of the onion. See Figure 3 (b). The fact that the presented group $\Pi(\mathcal{P}_{HO})$ of germs of words for (X, wt) is not an omega-group is reflected in this confusing state of affairs for the fundamental group $\pi_1(HO)$: (a) Each loop on a single ring is inessential in the presence of the hemispherical onion layer that it bounds in the harmonic onion HO, so that each of these generator loops represents a trivial path class of $\pi_1(HO)$; (b) order-type path-products of the inessential generator loops x_k in HO represent all path classes of loops; and (c) some of these order-type path-products are essential, because they traverse infinitely many of the rings of vanishing diameter and a path null-homotopy can cover at most finitely many of the hemispherical layers. Because the path class of an essential order-type path-products of the generator loops cannot equal the corresponding order-type product of the path classes of these inessential generator loops, we see again that order-type products are not well-defined in this group.

3.5 Harmonic Archipelago Group

The harmonic archipelago group $\Pi(\mathcal{P}_{HA})$ *is a non-cyclic group that paradoxically can be presented entirely by generators that become equal in* $\Pi(\mathcal{P}_{HA})$.

The weighted presentation $\mathcal{P}_{HA}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := (x_k \cdot x_{k+1}^{-1})^{\pm 1}, r\text{-}wt\ 1) : k \geq 1\} >,$$

whose consecutive generators x_k and x_{k+1} are equated by a sequence of relators r_k (as in Example 3.3, but this time) with constant relator-weights, presents a non-cyclic, in fact uncountable, group. Because each relator $r_k^{\pm 1} = (x_k \cdot x_{k+1}^{-1})^{\pm 1}$ belongs to the weighted normal closure $N = N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$, the generators determine the same element

$$x_1 \cdot N = x_2 \cdot N = \ldots = x_k \cdot N = \ldots$$

in the presented group $\Pi(\mathcal{P}_{HA})$. Despite the generators becoming equal in the presented group $\Pi(\mathcal{P}_{HA})$, it does not follow that this presented group is cyclic.

The constant relator-weights assure that any word (λ, r) for $(R, r\text{-}wt)$ is a finite word, i.e., $r(i) = 1_R$ for all but finitely many intervals $i \in \mathcal{I}_\omega$. Thus, an arbitrary weighted consequence $\lambda\langle w \cdot r\rangle \cdot (\lambda\langle w\rangle)^{-1}$ of $(R, r\text{-}wt)$ in $(\Omega(X, wt), w\text{-}wt)$ equals a finite consequence

$$((w_1 \cdot r_{k(1)}^{\epsilon(1)} \cdot (w_2 \cdot r_{k(2)}^{\epsilon(2)}) \cdot \ldots \cdot (w_k \cdot r_{k(m)}^{\epsilon(m)})) \cdot (w_1 \cdot w_2 \cdot \ldots \cdot w_m)^{-1}$$

in $\Omega(X, wt)$ of relators $r_k^{\pm 1} = (x_k \cdot x_{k+1}^{-1})^{\pm 1} \in R$. Thus the weighted normal closure $N(R, r\text{-}wt)$ is the (unweighted) ordinary normal closure $N_{\Omega(X,wt)}(R)$ in the group $\Omega(X, wt)$ of the set R of relators.

So the presented group $\Pi(\mathcal{P}_{HA})$ is the quotient group $\Omega(X, wt)/N_{\Omega(X,wt)}(R)$ of cosets $(\omega, x) \cdot N_{\Omega(X,wt)}(R)$, called **traces**, of reduced order-type words (ω, x) for the weighted alphabet (X, wt); two reduced words (ω, x) and (ω', x') represent the same *trace* if and only if they become similar when finitely many of the generators $x_k, k \geq 1$, are identified. This group of traces of reduced words is an uncountable, hence non-cyclic, group.

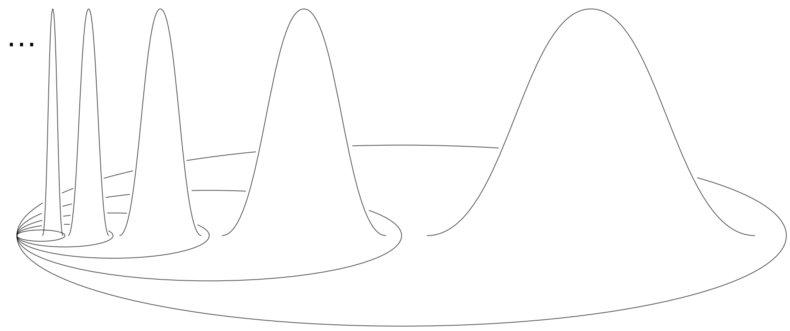

Figure 4: Harmonic Archipelago

The realization of the weighted presentation $\mathcal{P}_{HA}$ is the *harmonic archipelago*, the metric cell-complex HA constructed from the Hawaiian earring sequence of rings of decreasing diameter $\frac{1}{k}$ (corresponding to the generators of vanishing weight) by adjoining mountainous lunar crescents of constant unit elevation between the consecutive rings (corresponding to the relators of constant weight 1). See Figure 4. The generator loops on any two rings in the supporting Hawaiian earring HE are homotopic over the intervening mountainous crescents in the harmonic archipelago HA. For compactness reasons, a path null-homotopy can cover at most finitely many of the mountainous crescents, and so there exist uncountably many homotopically distinct essential loops arising from different order-type traverses of the rings.

The presented group $\Pi(\mathcal{P}_{HA})$ of traces of words for the weighted alphabet (X, wt) is not an omega-group, for much the same reasons as for the group $\Pi(\mathcal{P}_{HO})$ of germs of words for the weighted alphabet (X, wt) in Example 3.4. In fact, neither fundamental group $\pi_1(HO)$ nor $\pi_1(HA)$ acquires even a weight function; any loop in HO or HA is path-homotopic to a loop of arbitrarily small diameter. They are groups of *essential infinitesimals loops.*

Conclusion So the group $\Pi(\mathcal{P}) = \Omega(X, wt)/N(R, r\text{-}wt)$ presented by a weighted presentation $\mathcal{P}$ can fail to be an omega-group. Although the members $x \in X$ generate the free omega-group $(\Omega(X, wt), w\text{-}wt)$ in the sense that their order-type words constitute $\Omega(X, wt)$, their cosets $x \cdot N(R, r\text{-}wt)$ need not generate the presented group $\Pi(\mathcal{P})$ because non-finite order-type products of these cosets may not be well-defined. However, the structure of such a presented group is usually only fully understood when viewed as a quotient of the free omega-group $\Omega(X, wt)$ modulo the weighted normal closure $N(R, r\text{-}wt)$.

According to Theorem 1.4, a presented group $\Pi(\mathcal{P})$ can fail to be an omega-group only if the weighted normal closure $N(R, r\text{-}wt)$ of the weighted relator set $(R, r\text{-}wt)$ fails to be a normal omega-subgroup of the free omega-group $(\Omega(X, wt), w\text{-}wt)$. In Example 3.4, the reverse harmonic products $w_k, k \geq 1$, have word-weight $\frac{1}{k}$ and belong to the same non-trivial coset of the weighted normal closure $N(R, r\text{-}wt)$. The failure of this non-trivial coset to have a member of minimum word-weight alone shows that the weighted normal closure $N(R, r\text{-}wt)$ is not a normal omega-subgroup of the free omega-group.

3.6 Harmonic Projective Plane Group

The harmonic projective plane group $\Pi(\mathcal{P}_{HP})$ *is an omega-group that is a* free weighted-product *of a sequence of 2-cycles of vanishing weights.*

The weighted presentation $\mathcal{P}_{HP}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := x_k^{\pm 2}, r\text{-}wt\ \tfrac{1}{k}) : k \geq 1\} >$$

presents the quotient group $\Omega(X, wt)/N(XX, wt)$ of the free omega-group $\Omega(X, wt)$ with weighted basis (X, wt), modulo the weighted normal closure $N(XX, r\text{-}wt)$ of the the weighted relator set (XX, wt) of squares xx of the basis members $x \in X$. The realization of the weighted presentation $\mathcal{P}_{HP}$ is the

harmonic projective plane HP obtained by rotating a planar copy of the Hawaiian earring about its axis of symmetry and then identifying antipodal points in each of the resulting 2-spheres. The fundamental group $\pi_1(HP)$ is seen to be an omega-group by use of Theorem 2.11; it is a *free weighted-product* ([B-S97(2)]) of a sequence of 2-cycles of vanishing weight.

3.7 Projective Telescope Group

The projective telescope group $\Pi(\mathcal{P}_{PT})$ *is a non-abelian omega-group that paradoxically can be presented entirely by generators* x_k $(k \geq 1)$ *that become successive squares* $x_{k+1} = x_k^2$ $(k \geq 1)$ *of one another in* $\Pi(\mathcal{P}_{PT})$*.*

The weighted presentation $\mathcal{P}_{PT}$:

$$< \{(x_k^{\pm 1}, wt\ \tfrac{1}{k}) : k \geq 1\} : \{(r_k^{\pm 1} := (x_{k+1}^{-1} \cdot x_k^2,)^{\pm 1}, r\text{-}wt\ \tfrac{1}{k}) : k \geq 1\} >$$

presents the quotient group $\Omega(X, wt)/N(R, r\text{-}wt)$ of the *free omega-group* $\Omega(X, wt)$ with *weighted basis* $(X, wt) = \{(x_k^{\pm 1}, wt\ \frac{1}{k}) : k \geq 1\}$, modulo the *weighted normal closure* $N(R, r\text{-}wt)$ in $\Omega(X, wt)$ of the *weighted relator set* $(R, r\text{-}wt) = \{(r_k^{\pm 1} := (x_{k+1}^{-1} \cdot x_k^2)^{\pm 1}, r\text{-}wt\ \frac{1}{k}) : k \geq 1\}$, comprised of relators of vanishing weight that equate each generator with the square of its predecessor. This algebraic description expresses the group elements as cosets $(\omega, x) \cdot N(R, r\text{-}wt)$ of reduced order-type words (ω, x) for the weighted alphabet (X, wt); two reduced words (ω, x) and (ω', x') represent the same coset if and only if they are *similar modulo* finitely many applications of each relation $x_{k+1} = x_k^2, k \geq 1$. In [B-S97(2)], it is shown using Theorem 2.11 that this quotient group is an *omega-group* and also that its *omega-abelianization* is the group of 2-adic integers.

The metric realization of this presentation is the the *projective telescope* PT obtained by an iteration of the procedure of replacing a disc d_{k+1} of diameter $\frac{1}{k+1}$ in a projective plane P_k of diameter $\frac{1}{k}$ by a projective plane P_{k+1} of diameter $\frac{1}{k+1}$, beginning with a projective plane P_1 of diameter 1. There is a copy of the Hawaiian earring HE in the projective telescope PT and the generator loop x_k on the k-th ring is path homotopic in PT to the power $x_1^{2^k}$ of the generator loop of x_1 on the first ring. So any two generator loops x_k and x_j are path-homotopic in PT. Yet a similarity square argument shows that the loop that traverses all the even-indexed generator loops $x_{2k}, k \geq 1$, in succession does not commute up to path-homotopy with the generator loop x_1. Thus the fundamental group $\pi_1(PT)$ is non-abelian.

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