Improving Finiteness Properties for Metabelian Groups

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Abstract

We show that any finitely generated metabelian group can be embedded in a metabelian group of type F_3. The proof builds upon work of G. Baumslag [4], who independently with V. R. Remeslennikov [10] proved that any finitely generated metabelian group can be embedded in a finitely presented one. We also rely essentially on the Sigma theory of R. Bieri and R. Strebel [7], who introduced this geometric theory to detect finite presentability of metabelian groups. Within the context of Sigma theory, we also prove that if n is a positive integer and Q is a finitely generated abelian group, then any finitely generated $\mathbb{Z}Q$-module can be embedded in a module that is n-tame. For $n = 3$, this tameness result combines with a recent theorem of Bieri and J. Harlander [6] to imply the F_3 embedding theorem.

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1 Metabelian Groups

This paper is about finiteness properties of metabelian groups. The story begins in the 1970s with a series of papers by G. Baumslag and V. R. Remeslen-

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nikov, who independently investigated finitely generated and finitely presented metabelian groups and showed that the theory of these groups is more complex than one might expect. They exhibited examples of

- a finitely generated metabelian group that is not finitely presented but which has trivial multiplicator [2], and
- a finitely presented torsion-free metabelian group whose derived subgroup is free abelian of infinite rank [3, 10].

Well known for many years, H. Hopf's description of the multiplicator of a quotient $G = F/R$ a free group F as $H_2(G) = (R \cap [F,F])/[R,F]$ shows that every finitely presented group has finitely generated multiplicator. Baumslag's example in [2] was the first example of any sort to indicate that the converse is false, thus demonstrating that finer invariants would be needed to detect finite presentability among finitely generated groups. The metabelian example from [3, 10] showed that a finitely presented torsion-free group can contain finitely generated subgroups of arbitrarily high cohomological dimension. Baumslag and Remeslennikov also proved the following general embedding theorem, which is our actual point of departure.

Theorem 1.1 [4, 10] *Every finitely generated metabelian group can be embedded in a finitely presented metabelian group.*

Finite generation and finite presentability are the first two in a hierarchy of increasingly strong finiteness properties of groups. A group G is of *type* F_n if there is a connected aspherical CW complex with fundamental group isomorphic to G (that is, an Eilenberg-Maclane complex of type $K(G,1)$) with finite n-skeleton. Type F_1 is equivalent to finite generation and type F_2 is equivalent to finite presentability. The main general result of this paper is the following.

Theorem 1.2 *Every finitely generated metabelian group can be embedded in a metabelian group of type F_3.*

Our eventual aim is to improve this result to enable embeddings in metabelian groups of type F_n for arbitrarily large n and we make steps in that direction in this paper. The example from [3, 10] shows that this is the best one can hope for within the class of metabelian groups since a result of R. Bieri and

J. R. J. Groves implies that if a metabelian group G admits a $K(G, 1)$ with finitely many cells in each dimension (so that G is of type F_∞), then there is a uniform bound on the rank of the free abelian subgroups of G. That this restriction on possbile embeddings also applies in the more general context of soluble groups follows from subsequent work of P. H. Kropholler [9].

Our proof of Theorem 1.2 does not rely on Theorem 1.1 directly, but we follow the framework of Baumslag's proof in [4] quite closely. We do rely directly on the fact that every finitely generated metabelian group can be embedded in a finitely generated split metabelian group. We paraphrase this result from [4].

Lemma 1.3 [4, Lemma 3] *Every finitely generated metabelian group G can be embedded in a finitely generated split metabelian group of the form $M \rtimes Q$ where $Q = G^{ab}$ is a finitely generated abelian group and M is a finitely generated $\mathbb{Z}Q$-module.*

This reduces embedding problems for a finitely generated metabelian group to that of embedding a finitely generated module M over a finitely generated abelian group Q in a module $\overline{M}$ over a finitely generated abelian overgroup $\overline{Q}$ of Q.

In order to embed a finitely generated split metabelian group in a finitely presented one, Baumslag made essential and systematic use of *localization* techniques that he adapted from communtative ring theory; we will summarize the construction in §3. In order to study higher finiteness properties, we invoke the *Sigma theory* that was introduced by R. Bieri and R. Strebel in [7]. At its inception, the Sigma theory considered modules M over group rings of finitely generated abelian groups Q and led to an understanding of when extensions of M by Q are finitely presented. The theory continues to evolve and there are many papers on the subject; see [1, 5, 6, 7, 11] and the references cited there. Strebel's survey in [11] is an instructive overview of the early stages of the theory.

Most of the work in this paper involves computations within the Sigma theory. We summarize those portions of the theory that we need in §2, focusing on the concept of *n-tameness* for $\mathbb{Z}Q$-modules. There is considerable evidence to support a conjectural definitive connection between tameness of modules and higher finiteness properties F_n of metabelian groups and we rely on recent results of this sort to deduce Theorem 1.2 from our computations. The heart of the paper is in §4 where we show that Baumslag's localization techniques can be pushed to improve finiteness properties beyond finite presentability. Our main result within the Sigma theory is stated in §2 as Theorem 2.3; it shows that any finitely generated module can be embedded in one that is n-tame, this for any given finite n. Our main general result, Theorem 1.2, follows from the case $n = 3$. Projected improvements of Theorem 1.2 will require additional work on the role of tameness in determining the finiteness properties of a very specific collection of metabelian groups. The concluding §5 introduces the idea of an *essential decomposition* for a module and uses this to complete the proofs of the main results, namely Theorems 1.2 and 2.3.

2 Sigma Theory

These geometric considerations were introduced by Bieri and Strebel [7] for the purpose of detecting finite presentability of finitely generated metabelian groups. For a finitely generated abelian group Q, $V(Q)$ denotes the set of real-valued homomorphisms (or characters) from Q to the additive group of real numbers: $V(Q) = \mathrm{Hom}(Q, \mathbb{R})$. So $V(Q)$ is a real vector space with dimension equal to the torsion-free rank $\mathrm{rk}_{\mathbb{Z}}(Q)$ of Q. With the discrete topology on Q, the compact open topology on $V(Q)$ is homeomorphic to the Euclidean space of the same dimension.

Let M be a $\mathbb{Z}Q$-module. Given a character $\chi \in V(Q)$, there is the submonoid Q_χ of Q consisting of those $q \in Q$ for which $\chi(q) \geq 0$, and there is the subring $\mathbb{Z}Q_\chi$ of $\mathbb{Z}Q$. The *Sigma set* of the $\mathbb{Z}Q$-module M is

$$\Sigma(M, Q) = \{\chi \in V(Q) : M \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module}\}$$

and the *Sigma complement* is

$$\Sigma(M, Q)^c = V(Q) - \Sigma(M, Q).$$

Avoiding the zero character, the Sigma set and its complement can each be viewed as unions of positive rays in $V(Q)$, so one can view these sets in the sphere $S(Q) \approx S^{\mathrm{rk}_{\mathbb{Z}}(Q)-1}$ consisting of equivalence classes $[\chi]$ of characters, where two characters are identified if each is a positive scalar multiple of the other. Two advantages of this perspective include the availability of compactness arguments and the fact that the *discrete* characters on Q, defined as those having infinite cyclic image in $\mathbb{R}$, determine a dense subset of the sphere $S(Q)$.

Bieri and Strebel [7] discovered that there is a deep connection between the geometry of the Sigma complement $\Sigma(M,Q)^c$ and the finiteness properties of groups given as extensions of M by Q. To formulate this connection, the complement of every codimension one subspace of the vector space $V(Q)$ consists of two convex open subspaces of the Euclidean space $V(Q)$, called *open half-spaces*. These open half-spaces determine open hemispheres in the sphere $S(Q)$. For a positive integer k, the $\mathbb{Z}Q$-module is said to be *k-tame* if each k-element subset of the Sigma complement $\Sigma(M,Q)^c$ is contained in an open half-space of $V(Q)$. Thus, M is 1-tame if and only if $0 \in \Sigma(M,Q)$, which amounts to saying that M is finitely generated as a $\mathbb{Z}Q$-module. Higher tameness properties are unaffected if we view the Sigma complement in the sphere $S(Q)$ and formulate the property in terms of open hemispheres. For example, 2-tameness amounts to saying that $\Sigma(M,Q)^c$ contains no antipodal points in the sphere $S(Q)$. Note that $(k+1)$-tame implies k-tame. The connection between tameness and finiteness properties begins with the following definitive result of Bieri and Strebel.

Theorem 2.1 [7] *The following are equivalent for a group G with an abelian normal subgroup M such that $Q = G/M$ is finitely generated and abelian.*

1. *The group G is finitely presented.*
2. *The split extension $M \rtimes Q$ is finitely presented.*
3. *The $\mathbb{Z}Q$-module M is 2-tame.*

The F_n (or FP_n) conjecture for metabelian groups asserts that if G is a group with an abelian normal subgroup M such that $G/M = Q$ is finitely generated and abelian, then G is of type F_n if and only if M is n-tame as a $\mathbb{Z}Q$-module. The result of Bieri and Strebel above verifies the conjecture in the case $n = 2$ and provides a geometric invariant to distinguish finitely presented metabelian groups among finitely generated ones. A deep result of H. Åberg provides that the conjecture holds for all n when the group G has finite Prüfer rank, that is, when there is a uniform bound on the number of elements required to generate the finitely generated subgroups of G. We will rely on the fact that the split case of the conjecture for $n = 3$ has been completed by Bieri and Harlander.

Theorem 2.2 [6] *If Q is a finitely generated abelian group and M is a $\mathbb{Z}Q$-module, then the split extension $G = M \rtimes Q$ is of type* F_3 *if and only if the $\mathbb{Z}Q$-module M is 3-tame.*

In terms of the Sigma theory, the main result of this paper is the following.

Theorem 2.3 *Given a positive integer n, if Q is a finitely generated abelian group and M is a finitely generated $\mathbb{Z}Q$-module, then there is a finitely generated abelian overgroup $\overline{Q}$ of Q and an n-tame $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that contains M as a $\mathbb{Z}Q$-submodule.*

The theorem of Bieri and Harlander thus implies the following general embedding result, which sharpens Theorem 1.2 slightly.

Corollary 2.4 *Every finitely generated metabelian group can be embedded in a split metabelian group of type F_3.*

Proof: Lemma 1.3 shows that a given finitely generated metabelian group G can be embedded in a split one of the form $M \rtimes Q$ where $Q = G^{ab}$ is finitely generated abelian and M is a finitely generated $\mathbb{Z}Q$-module. Theorem 2.3 provides a finitely generated abelian overgroup $\overline{Q}$ of Q and a 3-tame $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that contains M as a $\mathbb{Z}Q$-submodule. Then $M \rtimes Q$, and hence G, embeds in $\overline{M} \rtimes \overline{Q}$, which is of type F_3 by Theorem 2.2. •

3 Localization

In showing how to embed finitely generated split metabelian groups in finitely presented ones, Baumslag [4] made essential use of a localization construction from commutative ring theory. We briefly recall the details. Let Q be a finitely generated abelian group and let M be a $\mathbb{Z}Q$-module. Consider a subset S of the group ring $\mathbb{Z}Q$ with the following properties:

- S is unital, in that $1 \in S$;
- S is multiplicatively closed, in that $S \cdot S \subseteq S$;
- S acts freely on M, in that if $s \in S$ and $m \in M$ are such that $sm = 0$, then $m = 0$;
- S is generated as a multiplicative submonoid of $\mathbb{Z}Q$ by a subset $\mathbf{y}$ of S.

Lemma 3.1 *With the notation as above, let $\overline{Q}$ be the direct product of Q with the free abelian group with basis consisting of elements z_y, $y \in \mathbf{y}$. There is a $\mathbb{Z}\overline{Q}$-module $\overline{M}$ such that*

1. *M is a $\mathbb{Z}Q$-submodule of $\overline{M}$: $M \leq \mathrm{Res}^{\mathbb{Z}\overline{Q}}_{\mathbb{Z}Q}(\overline{M})$,*

2. *any $\mathbb{Z}Q$-generating set for M determines a $\mathbb{Z}\overline{Q}$-generating set for $\overline{M}$, and*

3. *the annihilator $\mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ is the smallest ideal in $\mathbb{Z}\overline{Q}$ that contains $\mathrm{Ann}_{\mathbb{Z}Q}(M)$ and all elements of the form $z_y - y$, $y \in \mathbf{y}$.*

Proof: Define an equivalence relation on $S \times M$ by declaring that

$$(s, m) \sim (s', m') \Leftrightarrow sm' = s'm.$$

(The fact that $sm = 0$ only if $m = 0$ is required to verify that this relation is transitive.) One checks that $\overline{M}$ becomes a $\mathbb{Z}\overline{Q}$-module when addition is defined by

$$(s, m) + (s', m') = (ss', sm' + s'm)$$

and actions are determined as follows.

$$\begin{aligned} \lambda(s, m) &= (s, \lambda m) \quad (\lambda \in \mathbb{Z}Q) \\ z_y(s, m) &= (s, ym) \quad (y \in \mathbf{y}) \\ z_y^{-1}(s, m) &= (sy, m) \quad (y \in \mathbf{y}) \end{aligned}$$

The map $m \mapsto (1, m)$ embeds M in $\overline{M}$ as a $\mathbb{Z}Q$-submodule and the fact that $(sy^n, m) = z_y^{-n}(s, m)$ can be used to show that any $\mathbb{Z}Q$-generating set for M determines a $\mathbb{Z}\overline{Q}$-generating set for $\overline{M}$. Let I be the ideal of $\mathbb{Z}\overline{Q}$ generated by $\mathrm{Ann}_{\mathbb{Z}Q}(M)$ and all elements of the form $z_y - y$, $y \in \mathbf{y}$. The actions make it clear that $z_y - y \in \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ and the fact that $(s, 0) = (1, 0)$ is the additive identity in $\overline{M}$ shows that $\mathrm{Ann}_{\mathbb{Z}Q}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$, so that $I \subseteq \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$. To establish the reverse inclusion, let $\xi \in \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$. Multiplying ξ by a unit made up as a suitable product of non-negative powers of the elements z_y (that is, by "clearing denominators"), we may assume that ξ is expressible as a finite sum $\xi = \sum_i \lambda_i \overline{\sigma}_i$ where $\lambda_i \in \mathbb{Z}Q$ and each $\overline{\sigma}_i$ is a finite product of non-negative powers of the basis elements z_y, $y \in \mathbf{y}$: $\overline{\sigma}_i = \prod_{\mathbf{y}} z_y^{n_y}$ where $n_y \geq 0$. Associated to each $\overline{\sigma}_i = \prod_{\mathbf{y}} z_y^{n_y}$, we have the element $\sigma_i = \prod_{\mathbf{y}} y^{n_y} \in S$. Note that $\overline{\sigma}_i(s, m) = (s, \sigma_i m)$ since the exponents n_y are all non-negative. Now given arbitrary $m \in M$ and working in $\overline{M}$, we have

$$(1,0) = \xi(1,m) = (1, \sum_i \lambda_i \sigma_i m)$$

which implies that $\sum_i \lambda_i \sigma_i \in \mathrm{Ann}_{\mathbb{Z}Q}(M) \subseteq I$. Working in the quotient ring $\mathbb{Z}\overline{Q}/I$ we have $z_y + I = y + I$ for all $y \in \mathbf{y}$ and hence $\overline{\sigma}_i + I = \sigma_i + I$ for all i. This implies that

$$\xi + I = \sum_i \lambda_i \overline{\sigma}_i + I = \sum_i \lambda_i \sigma_i + I = I$$

and so $\xi \in I$. •

The module $\overline{M}$ constructed in Lemma 3.1 is usually denoted by $\overline{M} = S^{-1}M$ in the literature, and is viewed as a module over a localized version $S^{-1}\mathbb{Z}Q$ of the group ring $\mathbb{Z}Q$. It is more convenient for our purposes to focus on the status of $\overline{M}$ as a module over the expanded group ring $\mathbb{Z}\overline{Q}$.

A nonconstant polynomial with integer coefficients is called *special* if the leading and constant coefficients are both 1. When q is an element of infinite order in a finitely generated abelian group Q, a special polynomial $p(X)$ uniquely determines an element $p(q) \in \mathbb{Z}Q$ that is neither a unit nor a zero divisor in the group ring. Baumslag used the fact that the group ring $\mathbb{Z}Q$ is noetherian to show that for any element q of infinite order in Q and any finitely generated $\mathbb{Z}Q$-module M, there is a special polynomial $p(X)$ such that $p(q)$ acts freely on M.

Lemma 3.2 [4, Lemma 7] *Suppose that we are given an element q of infinite order in a finitely generated abelian group Q. If M is a finitely generated $\mathbb{Z}Q$-module, then there is a special polynomial $p(X)$ such that if $m \in M$ and $p(q)m = 0$, then $m = 0$.*

Given $q \in Q$, M, and $p(X)$ as in Lemma 3.1, the multiplicative submonoid S of $\mathbb{Z}Q$ generated by $p(q)$ acts freely on M so by Lemma 3.1 we can embed M in the module $S^{-1}M = \overline{M}$ over $\mathbb{Z}\overline{Q}$ where $\overline{Q} = Q \times \langle z \rangle$. The action of $p(q)$ on M is then invertible in the larger module $\overline{M}$. For future reference, we shall say that the module $\overline{M}$ is obtained from M by a *special localization in the q direction*. More generally, we will consider simultaneous special localizations in directions q_i taken from linearly independent subsets of Q.

4 Improving Tameness

In this section we show how to use localization to embed finitely generated modules over finitely generated abelian groups in modules with better tameness properties. Our approach relies on a general description of the Sigma set, due to Bieri and Strebel, that involves module centralizers:

$$\mathrm{Cent}_{\mathbb{Z}Q}(M) = \{\lambda \in \mathbb{Z}Q : \lambda m = m \quad \forall m \in M\}.$$

The support of an element $\lambda = \sum \lambda(q)q \in \mathbb{Z}Q$ is the finite set of elements $q \in Q$ for which the $\mathbb{Z}$-coefficient $\lambda(q)$ is nonzero. The following lemma implies that the nonzero elements of $\Sigma(M, Q)$ determine an open subset of the sphere $S(Q)$.

Lemma 4.1 [7, Proposition 2.1] *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module. A nonzero character $0 \neq \chi \in V(Q)$ lies in $\Sigma(M, Q)$ if and only if there is a centralizing element $c \in \mathrm{Cent}_{\mathbb{Z}Q}(M)$ such that $\chi(q) > 0$ for all q in the support of c.*

Since $\mathrm{Cent}_{\mathbb{Z}Q}(M) = 1 + \mathrm{Ann}_{\mathbb{Z}Q}(M)$, where $\mathrm{Ann}_{\mathbb{Z}Q}(M)$ is the annihilator of M, Lemma 4.1 shows that the Sigma set of the $\mathbb{Z}Q$-module M depends only on the annihilator of M, an ideal of $\mathbb{Z}Q$. We first use Lemma 4.1 to investigate how the Sigma complement of a $\mathbb{Z}Q$-module M is related to modules over subgroups of Q.

Lemma 4.2 *Let A be a subgroup of a finitely generated abelian group Q. Let M be a finitely generated $\mathbb{Z}Q$-module and let M_A be a finitely generated $\mathbb{Z}A$-module such that $\mathrm{Ann}_{\mathbb{Z}A}(M_A) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$. If $\chi \in \Sigma(M, Q)^c$, then either $\chi|_A = 0$ or $\chi|_A \in \Sigma(M_A, A)^c$.*

Proof: Assume that $\mathrm{Ann}_{\mathbb{Z}A}(M_A) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$ and $0 \neq \chi|_A \in \Sigma(M_A, A)$. It suffices to show that $\chi \in \Sigma(M, Q)$. By Lemma 4.1, there is a centralizing element $c \in \mathrm{Cent}_{\mathbb{Z}A}(M_A)$ such that $\chi(a) > 0$ for all a in the support of c. But $c \in \mathrm{Cent}_{\mathbb{Z}A}(M_A) = 1 + \mathrm{Ann}_{\mathbb{Z}A}(M_A) \subseteq 1 + \mathrm{Ann}_{\mathbb{Z}Q}(M) = \mathrm{Cent}_{\mathbb{Z}Q}(M)$ and so $\chi \in \Sigma(M, Q)$ by Lemma 4.1. •

In the setting of Lemma 4.2, it is worth noting that if M_A is the $\mathbb{Z}A$-submodule of M spanned by a $\mathbb{Z}Q$-generating set of M, then $\mathrm{Ann}_{\mathbb{Z}A}(M_A) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$. Our next objective is to examine how special localization affects the Sigma complement.

Lemma 4.3 *Suppose that q and z are elements of a finitely generated abelian group Q and that M is a finitely generated $\mathbb{Z}Q$-module. Let $p(X)$ be a special polynomial of degree $d \geq 1$ such that $z - p(q) \in \operatorname{Ann}_{\mathbb{Z}Q}(M)$. If χ is an element of the Sigma complement $\Sigma(M,Q)^c$ of M, then exactly one of the following conclusions applies.*

1. *If $\chi(q) = 0$, then $\chi(z) \geq 0$.*
2. *If $\chi(q) > 0$, then $\chi(z) = 0$.*
3. *If $\chi(q) < 0$, then $\chi(z) = d\chi(q)$.*

Proof: We have annihilating elements $z - p(q)$, $-z^{-1}(z - p(q))$, and $q^{-d}(z - p(q))$ in $\operatorname{Ann}_{\mathbb{Z}Q}(M)$. These determine centralizing elements c_1, c_2, and $c_3 \in \operatorname{Cent}_{\mathbb{Z}Q}(M)$ with supports contained in the following lists.

$$\begin{array}{ll} c_1 : & z, q^i, i = 1, \ldots, d \\ c_2 : & z^{-1}q^i, i = 0, \ldots, d \\ c_3 : & q^{-d}z, q^{-i}, i = 1, \ldots, d \end{array}$$

Since the polynomial $p(X)$ is special, the elements z^{-1} and $z^{-1}q^d$ are in the support of c_2. Lemma 4.1 implies that each c_k has a support element with nonpositive value under the character χ.

Suppose first that $\chi(q) = 0$. The centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some i, so that $\chi(z) \geq 0$.

Suppose now that $\chi(q) > 0$. The centralizing element c_1 implies that $\chi(z) \leq 0$ and the centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some $i = 0, \ldots, d$, that is, $\chi(z) \geq \min\{i\chi(q) : i = 0, \ldots, d\} = 0$. Thus $\chi(z) = 0$ in this case.

Suppose finally that $\chi(q) < 0$. The centralizing element c_3 implies that $\chi(q^{-d}z) \leq 0$, or rather, $\chi(z) \leq d\chi(q)$. The centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some $i = 0, \ldots, d$, that is $\chi(z) \geq \min\{i\chi(q) : i = 0, \ldots, d\} = d\chi(q)$. We conclude that $\chi(z) = d\chi(q)$ in this case. •

We say that an element q of infinite order in Q is an *essential direction* for M if $\chi(q) \neq 0$ for all $\chi \in \Sigma(M,Q)^c$. As a foundational example, note that if Q has torsion-free rank one, then every element of infinite order in Q is essential for every finitely generated $\mathbb{Z}Q$-module. On the other hand, the free cyclic module over the free abelian group of rank two has no essential directions. This follows from the fact that the group ring $\mathbb{Z}\mathbb{Z}^2$ has no zero divisors, so that $\Sigma(\mathbb{Z}\mathbb{Z}^2, \mathbb{Z}^2)^c = V(\mathbb{Z}^2) - \{0\}$ by Lemma 4.1. Our main innovation is to note that localization in an essential direction improves tameness.

Lemma 4.4 *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module with essential direction $q \in Q$. Let $p(X)$ be a special polynomial of degree $d \geq 1$. Let $\overline{Q} = Q \times \langle z \rangle$ be the direct product of Q with the infinite cyclic group generated by z and suppose that $\overline{M}$ is a finitely generated $\mathbb{Z}\overline{Q}$-module such that $z - p(q) \in \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ and $\operatorname{Ann}_{\mathbb{Z}Q}(M) \subseteq \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$. We conclude the following.*

1. *If M is k-tame, then $\overline{M}$ is $(k+1)$-tame.*
2. *$\overline{M}$ has an essential direction $\overline{q} \in \overline{Q}$.*

Proof: Assume that M is k-tame. Let $\overline{\chi}_1, \ldots, \overline{\chi}_{k+1} \in \Sigma(\overline{M}, \overline{Q})^c$ and suppose that we are given non-negative real scalars $t_1, \ldots, t_{k+1}$ such that $\sum_i t_i \overline{\chi}_i = 0$. We must show that $t_i = 0$ for all $i = 1, \ldots, k+1$. For each i, let $\chi_i = \overline{\chi}_i|_Q$.

Consider the case when $\chi_i \neq 0$ for all i. Then $\chi_i \in \Sigma(M, Q)^c$ by Lemma 4.2 so $\chi_i(q) \neq 0$ for all i since q is essential for M. Let $q^* = qz^{-2} \in \overline{Q}$. For any given i, if $\chi_i(q) > 0$, then $\overline{\chi}_i(q^*) = \chi_i(q)$, while if $\chi_i(q) < 0$, then $\overline{\chi}_i(q^*) = (1-2d)\chi_i(q)$ by Lemma 4.3. From this we conclude that $\overline{\chi}_i(q^*) > 0$ for all i. Then the fact that $0 = \sum_i t_i \overline{\chi}_i(q^*)$ implies that $t_i = 0$ for all i.

Now suppose that $\ell \leq k$ and that $\chi_i \neq 0$ if and only if $i \leq \ell$. Then $0 = \sum_{i=1}^{\ell} t_i \chi_i$ where $\chi_i \in \Sigma(M, Q)^c$ for $i = 1, \ldots, \ell$ by Lemma 4.2. The fact that M is k-tame thus implies that $t_i = 0$ for those i. We thus conclude that $\sum_{i=\ell+1}^{k+1} t_i \overline{\chi}_i = 0$. Given $\ell + 1 \leq i \leq k+1$, Lemma 4.3 shows that $\overline{\chi}_i(z) \geq 0$. But since $\overline{M}$ is finitely generated, that is, 1-tame as a $\mathbb{Z}\overline{Q}$-module, and $\overline{Q} = Q \times \langle z \rangle$, we must have $\overline{\chi}_i(z) \neq 0$. Thus $\overline{\chi}_i(z) > 0$ for $i = \ell + 1 \ldots, k+1$. Then the fact that $0 = \sum_{\ell+1}^{k+1} t_i \overline{\chi}_i(z)$ implies that $t_i = 0$ for all i. Thus, $\overline{M}$ is $(k+1)$-tame.

To see that $\overline{M}$ has an essential direction, set $\overline{q} = qz$ and let $\overline{\chi} \in \Sigma(\overline{M}, \overline{Q})^c$. If $\overline{\chi}(q) = 0$, then Lemma 4.2 implies that $\overline{\chi}|_Q = 0$ since q is essential for M. The fact that $\overline{M}$ is finitely generated implies that $\overline{\chi} \neq 0$, so we must have $\overline{\chi}(\overline{q}) = \overline{\chi}(z) \neq 0$ since $\overline{Q}$ is generated by Q and z. Next, suppose that $\overline{\chi}(q) > 0$. Then Lemma 4.3 shows that $\overline{\chi}(\overline{q}) = \overline{\chi}(q) \neq 0$. Finally, if $\overline{\chi}(q) < 0$ then Lemma 4.3 shows that $\overline{\chi}(\overline{q}) = (1+d)\overline{\chi}(q) \neq 0$. Thus we find that $\overline{q} \in \overline{Q}$ is essential for $\overline{M}$. •

5 Essential Decompositions

Lemma 4.4 shows that if M is a finitely generated $\mathbb{Z}Q$-module with an essential direction, then special localization in that direction (Lemma 3.1) produces a new module with an essential direction and with improved tameness. As noted previously, however, not every finitely generated module possesses an essential direction.

We circumvent this difficulty with the following more general concept. A *k-essential decomposition* for a finitely generated $\mathbb{Z}Q$-module M consists of a direct product decomposition $Q = Q_1 \times \cdots \times Q_r$ of Q, together with, for each $i = 1, \ldots, r$, a $\mathbb{Z}Q_i$-submodule M_i of M such that

- $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$,
- M_i is k-tame, and
- M_i has an essential direction $q_i \in Q_i$.

Lemma 5.1 *Let Q be a finitely generated abelian group. If a finitely generated $\mathbb{Z}Q$-module M possesses a k-essential decomposition, then M is k-tame.*

Proof: Suppose that we are given characters $\chi_1, \ldots, \chi_k \in \Sigma(M, Q)^c$ and non-negative real scalars $t_1, \ldots, t_k$ such that $\sum_j t_j \chi_j = 0$. In order to prove that M is k-tame, it suffices to show that $t_j = 0$ for all j. Given $1 \leq i \leq r$, upon restriction to Q_i we find that $\sum_j t_j \chi_j|_{Q_i} = 0$. By Lemma 4.2, those χ_j that do not vanish on Q_i restrict to characters $\chi_j|_{Q_i} \in \Sigma(M_i, Q_i)^c$, and so the fact that M_i is k-tame implies that

$$\{t_j : \chi_j|_{Q_i} \neq 0\} \subseteq \{0\}.$$

On the other hand, since M is finitely generated, that is, 1-tame as a $\mathbb{Z}Q$-module, we know that $\chi_j \neq 0$ for all j. This means that for each $j = 1, \ldots, k$, there exists $1 \leq i \leq r$ such that $\chi_j|_{Q_i} \neq 0$, and hence that $t_j = 0$. •

Lemma 5.2 *If Q is a finitely generated abelian group, then every finitely generated $\mathbb{Z}Q$-module M admits a 1-essential decomposition.*

Proof: Given Q and M, we can decompose Q as a direct product $Q = Q_1 \times \cdots \times Q_r$ where each Q_i has torsion-free rank one. Choose a finite generating set $\mathbf{x}$ for the $\mathbb{Z}Q$-module M and for each i, let M_i be the $\mathbb{Z}Q_i$-submodule of M generated by $\mathbf{x}$: $M_i = \mathbb{Z}Q_i \cdot \mathbf{x}$. Then $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$ and M_i is 1-tame. In addition, any element q_i of infinite order in Q_i is essential for M_i since $0 \notin \Sigma(M_i, Q_i)^c$ and no nonzero character on the virtually infinite cyclic group Q_i can vanish on q_i. •

Lemma 5.3 *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module that admits a k-essential decomposition. There is a finitely generated abelian overgroup $\overline{Q}$ of Q and a $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that contains M as a $\mathbb{Z}Q$-submodule and which admits a $(k+1)$-essential decomposition.*

Proof: Given the essential data $Q = Q_1 \times \cdots \times Q_r$, M, M_i, and $q_i \in Q_i$, Lemma 3.2 allows us to select special polynomials $p_1(X), \ldots, p_r(X)$ such that for $i = 1, \ldots, r$, $p_i(q_i)$ acts freely on M. We form the direct product $\overline{Q} = Q \times \langle z_1 \rangle \times \cdots \times \langle z_r \rangle$ and the unital multiplicative submonoid of $\mathbb{Z}Q$ generated by the $p_i(q_i)$. Since S acts freely on M, Lemma 3.1 provides that the simultaneous special localization $\overline{M} = S^{-1}M$ is a $\mathbb{Z}\overline{Q}$-module that contains M as a $\mathbb{Z}Q$-submodule and is $\mathbb{Z}\overline{Q}$-generated by any given finite $\mathbb{Z}Q$-generating set $\mathbf{x}$ for M. In addition, $\mathrm{Ann}_{\mathbb{Z}Q}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ and $z_i - p_i(q_i) \in \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ for all i. For $i = 1, \ldots, r$, we set $\overline{Q}_i = Q_i \times \langle z_i \rangle$ and let $\overline{M}_i = \mathbb{Z}\overline{Q}_i \cdot \mathbf{x}$ be the $\mathbb{Z}\overline{Q}_i$-submodule of $\overline{M}$ generated by $\mathbf{x}$. We have that $z_i - p_i(q_i) \in \mathrm{Ann}_{\mathbb{Z}\overline{Q}_i}(\overline{M}_i)$ for all i. Since $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$, we know that $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i)$ annihilates $\mathbf{x}$, and hence $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \mathrm{Ann}_{\mathbb{Z}\overline{Q}_i}(\overline{M}_i) \subseteq \mathrm{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ for all i. By Lemma 4.4, each $\overline{M}_i$ is $(k+1)$-tame and has an essential direction $\overline{q}_i \in \overline{Q}_i$. •

Lemmas 5.2 and 5.3 constitute an inductive proof of the following result, which combines directly with Lemma 5.1 to complete the proof of Theorem 2.3.

Theorem 5.4 *Let n be a positive integer. If Q is a finitely generated abelian group and M is a finitely generated $\mathbb{Z}Q$-module, then there is a finitely generated abelian overgroup $\overline{Q}$ of Q and a $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that contains M as a $\mathbb{Z}Q$-submodule and which admits an n-essential decomposition.*

To illustrate the foregoing analysis, suppose that Q is a finitely generated abelian group of torsion-free rank one and that M is a finitely generated $\mathbb{Z}Q$-module. We may view the Sigma complement in the sphere $S(Q)$, which consists of exactly two points corresponding to discrete characters. Upon localization in an essential direction, we obtain a finitely generated abelian overgroup $\overline{Q}$ of Q with torsion-free rank two and a finitely generated 2-tame $\mathbb{Z}\overline{Q}$-module $\overline{M}$ with an essential direction. Viewed in the sphere $S(\overline{Q})$, Lemma 4.3 shows that $\Sigma(\overline{M}, \overline{Q})^c$ consists of at most three points, each of which comes from a discrete character. Continuing this process, after $n-1$ successive special localizations, we end up with an n-tame module over a finitely generated abelian group of torsion-free rank n whose Sigma complement, when viewed in the $(n-1)$-sphere, is contained in a set of $n+1$ discrete points.

References

[1] H. Åberg, *Bieri-Strebel valuations (of finite rank)*, Proc. London Math. Soc. (3) **52** (1986), 269-304.

[2] G. Baumslag, *A finitely generated, infinitely related group with trivial multiplicator*, Bull. Austral. Math. Soc. **5** (1971), 131-136.

[3] G. Baumslag, *A finitely presented group with free abelian derived subgroup of infinite rank*, Proc. Amer. Math. Soc. **35** (1972), 61-62.

[4] G. Baumslag, *Subgroups of finitely presented metabelian groups*, J. Austral. Math. Soc. **16** (1973), 98-110.

[5] R. Bieri and J. R. J. Groves, *Metabelian groups of type (FP)$_\infty$ are virtually of type (FP)*, Proc. London Math. Soc. (3) **45** (1982), 365-384.

[6] R. Bieri and J. Harlander, *On the FP$_3$-conjecture for metabelian groups*, J. London Math. Soc. (2) **64** (2001), 595-610.

[7] R. Bieri and R. Strebel, *Valuations and finitely presented metabelian groups*, Proc. London Math. Soc. (3) **41** (1980), 439-464.

[8] K. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics **87** (Springer-Verlag, 1982).

[9] P. H. Kropholler, *Soluble groups of type FP_∞ have finite torsion-free rank*, Bull. London Math. Soc. **25** (1993), 558-566.

[10] V. R. Remeslennikov, *On finitely presented groups*, Proc. Fourth All-Union Symposium on the Theory of Groups, Novosibirsk (1973), 164-169.

[11] R. Strebel, *Finitely presented soluble groups*, in Group Theory: Essays for Philip Hall, edited by K. W. Gruenberg and J. E. Roseblade (Academic Press, 1984), 257-314.