Universal Path Spaces

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Abstract

This paper examines a theory of *universal path spaces* that properly includes the covering space theory of connected, path connected, semi-locally simply connected spaces. The latter hypothesis that each point in the base space has a relatively simply connected open neighborhood—necessary and sufficient for the existence of a simply connected covering space—is abandoned, thus admitting as base even those spaces that contain arbitrarily small essential loops at wild points. When the base space is a *wild metric 2-complex*, the universal path space is simply connected if and only if the fundamental group is an *omega-group*—a group whose elements acquire non-negative real weights and form countable products of all order-type whenever their weights vanish as their appearance in the order-type deepens. Then the endpoint projection enjoys all of the standard lifting properties of covering projections; in particular, the fundamental group of the base space is isomorphic to the group of equivariant self-homeomorphisms of the universal path space. But the properly discontinuous action of the fundamental group on the *discrete fibers* over the *tame points* of the base space converge to continuous actions on *Cantor-like fibers* (i.e., totally disconnected and perfect ones) over the *wild points* of the base space. The standard features of covering space theory are thus engulfed by the richer features of universal path space theory for a variety of base spaces whose wild local topology prevents the application of traditional covering space theory.

Over an arbitrary pointed topological space there is the *universal path space* with its *endpoint projection* map. (See Section 2) The fundamental group of the base acts as a group of homeomorphisms on the universal path space (Theorem 2.5); and the action is one of isometries when the base space is metrized (Theorem 2.13). In general, the fibres of the endpoint projection consist of indiscrete connected components (Corollary 2.10). This includes the case of discrete fibres in ordinary simply connected covering spaces, but also allows for the much richer topological structure of totally disconnected perfect fibres.

The simplest nontrivial example has as base space the *Hawaiian earring* HE, the metric one-point union of a countable sequence of circles whose diameters limit on zero. We view the Hawaiian earring as a *wild metric 1-complex* whose 1-cells limit on the sole 0-cell union point. The universal path space of the Hawaiian earring is described in Section 5.

Theorem A *The universal path space* $\widehat{HE}$ *of the Hawaiian earring* HE *is a contractible metric one-complex with a Cantor-like vertex set, a* wild metric tree. *Its vertex set, the fibre over the union point* $z_0 \in HE$*, is a totally disconnected perfect metrized copy of the uncountable* free omega-group $\pi_1(HE, z_0)$. *This group acts freely on* $\widehat{HE}$*, and properly discontinuously off the vertex set, by equivariant cellular isometries with orbit space* HE.

We show (Corollary 4.7) that simple connectivity of the universal path space is equivalent to the *unique path lifting* property for the endpoint projection. In the presence of unique path lifting, the endpoint projection enjoys all of the crucial lifting properties of ordinary covering projections (Theorem 4.6), and so the action of the fundamental group of the base as equivariant homeomorphisms on the universal path space provides a comprehensive geometric record of the algebraic structure of the fundamental group of the base. The combinatorial structure of the wild metric tree $\widehat{HE}$ and its equivariant homeomorophisms is a direct manifestation of the fact that the fundamental group $\pi_1(HE, z_0)$ is a *free omega-group* (see [S97, Corollary 2.4] and [B-S97(1), Theorem 1.4]) with countable *weighted basis* in natural one-to-one correspondence with the 1-cells of the Hawaiian earring. More general omega-groups arise as fundamental groups of *wild metric 2-complexes*, in which 1-cells and 2-cells may limit on a wild 0-cell. The following result from Section 7 indicates the intrinsic connection between universal path space theory and omega-group theory.

Theorem B *The universal path space* $\widehat{Z}$ *of a wild metric 2-complex* Z *with single 0-cell* z_0 *is simply connected if and only if the fundamental group* $\pi_1(Z, z_0)$ *is an omega-group. In this case, the fundamental group acts freely, and properly discontinuously off the fiber* $p^{-1}(z_0)$*, by isometries on* $\widehat{Z}$ *with orbit space* Z*.The fibre* $p^{-1}(z_0)$ *is a metrized copy of the fundamental group that is either a discrete space or a Cantor-like space, according as* Z *is semi-locally simply connected or not.*

Practical criteria for the fundamental group of a wild metric 2-complex to be an omega-group are developed in [B-S97(2), S97]; a number of examples are described below. Thus the standard features of covering space theory are made available to a variety of base spaces whose wild local topology prevents application of traditional covering space theory.

1 Wild Metric Complexes and Omega-Groups

This introductory section is devoted to brief overviews of omega-groups and wild metric 2-complexes.

1.1 Omega-groups

The category of omega-groups was introduced by Sieradski [S97]; it is conceptually based upon the replacement of finite words for an alphabet by *order-type words* for a *weighted* *alphabet*. We begin a quick tour with the terminology of *weighted groups* (G, wt) and their associated *word groups* $\Omega(G, wt)$.

A **weighted set** is a pair (X, wt) where X is a set with an involution $x \mapsto x^{-1}$ (called *inversion*) with a distinguished fixed point 1_X (called the *identity*) and where $wt : X \to \mathbf{R}$ is a non-negative real-valued function on X such that, for each $x \in X$, $wt(x^{-1}) = wt(x)$ and $wt(x) = 0 \Leftrightarrow x = 1_X$. When the identity element 1_X is the unique fixed point of the inversion involution, the weighted set (X, wt) is called a **weighted alphabet**.

An **order-type** ω is a closed nowhere dense subset of the unit interval $I = [0, 1]$ that contains 0 and 1. The open dense complement $I - \omega$ is the union of a countable collection $\mathcal{I}_\omega$ of disjoint open intervals $i = (a_i, b_i)$, called **complementary intervals** of ω. The **length** of each complementary interval $i = (a_i, b_i) \in \mathcal{I}_\omega$ is denoted by $l(i) = b_i - a_i$. For example, there is the **trivial** order-type $\tau = \{0, 1\}$ whose sole interval $i = (0, 1)$ has length 1. The standard ordering on the unit interval determines a linear ordering on $\mathcal{I}_\omega$.

An **order-type word** for the weighted set (X, wt) is a pair (ω, x) where ω is an order-type and $x : \mathcal{I}_\omega \to X$ is a **labeling function** that satisfies the **weight restriction**: $wt(x(i)) \to 0$ as $l(i) \to 0$. Any $x \in X$ can be used to label the sole interval of the trivial order-type τ to form a **monosyllabic word** (τ, x). For order-type words, there are natural notions of **inversion** $(\omega, x)^{-1}$, **product** $(\omega, x) \cdot (\omega', x')$ by concatenation, **initial subwords** (ω_p, x_p) preceding order-type points $p \in \omega$, and **subwords** (ω_{pq}, x_{pq}) between order-type points $p, q \in \omega$. An **identity word** is any order-type word $(\omega, 1_X)$ in which each complementary interval is labeled by the identity element 1_X. The **word-weight** of an order-type word (ω, x) is the maximum weight $wt(x(i))$ $(i \in \mathcal{I}_\omega)$ of the labels on the complementary intevals of ω; it is finite by the weight restriction on order-type words.

Two order-type words (ω, x) and (ω', x') are **similar**, written $(\omega, x) \overset{s}{\sim} (\omega', x')$, if there is a pairing of those complementary intervals for the product word $(\lambda, y) = (\omega, x) \cdot (\omega', x')^{-1}$ having non-identity labels such that the following two conditions are true: (i) paired intervals $i \leftrightarrow j$ have non-identity inverse labels $y(i) = y(j)^{-1}$, and (ii) any two pairs $i \leftrightarrow j$ and $k \leftrightarrow l$ of paired intervals are either nested $(i < k < l < j)$ or disjoint $(i < j < k < l)$. For example, the empty pairing shows that any two given identity words are similar. A pairing of concentric arcs shows that any inverse product $(\omega, x) \cdot (\omega, x)^{-1}$ is similar to an identity word. Sieradski showed [S97, Lemma 1.1] that similarity is an equivalence relation on the set of order-type words (ω, x) for (X, wt) and that the set $\Omega(X, wt)$ of equivalence classes $[(\omega, x)]$ is a group under concatenation of order-type words. The group $\Omega(X, wt)$ is the **word group** on the weighted set (X, wt). See [S97], Sections 1 and 2, for the details.

The word group $\Omega = \Omega(X, wt)$ on a weighted alphabet (X, wt) has as subgroup the ordinary free group $F = F(X)$ of finite reduced words in the (unweighted) alphabet X. These groups coincide when the weights $wt(x)$ of all $x \in X$ are bounded below by some $\epsilon > 0$, but otherwise, $F = F(X)$ is a tiny portion of $\Omega = \Omega(X, wt)$ and, indeed, then Ω is isomorphic to no free group ([S97, Corollary 2.6]).

An order-type word (ω, x) is **reduced** if either it is the identity word $(\tau, 1_X)$ based on the trivial order-type $\tau : 0 < 1$, or else it has no subword that is similar to an identity word. As seen in [S97, Lemmas 2.1-2.3], each order-type word for (X, wt) is similar to a reduced order-type word and two reduced order-type words are are similar if and only if they are **associated** in the sense that there is an order-preserving bijection of the unit interval that matches the underlying order-types and the complementary labeled intervals. Up to association of order-type words, the word group $\Omega(X, wt)$ may therefore be viewed as the set of all reduced order-type words for (X, wt). The **word-weight** $w\text{-}wt(u)$ of $u \in \Omega(X, wt)$ is the word-weight of the reduced representative of u; this is the minimum of the word-weights of order-type words that represent the similiarity class u.

There is an alternative pictorial version of the similarity relation for order-type words. Two order-type words (ω, x) and (ω', x') are similar precisely when they can be viewed as decoration on the top and bottom edges of a unit square in which their complementary intervals that are labeled by non-identity letters $x(i) \neq 1_X \neq x'(i')$ can be simultaneously paired (connected) by a family of disjoint arcs α, each of which either joins a labeled interval of the word (ω, x) to an identically labeled interval of the word (ω', x') or joins inversely labeled intervals from the same word (ω, x) or (ω', x'). Such a display S of arcs is a **similarity square** for the words (ω, x) and (ω', x'); we write $S : (\omega, x) \ \sim \ (\omega', x')$. An arc whose two ends are at identically labeled intervals in the two opposing words, say $\cdot x(i) \cdot$ and $\cdot x'(i') \cdot$ where $x(i) = x'(i')$, is called an **associating arc**; an arc whose two ends are at inversely labeled intervals in the same word, say $\cdot x(j) \cdot$ and $\cdot x(k) \cdot$ where $x(j) = x(k)^{-1}$ or $\cdot x'(j') \cdot$ and $\cdot x'(k') \cdot$ where $x'(j') = x'(k')^{-1}$, is called a **cancelling arc**. A similarity square consisting entirely of associating arcs is an **associativity square**. Note that the two ends of a cancelling arc in a similarity square $S : (\omega, x) \ \sim \ (\omega', x')$ bound a subword of either (ω, x) or (ω', x') that is similar to any identity word. Thus, if the word (ω, x) is reduced, then any cancelling arc in S must join two complementary intervals of the word (ω', x'). Any similarity square relating reduced words (ω, x) and (ω', x') is therefore an associativity square. See [B-S97(1)] for further illustration.

A **weighted group** (G, wt) is a group G together with a nonnegative real-valued function $wt : G \to \mathbf{R}$ such that $wt(g^{-1}) = wt(g)$, $wt(g) = 0 \Leftrightarrow g = 1_G$, and $wt(g \cdot g') \leq max\{wt(g),\ wt(g')\}$ for all $g, g' \in G$. A weighted group (G, wt) has the **weight-derived metric** defined by $m(g, g') = wt(g^{-1}g')$. For example, word-weight makes the word group $\Omega(X, wt)$ into a weighted group $(\Omega(X, wt), w\text{-}wt)$ with its own word-weight-derived metric.

If (Z, z_0) is a pointed metric space with fundamental group $G = \pi_1(Z, z_0)$, then one has the subgroups $G^{(n)} \leq G$, $n \in \mathbf{Z}_+$, consisting of all elements for which there is a representative loop whose trajectory lies within the ball of radius $1/n$ about the basepoint z_0. There is a real-valued function on G whose value on a path class is less than or equal to $1/n$ if and only if that class is an element of the subgroup $G^{(n)}$. This is a weight function on $G = \pi_1(Z, z_0)$ precisely when the **infinitesimal subgroup** $\bigcap_n G^{(n)}$ is trivial, which is to say that the path homotopy class $\widehat{z_0} = [z_0^*]$ of the constant loop z_0^* at z_0 is the only class that contains loops with trajectories of arbitrarily small diameter.

If (G, wt) is any weighted group, then there is the word group $\Omega(G, wt)$. The correspondence of each $g \in G$ with the similarity class of the monosyllabic word (τ, g) injects the group G as a subset (*not* a subgroup!) of the word group. A **word evaluation** on a weighted group (G, wt) is a group homomorphism $\langle\ \rangle : \Omega(G, wt) \to G$, denoted by $[(\omega, g)] \to \omega\langle g\rangle$, such that $\tau\langle g\rangle = g$ for all $g \in G$. Such a word evaluation retracts the word group $\Omega(G, wt)$ onto the non-subgroup G, evaluating countable products of group elements of all order-types in a manner that extends the binary group operation in G.

An **omega-group** consists of a weighted group (G, wt) together with a word evaluation $\langle\ \rangle : \Omega(G, wt) \to G$ satisfying the *Weight, Multiplication, and Factorization Axioms* in [S97]. The Weight Axiom requires that weight of the evaluation $\omega\langle g\rangle \in G$ is dominated by the the word-weight of the word (ω, g). The Multiplication Axiom is a generalized associativity axiom for countable order-type products, and the Factorization Axiom, conversely, regulates countable order-type factorizations. The reader is referred to [S97] for their precise formulations. According to [S97, Theorem 1.12] and [B-S97(3)], the Multiplication and Factorization Axioms combine into the following *Uniqueness Property* in the presence of the other axioms for omega-groups.

Uniqueness Property *Let (ω, g) be an order-type word for the omega-group (G, wt). There is a unique continuous function $h : \omega \to G$ in the weight-derived metric on G with $h(0) = 1_G$ and $h(a_i) \cdot g(i) = h(b_i)$ in G for each complementary interval $i = (a_i, b_i) \in \mathcal{I}_\omega$, namely $h(p) = \omega_p\langle g_p\rangle$, the evaluation of the initial subword (ω_p, g_p) for each $p \in \omega$.*

The Uniqueness Property implies that the word evaluation $\langle\ \rangle : \Omega(G, wt) \to G$ for an omega-group is uniquely determined by the underlying weighted group (G, wt). When an omega-group is given as the fundamental group of a wild metric complex as in Theorem B, the Uniqueness Property plays a crucial role in the verification of unique path lifting for the endpoint projection of the universal path space; see Theorem 7.2.

The word group on any weighted alphabet turns out to be an example of an omega-group, indeed, one that is free in a categorical sense. See [S97]. For any weighed alphabet (X, wt), the word-weighted word group $(\Omega(X, wt), w\text{-}wt)$ is called the **free omega-group** with **weighted basis** (X, wt); its word evaluation $\langle\ \rangle : \Omega(\Omega(X, wt), w\text{-}wt) \to (\Omega(X, wt), w\text{-}wt)$ converts a word of words into a composite word.

1.2 Wild metric 2-complexes

The Hawaiian earring HE has a metric topology and a canonical 1-dimensional cellular decompostition in which the 1-cells limit on the single 0-cell. The open 1-cells of HE are open in this metric topology, which is nevertheless much coarser than the weak topology that is determined by the closed 1-cells. The Hawaiian earring is a *wild metric 1-complex* in the sense that it supports essential based loops of arbitrarily small diameter. More generally, each weighted alphabet (X, wt) is realized by a metric space $Z = Z(X, wt)$ built as the 1-point union of Euclidean 1-spheres $S^1_{x^{\pm 1}} \subseteq E^2_{x^{\pm 1}}$ sharing a common basepoint z_0, each one indexed by a non-identity element pair $x^{\pm 1} \in X$ and scaled to have diameter,

m-diam $(S^1_{x^{\pm 1}})$, equal to the generator weight $wt(x^{\pm 1})$. The metric m on Z is inherited from the max metric on the weak product of the Euclidean planes $E^2_{x^{\pm 1}}, x^{\pm 1} \in X$. The fundamental isomorphism $\Omega(X, wt) \cong \pi_1(Z(X, wt), z_0)$ established in [B-S97(1), Theorem 1.4] extends the usual interpretation of an ordinary free group as the fundamental group of a bouquet of circles with the weak topology.

Combinatorial terminology from [S97] will aid in the discussion of 2-dimensional examples. A **weighted presentation** $\mathcal{P} = < (X, wt) : (R, r\text{-}wt) >$ consists of a **weighted generator alphabet** (X, wt) and a **weighted relator set** $(R, r\text{-}wt)$. The latter is comprised of order-type words $r = (\gamma_r, y_r)$ from the free omega-group $(\Omega(X, wt), w\text{-}wt)$ with weighted-basis (X, wt) and these relator words are assigned **relator-weights** $r\text{-}wt(r) \geq w\text{-}wt(r)$ independently of their word-weights $w\text{-}wt(r)$. For any weighted presentation $\mathcal{P}$, the **weighted normal closure** of the weighted relator set $(R, r\text{-}wt)$ in the free omega-group $(\Omega(X, wt), w\text{-}wt)$ is the group $N(R, r\text{-}wt)$ of all **weighted consequences**, $\omega\langle w \cdot r\rangle \cdot (\omega\langle w\rangle)^{-1} \in \Omega(X, wt)$, associated with order-types ω and pairs of words (ω, w) and (ω, r) for the two weighted sets $(\Omega(X, wt), w\text{-}wt)$ and $(R, r\text{-}wt)$, respectively. The weighted presentation $\mathcal{P}$ **presents** the quotient group $\Pi(\mathcal{P}) = \Omega(X, wt)/N(R, r\text{-}wt)$.

As described in [B-S97(1)], each weighted presentation $\mathcal{P}$ is realized by a cellular metric space $Z(\mathcal{P})$ whose 1-skeleton is the metric bouquet $Z(X, wt)$. The realization $Z(\mathcal{P})$ has 2-cells attached via the relator words $r \in R \subseteq \Omega(X, wt) \cong \pi_1(Z(X, wt), z_0)$ whose diameters are regulated by the relator-weights $r\text{-}wt(r)$. The metric topology on $Z(\mathcal{P})$ is not the weak topology with respect to the closed cells, as the 1-cells and 2-cells may limit on the 0-cell when the weights are not bounded away from zero. But $Z(\mathcal{P})$ has closed skeletons, the k-cells are open in the k-skeleton for $k = 1, 2$, and the usual technique of radially expanding central portions of the 2-cells defines a deforming homotopy of the identity map relative the 1-skeleton. We call such a space a **wild metric complex**. Just as an ordinary group presentation presents the fundamental group of a 2-complex modeled on its generators and relators, each weighted presentation $\mathcal{P}$ presents the fundamental group $\pi_1(Z(\mathcal{P}), z_0)$ of its metric realization $Z(\mathcal{P})$ [B-S97(1), Theorem 2.5].

Example 1.1 *The harmonic archipelago bounded by the Hawaiian earring.*

The **disc of lunar crescents** DC is the wild metric cellular decomposition of the unit disc whose 1-skeleton is a planar copy of the Hawaiian earring and with lunar crescents between consecutive rings as 2-cells. The **harmonic archipelago** HA is the metric subspace of Euclidean 3-space produced from DC by raising an interior portion of each lunar crescent to form a mountain of unit elevation. See Figure 1.

Notice that the disc of crescents DC and the harmonic archipelago HA have identical cellular decompositions and even isometric 1-skeletons. The disc DC with its Euclidean metric topology is contractible, but the harmonic archipelago HA is not even simply connected. Based loops in the Hawaiian earring are described by order-type words (ω, x) for the weighted generator alphabet $\{(x_n^{\pm 1},\ wt\ \frac{1}{n}) : n \in \mathbf{Z}\}$. Compactness considerations

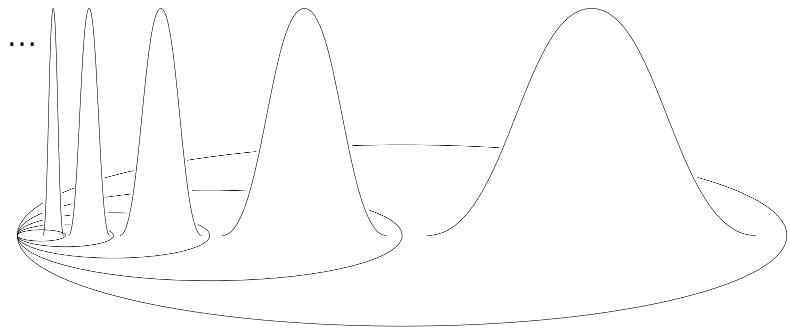

Figure 1: Harmonic Archipelago

show that any loop deforms continuously over at most finitely many of the mountainous crescents in HA, and it follows that $\pi_1(HA, z_0)$ is nontrivial, in fact uncountable. Deforming over the two-cells of HA, each based loop in HA is path homotopic to a based loop of arbitrarily small diameter, so that $G = \pi_1(HA, z_0)$ is **infinitesimal** in the sense that it is equal to the infinitesimal subgroup $\bigcap_n G^{(n)}$. The group $\pi_1(HA, z_0)$ is not an omega-group, nor even a weighted group.

The harmonic archipelago is a metric model of the weighted presentation $\mathcal{P}_{HA}$:

$$< \{(x_n, wt\ \tfrac{1}{n}) : n \geq 1\} : \{(r_n := x_n \cdot x_{n+1}^{-1}, r\text{-}wt\ 1) : n \geq 1\} >, \tag{1}$$

in which the constant relator-weights $r\text{-}wt(r_n) = 1$ manifest the unit elevation of the mountainous 2-cells with crescent-shaped boundaries. The great distinction between the trivial fundamental group $\pi_1(DC, z_0) = 1$ and the uncountable group $\pi_1(HA, z_0)$ is due to the different metric limiting behavior of their 2-cells. The weighted presentation $\mathcal{P}_{DC}$:

$$< \{(x_n, wt\ \tfrac{1}{n}) : n \geq 1\} : \{(r_n := x_n \cdot x_{n+1}^{-1}, r\text{-}wt\ \tfrac{1}{n}) : n \geq 1\} > \tag{2}$$

has the disc of crescents DC as metric model, and this weighted presentation presents the trivial group (not the infinite cylic group presented by its unweighted analog).

Example 1.2 *The harmonic projective plane on the Hawaiian earring.*

The weighted presentation $\mathcal{P}_{HP}$:

$$< \{(x_n, wt\ \tfrac{1}{n}) : n \geq 1\} : \{(r_n := x_n^2, r\text{-}wt\ \tfrac{1}{n}) : n \geq 1\} > \tag{3}$$

presents the quotient group $\Omega(X, wt)/N(XX, wt)$ of the free omega-group $\Omega(X, wt)$ on the weighted basis (X, wt), modulo the weighted normal closure $N(XX, wt)$ of the weighted relator set of squares xx of members $x \in X$. The metric realization HP of this weighted presentation can be created by the identification of antipodal points in each 2-sphere in the surface of revolution obtained by rotating the Hawaiian earring about its axis of symmetry. It is a metric 1-point union of copies of the real projective plane of vanishing diameters. In [B-S97(2)], the fundamental group of HP is seen to be an omega-group by use of [S97, Theorem 2.11]; this group is a *weighted free product* [B-S97(2)] of a sequence of copies of the finite cyclic group of order 2 of vanishing weights.

Example 1.3 *The projective telescope PT through the Hawaiian earring.*

The weighted presentation $\mathcal{P}_{PT}$:

$$< \{(x_n, wt\ \tfrac{1}{n}) : n \geq 1\} : \{(r_n := x_{n+1}^{-1} \cdot x_n^2, r\text{-}wt\ \tfrac{1}{n}) : n \geq 1\} > \tag{4}$$

presents the quotient group $\Omega(X, wt)/N(R, r\text{-}wt)$ of the free omega-group whose elements are cosets $(\omega, x)\cdot N(R, r\text{-}wt)$ of reduced order-type words (ω, x) for the weighted generating set (X, wt); two reduced words (ω, x) and (ω', x') represent the same coset if and only if they are similar modulo finitely many applications of each relation $x_{n+1} = x_n^2, n \geq 1$. This quotient group is an omega-group; see [B-S97(2)].

The metric realization of $\mathcal{P}_{PT}$ is the **projective telescope** PT obtained by iteratively replacing a disc d_{n+1} of diameter $\frac{1}{n+1}$ in a projective plane P_n of diameter $\frac{1}{n}$ by a projective plane P_{n+1} of diameter $\frac{1}{n+1}$, beginning with a projective plane P_1 of diameter 1. An analysis using similarity squares [B-S97(1)] shows that $\pi_1(PT, z_0)$ is nonabelian; it has the additive group $\mathbf{Z}_2$ of 2-adic integers as its *omega-abelianization* [B-S97(2)].

2 Universal Path Spaces

Let (Z, z_0) be a pointed topological space. Working in the fundamental $\pi(Z)$, we consider the set $\pi(Z)_{z_0}$ of path homotopy classes $[f]$ of paths $f : (I, 0) \to (Z, z_0)$ in Z that begin at z_0. To topologize $\pi(Z)_{z_0}$, we utilize the following notation.

2.1 Local subgroupoids and cosets

Given any open subspace $U \subseteq^{op} Z$, there is the **local subgroupoid**

$$\Pi_Z^U = \mathrm{Im}(\mathrm{inc}_\# : \pi(U) \to \pi(Z)) \tag{5}$$

and, given any path class $[f] \in \pi(Z)_{z_0}$, there is the **local (groupoid) coset**

$$[f] \cdot \Pi_Z^U = \{[h] \in \pi(Z)_{z_0} : \exists \epsilon\ : (I, 0) \to (U, f(1)) \text{ such that } [h] = [f][\epsilon]\}. \tag{6}$$

Consider the set Σ of local subgroupoids Π_Z^U of $\pi(Z)_{z_0}$ for all open subspaces $U \subseteq^{op} Z$:

$$\Sigma = \{\Pi_Z^U : U \subseteq^{op} Z\}. \tag{7}$$

Lemma 2.1 *The set $\pi(Z)_{z_0}/\Sigma$ of all local cosets $[f]\cdot\Pi_Z^U$ is a base for a topology on $\pi(Z)_{z_0}$.*

Proof: If $[f] \in [g] \cdot \Pi_Z^U \cap [h] \cdot \Pi_Z^V$, then $[f] \in [f] \cdot \Pi_Z^{U\cap V} \subseteq [g] \cdot \Pi_Z^U \cap [h] \cdot \Pi_Z^V$. □

2.2 Universal path space and endpoint projection

Using the base $\pi(Z)_{z_0}/\Sigma$ of local cosets $[f] \cdot \Pi_Z^U$ to topologize $\pi(Z)_{z_0}$, the resulting **universal path space** (at z_0) is denoted $\widehat{Z}$ and provided the basepoint $\widehat{z_0} = [z_0^*]$, the path class of the constant loop z_0^* at z_0. There is the **endpoint projection**

$$p = p_Z : \widehat{Z} \to Z \text{ given by } p([f]) = f(1). \tag{8}$$

(This is the standard construction of the universal covering space of a connected, locally path connected, semi-locally simply connected space Z; but here we make no such simplifying assumptions about the local topology of Z.)

Theorem 2.2 *Let (Z, z_0) be a pointed topological space.*

(a) The endpoint projection $p : \widehat{Z} \to Z$ is continuous.

(b) The image of the endpoint projection $p : \widehat{Z} \to Z$ is the path component of z_0 in Z. In particular, $p : \widehat{Z} \to Z$ is surjective if and only if Z is path connected.

(c) The endpoint projection $p : \widehat{Z} \to p(\widehat{Z})$ is an open surjection if and only if the path component $p(\widehat{Z})$ of z_0 in Z is locally path connected.

Proof: The theorem follows from these facts with some consideration: (1) the pre-image under p of an open subset $U \subseteq^{op} Z$ is $p^{-1}(U) = \bigcup\{[g] \cdot \Pi_Z^U : [g] \in \widehat{Z}\}$, (2) the image $P = p(\widehat{Z})$ is the path component of z_0 in Z, (3) for each $[f] \in \widehat{Z}$, the image of the basic open set $[f] \cdot \Pi_Z^U$ under p is the path component of $f(1)$ in U, and (4) if $z \in P$ and $z \in U \subseteq^{op} Z$, then the path component of z in U equals its path component in $P \cap U$ □

Corollary 2.3 *When Z is connected and locally path connected, the endpoint projection $p : \widehat{Z} \to Z$ is a continuous open surjection.* □

2.3 Change of basepoint and functorality

Left multiplication by a fixed element $[\alpha] \in \pi(Z)$ in the fundamental groupoid $\pi(Z)$ determines a function $T_{[\alpha]} : \pi(Z)_{\alpha(1)} \to \pi(Z)_{\alpha(0)}$, $T_{[\alpha]}([f]) = [\alpha][f]$ for all $[f] \in \pi(Z)_{\alpha(1)}$, relating the sets of path homotopy classes of paths that begin at the terminal point $\alpha(1)$ and the intitial point $\alpha(0)$. One notes that the constant path class induces the identity function, $T_{[z_0^*]} = 1_{\pi(Z)_{z_0}}$, and that if $\alpha(1) = \beta(0)$, then $T_{[\alpha]} \circ T_{[\beta]} = T_{[\alpha][\beta]}$. In particular, the function $T_{[\alpha]}$ is a bijection with inverse $T_{[\alpha]}^{-1} = T_{[\alpha]^{-1}}$. Because

$$T_{[\alpha]}([f] \cdot \Pi_Z^U) = ([\alpha][f]) \cdot \Pi_Z^U$$

for all $[f] \in \pi(Z)_{\alpha(1)}$ and $U \subseteq^{op} Z$, the function $T_{[\alpha]}$ is an open mapping from the universal path space based at the terminal point $\alpha(1)$ to that based at the initial point $\alpha(0)$. So the function $T_{[\alpha]} : \pi(Z)_{\alpha(1)} \to \pi(Z)_{\alpha(0)}$ is a **change of basepoint homeomorphism**. Thus:

Theorem 2.4 *The universal path space is independent of the choice of the basepoint within a path component of Z: Each path $\alpha : (I, 0, 1) \to (Z, z_0, z_1)$ in Z determines a homeomorphism $T_{[\alpha]} : \pi(Z)_{z_1} \to \pi(Z)_{z_0}$ between the universal path spaces at z_1 and z_0.* □

So the fundamental group $\pi_1(Z, z_0)$ acts on the universal path space $\widehat{Z} = \pi(Z)_{z_0}$:

Theorem 2.5 *For any pointed topological space (Z, z_0), the fundamental group $\pi_1(Z, z_0)$ acts by homeomorphisms on the universal path space $\widehat{Z} = \pi(Z)_{z_0}$ via left multiplication in the fundamental groupoid $\pi(Z)$:*

$$[g] * [f] = T_{[g]}([f]) = [g][f]$$

for all $[g] \in \pi_1(Z, z_0)$ and $[f] \in \widehat{Z}$. This action has the following properties.

(a) The action is free: If $[g] * [f] = [f]$, then $[g] = 1 = [z_0^*]$ in $\pi_1(Z, z_0)$.

(b) The action is p-equivariant: $p([g] * [f]) = p([f])$ for all $[g] \in \pi_1(Z, z_0)$ and $[f] \in \widehat{Z}$.

(c) The action is transitive on fibres of p: If $p([f]) = z$, then $p^{-1}(z) = \pi_1(Z, z_0) * [f]$. In particular, $p^{-1}(z_0) = \pi_1(Z, z_0)$. □

For functorality, there is the following

Theorem 2.6 *If $F : (Y, y_0) \to (Z, z_0)$ is a pointed continuous map, then there is a continuous function $\widehat{F} : \widehat{Y} \to \widehat{Z}$ given by $\widehat{F}([g]) = [F \circ g]$ for each $[g] \in \widehat{Y}$. In addition,*

(a) the $\widehat{\ }$ process is functorial: where defined, $\widehat{G \circ F} = \widehat{G} \circ \widehat{F}$ and $\widehat{1_Z} = 1_{\widehat{Z}}$;

(b) the $\widehat{\ }$ process respects the endpoint projections: $p_Z \circ \widehat{F} = F \circ p_Y$;

(c) the map $\widehat{F}$ restricts to the fibre $p^{-1}(y_0) = \pi_1(Y, y_0)$ to give the fundamental group homomorphism induced by F: $\widehat{F}|_{p_Y^{-1}(y_0)} = F_\# : \pi_1(Y, y_0) \to \pi_1(Z, z_0) = p_Z^{-1}(z_0)$; and

(d) the $\widehat{\ }$ process is compatible with change of basepoint: $\widehat{F}(T_{[\alpha]}([f])) = T_{[F\circ\alpha]}(\widehat{F}([f]))$.

Proof: To verify continuity of $\widehat{F}$, suppose that $[g] \in \widehat{Y}$ and $\widehat{F}([g]) = [F \circ g]$ is contained in the basic open subset $[f] \cdot \Pi_Z^U$ of $\widehat{Z}$. Using the open set $W = F^{-1}(U) \subseteq Y$, one checks that $[g] \in [g] \cdot \Pi_Y^W \subseteq^{op} \widehat{Y}$ and that $\widehat{F}([g] \cdot \Pi_Y^W) \subseteq [F \circ g] \cdot \Pi_Z^U = [f] \cdot \Pi_Z^U$. The remaining assertions are easily checked. □

The universal path space construction respects product spaces:

Theorem 2.7 *For any indexed family $\{(Z_\alpha, z_\alpha)\}$ of pointed spaces, the universal path space of the pointed product space $(\Pi_\alpha Z_\alpha, (z_\alpha))$ is canonically homeomorphic to the product of the universal path spaces of the factors: $\widehat{\Pi_\alpha Z_\alpha} \approx \Pi_\alpha \widehat{Z_\alpha}$.*

Proof: By functorality, Theorem 2.6, the projections $\pi_\alpha : \Pi_\alpha Z_\alpha \to Z_\alpha$ induce continuous maps $\widehat{\pi_\alpha} : \widehat{\Pi_\alpha Z_\alpha} \to \widehat{Z_\alpha}$, and hence a continuous map $\pi : \widehat{\Pi_\alpha Z_\alpha} \to \Pi_\alpha \widehat{Z_\alpha}$. It is routine to check that the map π is a continuous open bijection. □

2.4 Endpoint projection fibres

We now consider the fibres of the endpoint projection $p : \widehat{Z} \to Z$ defined on the universal path space $\widehat{Z}$. Given any point $z \in Z$, there is the **fibre**

$$p^{-1}(z) = \{[g] \in \widehat{Z} : p([g]) = g(1) = z\}.$$

The subspace topology that the fibre $p^{-1}(z)$ acquires from $\widehat{Z}$ has base

$$\{([h] \cdot \Pi_Z^U) \cap p^{-1}(z) : [h] \in \widehat{Z} \text{ and } U \subseteq^{op} Z\}.$$

To reformulate this base, we define for each open neighborhood U of z the **local subgroup**

$$\Pi_Z^U(z) = \mathrm{Im}(\mathrm{inc}_\# : \pi_1(U, z) \to \pi_1(Z, z)). \tag{9}$$

Let $\Sigma(z)$ denote the set of all local subgroups $\Pi_Z^U(z)$ of $\pi_1(Z, z)$ for the open neighborhoods U of z in Z:

$$\Sigma(z) = \{\Pi_Z^U(z) : z \in U \subseteq^{op} Z\}. \tag{10}$$

We give $\pi_1(Z, z)$ the **local subgroup topology** with base the set $\pi_1(Z, z)/\Sigma(z)$ of all cosets in $\pi_1(Z, z)$ of all local subgroups $\Pi_Z^U(z) \in \Sigma(z)$:

$$\pi_1(Z, z)/\Sigma(z) = \{[g] \cdot \Pi_Z^U(z) : [g] \in \pi_1(Z, z), z \in U \subseteq^{op} Z\}. \tag{11}$$

This set (11) of group cosets is a base for $\pi_1(Z, z)$: If $[f] \in [g] \cdot \Pi_Z^U(z) \cap [h] \cdot \Pi_Z^V(z)$, then $[f] \in [f] \cdot \Pi_Z^{U \cap V}(z) \subseteq [g] \cdot \Pi_Z^U(z) \cap [h] \cdot \Pi_Z^V(z)$. Given any fibre element $[f] \in p^{-1}(z) \subseteq \widehat{Z}$, multiplication in the fundamental groupoid $\pi(Z)$ provides a function

$$\phi_{[f]} : \pi_1(Z, z) \to p^{-1}(z)$$

given by $\phi_{[f]}([g]) = [f][g] \in \pi(Z)_{z_0}$ for all $[g] \in \pi_1(Z, z)$.

Lemma 2.8 *For each $[f] \in \widehat{Z}$ with $p([f]) = f(1) = z$, the map $\phi_{[f]} : \pi_1(Z, z) \to p^{-1}(z)$ is a homeomorphism from the local subgroup topology on $\pi_1(Z, z)$ to the subspace topology on the fibre $p^{-1}(z) \subseteq \widehat{Z} = \pi(Z)_{z_0}$ of the universal path space.*

Proof: Left cancellation in the groupoid $\pi(Z)$ implies that $\phi_{[f]}$ is injective. If $[h] \in p^{-1}(z)$, then $[f]^{-1}[h] \in \pi_1(Z, z)$ and $\phi_{[f]}([f]^{-1}[h]) = [h]$, showing that $\phi_{[f]}$ is surjective. The bijection $\phi_{[f]}$ is a homeomorphism because, for any an open neighborhood U of $z = f(1)$ in Z, $\phi_{[f]}$ matches the base coset $[g] \cdot \Pi_Z^U(z)$ in $\pi_1(Z, z)$ with the base set $([h] \cdot \Pi_Z^U) \cap p^{-1}(z)$ in $p^{-1}(z)$, via the correspondences $[f][g] = [h]$ and $[g] = [f]^{-1}[h]$. □

2.5 Subgroup topologies

We now investigate the properties of the subgroup topologies of the fibres of the universal path space. To simplify the notation, we formulate the concept for an abstract group.

For any group G, a non-empty family Σ of subgroups S of G is a **neighborhood family** provided: Given $S, S' \in \Sigma$, there exists $S'' \in \Sigma$ such that $S'' \subseteq S \cap S'$. (Equivalently, Σ is a *directed set* under the partial ordering of reverse inclusion.)

Let G/Σ denote the set of all cosets in G of subgroups from any neighborhood family Σ:

$$G/\Sigma = \{gS : g \in G, S \in \Sigma\}.$$

Because Σ is a neighborhood family of subgroups, G/Σ is a base for a topology on G, called the **subgroup topology**. The basic open sets gS, $g \in G$, $S \in \Sigma$, are both open and closed in the subgroup topology since the cosets of a given subgroup form a partition of G. The subgroup topology is homogeneous since left mulitplication by elements of G determine self-homeomorphisms of G. However the group G is not necessarily a topological group since right multiplication by a fixed element of G need not be continuous.

The intersection $S_\Sigma = \bigcap\{S : S \in \Sigma\}$ is the **infinitesimal subgroup** for the neighborhood family Σ. It is a closed subgroup in the subgroup topology on G with base G/Σ. Indeed, S_Σ is the closure of the identity $1_G \in G$; its coset gS_Σ is the closure of the element $g \in G$.

Theorem 2.9 *Let the group G have the subgroup topology determined by a neighborhood family Σ of subgroups of G.*

(a) The infinitesimal subgroup S_Σ is a maximal indiscrete subspace of G.

(b) Given $g \in G$, the connected component of g in G is the coset gS_Σ.

(c) The following are equivalent.

(i) G is Hausdorff;

(ii) *G satisfies the* T_0 *separation property;*

(iii) *The infinitesimal subgroup is trivial:* $S_\Sigma = \{1\}$*;*

(iv) *G is totally disconnected.*

(d) *The group G is discrete if and only if* Σ *contains the trivial group.*

(e) *When the group G is totally disconnected but not discrete, it is perfect in the topological sense that each element of G is an accumulation point of G.*

(f) *When the group G is not totally disconnected, it consists of nontrivial indiscrete components, which are the cosets* gS_Σ *of the infinitesimal subgroup.*

Proof: The collection $\{gS \cap S_\Sigma : g \in G, S \in \Sigma\}$ is a base for the subspace topology on the infinitesimal subgroup S_Σ. Because the cosets $gS, g \in G$, of any $S \in \Sigma$ partition G and because $S_\Sigma = \bigcap\{S : S \in \Sigma\}$, the only non-empty base members occur when $gS = S$ and so have the form $S \cap S_\Sigma = S_\Sigma$. Thus S_Σ has the indiscrete topology, and hence is connected. If F is a subset of G properly containing S_Σ, then there is an element $g \in F$ and a subgroup $S \in \Sigma$ such that $g \notin S$. Then the set $F \cap gS$ is a non-empty proper open and closed subset of F, so F is not connected. Thus S_Σ is the maximal connected subset of G that contains the identity, and (b) is proved. Since indiscrete spaces are connected, S_Σ is maximal among indiscrete subsets of G, which completes the proof of (a).

As for (c), the implication $(i) \Rightarrow (ii)$ is trivial. If G satisfies the T_0 property, then so does its indiscrete subspace S_Σ, which therefore can contain at most one point. Thus S_Σ is trivial, as in (iii). The implication $(iii) \Rightarrow (iv)$ follows directly from (b). Finally, the implication $(iv) \Rightarrow (i)$ is a general fact for topological spaces and is seen as follows: If points x and y fail to have disjoint open neighborhoods in a space X, then the point y lies in the intersection $\bigcap \overline{U}$ of the closures of all open neighborhoods U of x in X; the intersection $\bigcap \overline{U}$ is connected since x lies in the closure of each of non-empty open subset $V \subseteq^{op} X$ that meets $\bigcap \overline{U}$.

The group G is discrete if and only if each element g is contained in the base G/Σ, which happens if and only if Σ contains the trivial group. This proves (d).

When G is totally disconnected but not discrete, no member of the family Σ is trivial yet their intersection, the infinitesimal subgroup, is trivial. Thus each member of Σ is infinite, which implies that each element of G is an accumulation point of G. This proves (e). The statement (f) follows from (a), (b), and (c). □

2.6 Fibre topologies

The preceding analysis applies to each fibre $p^{-1}(z)$ of the endpoint projection $p : \widehat{Z} \to Z$. By Lemma 2.8, $p^{-1}(z)$ is homeomorphic to the fundamental group $\pi_1(Z, z)$, given the subgroup topology with respect to the neighborhood family $\Sigma(z)$ of local subgroups $\Pi_Z^U(z)$ of $\pi_1(Z, z)$. The infinitesimal subgroup in $\pi_1(Z, z)$ with respect to the neighborhood family $\Sigma(z)$ of local subgroups is denoted by $\Pi(z)$. Thus,

$$\Pi(z) = \bigcap\{\Pi_Z^U(z) : z \in U \subseteq^{op} Z\}$$

consists of all path classes of loops in Z based at the point z that can be represented by a loop within each open neighborhood of z.

To express the conclusions of Theorem 2.9, we devise the following terminology. The point z is a **regular point** of Z if the family $\Sigma(z)$ of local subgroups of $\pi_1(Z, z)$ contains the trivial group. This means that the point z has a relatively simply connected open neighborhood U in Z, i.e., $\mathrm{inc}_\# : \pi_1(U, z) \to \pi_1(Z, z)$ is trivial. The point z is a **nonsingular point** if the infinitesimal subgroup $\Pi(z)$ is trivial, i.e., only the trivial path class can be represented by a loop within each open neighborhood of z. The point z is a **singular point** if the infinitesimal subgroup $\Pi(z)$ is nontrivial, i.e., some non-trivial path class can be represented by an essential loop withing each open neighborhood of z.

Corollary 2.10 *Let the point z lie in the path component of z_0 in Z. There are just these three mutually exclusive possibilities for the non-empty fiber $p^{-1}(z)$:*

(a) it is discrete if and only if z is regular;

(b) it is totally disconnected and perfect if and only if z is nonsingular but not regular; and

(c) it consists of nontrivial indiscrete components if and only if z is singular. □

2.7 Separation and metrizability

We can now examine the Hausdorff and metric status of universal path spaces.

Theorem 2.11 *When Z is Hausdorff, its universal path space $\widehat{Z}$ is Hausdorff if and only if each point $z \in Z$ is nonsingular.*

Proof: Since the Hausdorff property is hereditary, Corollary 2.10(c) shows that the space $\widehat{Z}$ is Hausdorff only if each point of Z is nonsingular. Now suppose that Z is Hausdorff and that each point of Z is nonsingular. To show $\widehat{Z}$ Hausdorff, let $[f], [g] \in \widehat{Z}$ with $[f] \neq [g]$. If $f(1) \neq g(1)$, then there are disjoint open neighborhoods $U, V \subseteq^{op} Z$ such that $f(1) \in U$ and $g(1) \in V$. Then $[f] \cdot \Pi_Z^U$ and $[g] \cdot \Pi_Z^V$ are disjoint open neighborhoods of $[f]$ and $[g]$ in $\widehat{Z}$. If $f(1) = g(1) = z$, then $[f]^{-1}[g]$ is nontrivial in $\pi_1(Z, z)$. Since z is nonsingular, there is an open neighborhood U of z in Z such that $[f]^{-1}[g] \notin \Pi_Z^U(z)$. This implies that $[g] \cdot \Pi_Z^U$ and $[f] \cdot \Pi_Z^U$ are disjoint open neighborhoods of $[f]$ and $[g]$ in $\widehat{Z}$. $\square$

Assume that (Z, m) is a metric space. We define the **weight** of a path homotopy class $[f] \in \pi(Z)$ to be the greatest lower bound of the m-diameters of the images $g(I)$ of all paths $g : I \to Z$ that are path homotopic to f in Z:

$$wt([f]) = \text{glb}\{\text{diam}\, g(I) : [f] = [g]\}. \tag{12}$$

Note: $wt([f]) = wt([f]^{-1}) \geq 0$ and whenever the path product $[f][g]$ is defined the **multiplicative property** $wt([f][g]) \leq wt([f]) + wt([g])$ holds by path-diameter calculations.

Given a fixed basepoint $z_0 \in Z$, we define the distance d between path classes $[f]$ and $[g]$ in the universal path space $\widehat{Z} = \pi(Z)_{z_0}$ using the path class weight wt (12), as follows:

$$d([f], [g]) = wt([f]^{-1}[g]). \tag{13}$$

This distance function (13) is a symmetric non-negative function. Further, the triangle inequality $d([f], [h]) \leq d([f], [g]) + d([g], [h])$, for $[f]$, $[g]$, $[h] \in \widehat{Z}$, follows from the multiplicative property: $wt([f]^{-1}[h]) \leq wt([f]^{-1}[g]) + wt([g]^{-1}[h])$. Thus d is a pseudo-metric on the universal path space $\widehat{Z}$, called the **lifted pseudo-metric**.

Because the metric balls $B_\epsilon(z)$ are a neighborhood base of z in the metric space (Z, m), because any path $g : (I, 0) \to (Z, z)$ with diam $g(I) < \epsilon$ is contained in $B_\epsilon(z)$, and because any path g in $B_\epsilon(z)$ has diam $g(I) < 2\epsilon$, metric calculations in (Z, m) yield:

Lemma 2.12 *(a) If $[f], [g] \in \widehat{Z}$, then $d([f], [g]) = 0$ if and only if $f(1) = g(1) \in Z$ and $[f]^{-1}[g]$ lies in the infinitesimal subgroup $\Pi_{f(1)} \leq \pi_1(Z, f(1))$.*

(b) If $[f] \in \widehat{Z}$ and $\epsilon > 0$, then $B_\epsilon([f]) \subseteq [f] \cdot \Pi_Z^{B_\epsilon(f(1))} \subseteq B_{2\epsilon}([f])$. $\square$

Theorem 2.13 *Let (Z, m, z_0) be a pointed metric space.*

(a) The lifted pseudo-metric determines the universal path space topology on $\widehat{Z} = \pi(Z)_{z_0}$.

(b) The lifted pseudo-metric is preserved by change of basepoint homeomorphisms and the fundamental group $\pi_1(Z, z_0)$ acts on $\widehat{Z} = \pi(Z)_{z_0}$ by isometries.

(c) The universal path space $\widehat{Z}$ is metrizable if and only if each point in the path component of z_0 in Z is nonsingular, in which case its lifted pseudo-metric is a metric.

Proof: Lemma 2.12(b) shows that the pseudo-metric balls determine the universal path space topology for $\pi(Z)_{z_0} = \widehat{Z}$. In addition, given $[\alpha] \in \pi(Z)$ and $[g], [h] \in \pi(Z)_{\alpha(1)}$, we have $d(T_{[\alpha]}([f]), T_{[\alpha]}([g])) = d([f], [g])$, since $wt(([\alpha][f])^{-1}([\alpha][g])) = wt([f]^{-1}[g])$. This shows that the change of basepoint homeomorphisms are distance preserving. In particular, the fundamental group $\pi_1(Z, z_0)$ acts on $\widehat{Z} = \pi(Z)_{z_0}$ by isometries. By Corollary 2.10, the space $\widehat{Z}$ contains indiscrete copies of the infinitesimal subgroups $\Pi(z)$ for each point z in the path component of z_0 in Z. This shows that if $\widehat{Z}$ is metrizable, then each such point is nonsingular. Conversely, Lemma 2.12(a) shows that if each point z is nonsingular, then the pseudo-metric d is actually a metric that defines the topology of $\widehat{Z}$. □

2.8 Fibres over metric models

Returning to the wild metric complexes described in Section 1.2, the join point z_0 of the Hawaiian earring HE is nonsingular but not regular; all remaining points are regular. The fibres of the endpoint projection $p_{HE} : \widehat{HE} \to HE$ are therefore all discrete, except for the fibre $p_{HE}^{-1}(z_0)$ over the join point, which is a totally disconnected and perfect metrized copy of the fundamental group $\pi_1(HE, z_0)$. The lifted pseudo-metric on the universal path space is a metric and the Hawaiian earring group acts by isometries on the universal path space $\widehat{HE}$ with orbit space HE. In Section 5, we give a complete description of $\widehat{HE}$ as a wild metric tree with the totally disconnected perfect fiber as vertex set.

For the harmonic archipelago HA (Example 1.1), compactness considerations imply that the fundamental group is non-trivial, since any loop deforms continuously over at most finitely many of the mountainous crescents bounded by the consecutive rings. The fundamental group $\pi_1(HA, z_0) \cong \Pi(\mathcal{P}_{HA})$ is the quotient group $\Omega(X, wt)/N_{\Omega(X,wt)}(R)$ of the free omega-group $\Omega(X, wt)$ on the weighted generator alphabet (X, wt), modulo the weighted normal closure $N(R, r\text{-}wt)$. Because all the relator weights are 1, the latter is just the ordinary normal closure $N_{\Omega(X,wt)}(R)$ of the relator set R in the free omega-group $\Omega(X, wt)$. The cosets that comprise this quotient are called **traces** of words for the weighted alphabet (X, wt), and two reduced words (ω, x) and (ω', x') represent the same *trace* if and only if they become similar when finitely many of the generators $x_1, x_2, \ldots, x_n, \ldots$ are identified with one another. Since each local subgroup $\Pi_Z^U(z_0)$ of the fundamental group at the union point z_0 carries the entire fundamental group, the infinitesimal subgroup Π_{z_0} at z_0 is $\pi_1(HA, z_0)$. Thus the point z_0 is singular in HA and the fibre $p_{HA}^{-1}(z_0)$ in the universal path space $\widehat{HA}$ is an uncountable indiscrete space. The lifted pseudo-metric on $\widehat{HA}$ degenerates within this fibre, assigning it diameter zero. All non-basepoints $z_0 \neq z \in HA$ are regular, so the fibres $p_{HA}^{-1}(z) \subset \widehat{HA}$ over these points are discrete.

The basepoint z_0 in the harmonic projective plane HP (Example 1.2) is nonsingular but not regular; all remaining points are regular so the universal path space $\widehat{HP}$ is metrizable. Nonsingularity of the basepoint follows from the fact that HP is the *weak join* [M-M86] of a countable infinite family $\{P_n : n \in \mathbf{Z}_+\}$ of copies of the projective plane having metric diameter $\operatorname{diam}(P_n) = 1/n$. In this weak join, each open neighborhood of the join point z_0 contains all but finitely many of the constituent planes P_n. Collapsing the planes P_n, $n > N \in \mathbf{Z}_+$, to z_0 determines a map $HP \to P(N) = \bigvee_{n=1}^{N} P_n$ onto the finite one-point union of N projective planes with the weak topology; these maps induce a homomorphism

$$\theta : \pi_1(HP, z_0) \to \overleftarrow{\lim_{N \in \mathbf{Z}_+}} \pi_1(P_N, z_0)$$

into the inverse limit of finite free products of cyclic groups of order two. The main result of [M-M86] implies that θ is a monomorphism, which in turn implies that the infinitesimal subgroup based at z_0 is trivial, so the point z_0 is nonsingular. By Corollary 2.10, the vertex set $p_{HP}^{-1}(z_0)$ is a totally disconnected and perfect metrized copy of the fundamental group $\pi_1(HP, z_0)$. This fundamental group is seen to be an omega-group in [B-S97(2)].

The basepoint z_0 in the projective telescope PT (Example 1.3) is nonsingular (since $\pi_1(PT, z_0)$ is shown to be an omega-group in [B-S97(2)]) but not regular; all remaining points are regular so the universal path space $\widehat{PT}$ is metrizable. The fibre $p_{PT}^{-1}(z_0)$ is a totally disconnected and perfect metrized copy of the fundamental group $\pi_1(PT, z_0)$.

3 Orbit Spaces

Consider a pointed space (Z, z_0) and any subgroup K of the fundamental group $\pi_1(Z, z_0)$. The action of the fundamental group $\pi_1(Z, z_0)$ on the universal path space $\widehat{Z}$ restricts to an action of K on $\widehat{Z}$: $[k] * [f] = [k][f] \in \widehat{Z}$ for each $[f] \in \widehat{Z}$ and $[k] \in K$.

The **orbit space** $_K\widehat{Z}$ is defined to be the quotient space of $\widehat{Z}$ modulo this action of K. This is the groupoid coset space

$$_K\widehat{Z} = \{K \cdot [f] \subseteq \pi(Z)_{z_0} : [f] \in \widehat{Z}\}. \tag{14}$$

3.1 Intermediate endpoint projection

The action of K permutes the local (groupoid) cosets $\Pi_Z^U \subseteq \pi(Z)$ for each $U \subseteq^{op} Z$:

$$[k] * ([f] \cdot \Pi_Z^U) = ([k][f]) \cdot \Pi_Z^U.$$

So the quotient topology on the orbit space $_K\widehat{Z}$ (14) has as base the family of all double (groupoid) cosets:

$$K \backslash \pi(Z)_{z_0} / \Sigma = \{K \cdot [h] \cdot \Pi_Z^U : [h] \in \widehat{Z} \text{ and } U \subseteq^{op} Z\}.$$

Because the action of K is p-equivariant, the endpoint projection $p : \widehat{Z} \to Z$ factors continuously into the quotient **orbit map**

$$q_K : \widehat{Z} \to {_K\widehat{Z}}, q_K([f]) = K \cdot [f], \tag{15}$$

and the **intermediate endpoint projection**

$$p_K : {_K\widehat{Z}} \to Z, p_K(K \cdot [f]) = p([f]). \tag{16}$$

These orbit spaces and their endpoint projections include all covering projections of Z.

Lemma 3.1 *Let* $[g] \in \widehat{Z}$ *and let* U *be an open neighborhood of* $z = g(1) \in Z$. *The restriction of the intermediate endpoint projection* $p_K : {}_K\widehat{Z} \to Z$ *to the open double groupoid coset* $K \cdot [g] \cdot \Pi_Z^U \subseteq^{op} {}_K\widehat{Z}$ *is injective if and only if the local subgroup* $\Pi_Z^U(z)$ *is contained in the groupoid conjugate* $[g]^{-1} \cdot K \cdot [g]$.

Proof: Given paths $\epsilon, \epsilon' : (I, 0) \to (U, z)$, set $[h] = [g][\epsilon]$ and $[h'] = [g][\epsilon']$. Then

$$p_K(K \cdot [h]) = p_K(K \cdot [h']) \Leftrightarrow \epsilon(1) = \epsilon'(1) \Leftrightarrow [g]^{-1}[h'][h]^{-1}[g] = [\epsilon'][\epsilon]^{-1} \in \Pi_Z^U(z).$$

So p_K restricts to an injection on the double groupoid coset $K \cdot [g] \cdot \Pi_Z^U$ if and only if $\Pi_Z^U(z) \subseteq [g]^{-1} \cdot K \cdot [g]$. □

Theorem 3.2 *Assume that* (Z, z_0) *is connected and locally path connected pointed space, and let* $K \leq \pi_1(Z, z_0)$.

(a) The intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ *is a covering projection if and only if each point* $z \in Z$ *has an open neighborhood* $U \subseteq^{op} Z$ *such that for each path* $g : (I, 0, 1) \to (Z, z_0, z)$ *from* z_0 *to* z *in* Z, *the local subgroup* $\Pi_Z^U(z)$ *is contained in the groupoid conjugate* $[g]^{-1} \cdot K \cdot [g]$.

(b) The universal endpoint projection $p : \widehat{Z} \to Z$ *is a covering projection if and only if each point* $z \in Z$ *has an open neighborhood* $U \subseteq^{op} Z$ *that is* relatively simply connected in Z *in the sense that the local subgroup* $\Pi_Z^U(z)$ *is trivial.*

(c) The universal endpoint projection $p : \widehat{Z} \to Z$ *is a homeomorphism if and only if* Z *is simply connected.*

Proof: The universal endpoint projection p is a continuous open surjection by Theorem 2.3. This implies that the intermediate endpoint projection p_K is also a continuous open surjection. The result (a) follows from Lemma 3.1, using the fact that the pre-image $p_K^{-1}(U) = q_K(p^{-1}(U))$ of an open subset U of Z under the intermediate projection is the union of open double groupoid cosets

$$p_K^{-1}(U) = \bigcup\{K \cdot [g] \cdot \Pi_Z^U : [g] \in \widehat{Z},\ g(1) \in U\},$$

and that two such double groupoid cosets are either disjoint or else they coincide. The result (b) follows from (a) by taking K to be the trivial subgroup of $\pi_1(Z, z_0)$. Since $\widehat{Z} = [z_0^*] \cdot \Pi_Z^Z$, Lemma 3.1 further shows that the universal endpoint projection p is injective (and hence a homeomorphism) if and only if Z is simply connected. □

Corollary 3.3 *The universal path space of the product of a pointed space* (Z, z_0) *with the pointed unit interval* $(I, 0)$ *is canonically homeomorphic to the product of the universal path space* $\widehat{Z}$ *with the unit interval:* $\widehat{Z \times I} \approx \widehat{Z} \times I$.

Proof: This follows directly from Theorem 3.2(c) and Theorem 2.7 since the unit interval I is simply connected. □

3.2 Coset space topologies

In order to describe the fibres of the intermediate endpoint projections, we first develop the abstract situation. Consider a group G with the subgroup topology determined by a neighborhood family of subgroups Σ. For any subgroup K of G, the set $K\backslash G = \{Kg : g \in G\}$ of right cosets $Kg = \{kg : k \in K\}$ of K in G acquires a quotient space topology from the subgroup topology on G under the natural projection $r : G \to K\backslash G$.

Each group $S \in \Sigma$ acts on $K\backslash G$ by right multiplication. Denote the S-orbit in $K\backslash G$ of the coset $Kg \in K\backslash G$ by $K \cdot g \cdot S$:

$$K \cdot g \cdot S = \{Kgs \in K\backslash G : s \in S\} \subseteq K\backslash G.$$

This orbit is just the image of the double coset $KgS = \{kgs : k \in K \text{ and } s \in S\} \subseteq G$ under the quotient map $r : G \to K\backslash G$. Since the double coset $KgS = \bigcup_{s\in S} Kgs \subseteq^{op} G$ is a union of cosets, it is the full pre-image under r of the orbit $K \cdot g \cdot S$: $r^{-1}(K \cdot g \cdot S) = KgS$. This implies that the orbit $K \cdot g \cdot S$ is open in the quotient space $K\backslash G$, and it is routine to verify that the set

$$K\backslash G/\Sigma = \{K \cdot g \cdot S : g \in G, S\Sigma\}$$

of all such orbits is a base for the quotient space topology on $K\backslash G$. The basic open sets $K \cdot g \cdot S$ are also closed since the orbits of the S-action partition the coset space $K\backslash G$.

Recall that the intersection $S_\Sigma = \bigcap_{S\in\Sigma} S$ is the infinitesimal subgroup for the neighborhood family Σ. For each $g \in G$, there is the *infinitesimal subset* $\bigcap_{S\in\Sigma} KgS$ of G, which is larger than the double coset KgS_Σ in general. The infinitesimal subset $\bigcap_{S\in\Sigma} KgS$ is the union of those cosets Kx that lie in the image $r(\bigcap_{S\in\Sigma} KgS) = \bigcap_{S\in\Sigma} K \cdot g \cdot S \subseteq K\backslash G$, so we may also think of the infinitesimal subset as living in the coset space $K\backslash G$. From this viewpoint, one checks that the infinitesimal subset $\bigcap_{S\in\Sigma} K \cdot g \cdot S$ is the closure of the singleton $\{Kg\}$ in $K\backslash G$. The infinitesimal subsets need not be singletons in $K\backslash G$, even when the infinitesimal subgroup S_Σ is trivial. This is illustrated in Section 6 with the free omega-group $G = \Omega(X, wt)$ with weighted basis (X, wt) and the ordinary free group $K = F(X)$ with (unweighted) basis X. See Example 6.8, where it is shown that the coset space $F(X)\backslash\Omega(X, wt)$ contains a nontrivial indiscrete subspace.

Theorem 3.4 *Let G have the subgroup topology determined by a neighborhood family Σ of subgroups of G. Let the coset space $K\backslash G$ have the quotient topology.*

(a) Each infinitesimal subset $\bigcap_{S\in\Sigma} K \cdot g \cdot S$ is a maximal indiscrete subspace of the coset space $K\backslash G$.

(b) Given $Kg \in K\backslash G$, the connected component of Kg in the coset space $K\backslash G$ is the infinitesimal subset $\bigcap_{S\in\Sigma} K \cdot g \cdot S$.

(c) The following are equivalent.

(i) The coset space $K\backslash G$ is Hausdorff;

(ii) The subgroup K is a **totally closed** *subgroup of G in the sense that every coset Kg is closed in the subgroup topology on G;*

(iii) The coset space $K\backslash G$ satisfies the T_0 separation property;

(iv) Each infinitesimal subset $\bigcap_{S\in\Sigma} KgS$ is a single coset: $\bigcap_{S\in\Sigma} KgS = Kg$;

(v) The coset space $K\backslash G$ is totally disconnected.

Proof: A non-empty open subset of $\bigcap_{S\in\Sigma} K\cdot g\cdot S$ contains a non-empty subset of the form $V=(K\cdot g'\cdot S')\cap(\bigcap_{S\in\Sigma} K\cdot g\cdot S)$, where $g'\in G$ and $S'\in\Sigma$. Thus $(K\cdot g'\cdot S')\cap K\cdot g\cdot S'$ is non-empty so $K\cdot g'\cdot S' = K\cdot g\cdot S'$ because the S'-orbits partition the coset space $K\backslash G$. Therefore

$$V = K\cdot g\cdot S'\cap(\bigcap_{S\in\Sigma} K\cdot g\cdot S) = \bigcap_{S\in\Sigma} K\cdot g\cdot S.$$

This proves that $\bigcap_{S\in\Sigma} K\cdot g\cdot S$ is an indiscrete, and hence connected, subspace of the coset space $K\backslash G$.

Suppose next that F is a subset of the coset space $K\backslash G$ that properly contains the infinitesimal subset $\bigcap_{S\in\Sigma} K\cdot g\cdot S$. There is a coset $Kg'\in F$ and a subgroup $S\in\Sigma$ such that $Kg'\notin K\cdot g\cdot S$. Then $F\cap(K\cdot g'\cdot S)$ is an open and closed subset of F that contains Kg' but not Kg. This implies that F is not connected, and hence that F is not indiscrete. We conclude that the infinitesimal subset $\bigcap_{S\in\Sigma} K\cdot g\cdot S$ is the connected component of the coset space $K\backslash G$ that contains Kg, as in (b), and that the infinitesimal subset is a maximal indiscrete subspace of $K\backslash G$, which proves (a).

As for (c), the condition (ii) is equivalent to the assertion that each singleton $\{Kg\}$, $g\in G$, is closed in the coset space, i.e. that the coset space satisfies the T_1 separation property. The implications $(i)\Rightarrow(ii)$ and $(ii)\Rightarrow(iii)$ follow immediately. If G satisfies the T_0 property, then so does its indiscrete subspace $\bigcap_{S\in\Sigma} K\cdot g\cdot S$, which therefore can contain at most one point. Thus $\bigcap_{S\in\Sigma} K\cdot g\cdot S=\{Kg\}$ and hence $\bigcap_{S\in\Sigma} KgS = Kg$, as in (iv). If $\bigcap_{S\in\Sigma} KgS = Kg$, then $\bigcap_{S\in\Sigma} K\cdot g\cdot S=\{Kg\}$, so the implication $(iv)\Rightarrow(v)$ follows from (b). The implication $(v)\Rightarrow(i)$ is a general fact for topological spaces. □

3.3 Fibre topologies

We now return to a pointed space (Z,z_0) and a subgroup $K\leq\pi_1(Z,z_0)$ with the associated intermediate endpoint projection $p_K: {}_K\widehat{Z}\to Z$ on the orbit space ${}_K\widehat{Z}$ of the universal path space $\widehat{Z}$ over Z. Given any path $f:(I,0,1)\to(Z,z_0,z)$ in Z from the basepoint z_0 to a point $z\in Z$, Lemma 2.8 provides a homeomorphism $\phi_{[f]}:\pi_1(Z,z)\to p^{-1}(z)$ that identifies the fibre $p^{-1}(z)$ of the endpoint projection $p:\widehat{Z}\to Z$ with the local subgroup topology on $\pi_1(Z,z)$ that is given by the neighborhood family $\Sigma(z)$ of local subgroups at z. Conjugation in the fundamental groupoid $\pi(Z)$ provides a subgroup $[f]^{-1}\cdot K\cdot[f]\leq\pi_1(Z,z)$ so there is the quotient coset space $([f]^{-1}\cdot K\cdot[f])\backslash\pi_1(Z,z)$.

Lemma 3.5 *When $[f] \in \pi(Z)_{z_0}$ and $f(1) = z \in Z$, there is a homeomorphism*

$$\psi_{[f]} : ([f]^{-1} \cdot K \cdot [f]) \backslash \pi_1(Z, z) \to p_K^{-1}(z)$$

given by $\psi_{[f]}([f]^{-1} \cdot K \cdot [f][g]) = K \cdot [f][g]$, $[g] \in \pi_1(Z, z)$, which identifies the fibre $p_K^{-1}(z) \subseteq {}_K\widehat{Z}$ of the intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ (16) as a coset space.

Proof: The composite map $q_K \circ \phi_{[f]} : \pi_1(Z, z) \to p^{-1}(z) \to p_K^{-1}(z)$ makes the same identifications as the quotient map $r : \pi_1(Z, z) \to ([f]^{-1} \cdot K \cdot [f]) \backslash \pi_1(Z, z)$ onto the coset space, so induces a continuous bijection $\psi_{[f]} : ([f]^{-1} \cdot K \cdot [f]) \backslash \pi_1(Z, z) \to p_K^{-1}(z)$. The image under $\psi_{[f]}$ of a basic open set $([f]^{-1} \cdot K \cdot [f]) \cdot [g] \cdot \Pi_Z^U(z)$, $[g] \in \pi_1(Z, z)$, $U \subseteq^{op} Z$, is the basic open set $(K \cdot [f][g] \cdot \Pi_Z^U) \cap p_K^{-1}(z)$ for the subspace topology on the fibre $p_K^{-1}(z) \subseteq {}_K\widehat{Z}$, so the map $\psi_{[f]}$ is a homeomorphism. □

The analysis of coset space topologies in Theorem 3.4 thus applies to the fibres of the endpoint projection.

Corollary 3.6 *For each point $z \in Z$, the connected components of the fibre $p_K^{-1}(z)$ are indiscrete. The fibre $p_K^{-1}(z)$ is totally disconnected if and only if it is Hausdorff, which happens if and only if, for some, hence all, paths f from z_0 to z in Z, the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$ is totally closed in the local subgroup topology on $\pi_1(Z, z)$.*

3.4 Separation and metrizability

The results of Section 2.7 generalize to the orbit spaces of the universal path space.

Theorem 3.7 *When (Z, z_0) is a pointed Hausdorff space and $K \leq \pi_1(Z, z_0)$, the orbit space ${}_K\widehat{Z}$ is Hausdorff if and only if, for each path class $[f] \in \pi(Z)_{z_0} = \widehat{Z}$, the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, f(1))$.*

Proof: When the orbit space ${}_K\widehat{Z}$ is Hausdorff, then so are all of the fibres $p_K^{-1}(z)$, $z \in Z$. By Corollary 3.6, if f is any path in Z beginning at z_0, then the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, f(1))$. For the converse, suppose that $K \cdot [f]$ and $K \cdot [g]$ are distinct elements in the orbit space ${}_K\widehat{Z}$. If $f(1) \neq g(1)$ and U, V are disjoint open neighborhoods of $f(1), g(1)$ in Z, then $K \cdot [f] \cdot \Pi_Z^U, K \cdot [g] \cdot \Pi_Z^V$ are disjoint open neighborhoods of $K[f], K[g]$ in ${}_K\widehat{Z}$. Suppose that $f(1) = g(1) = z \in Z$. Since $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, z)$, the intersection $\bigcap_{\Pi_Z^U(z) \in \Sigma(z)} ([f]^{-1} \cdot K \cdot [f]) \cdot [f(1)^*] \cdot \Pi_Z^U(z)$ is equal to the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$. It follows that the intersection $\bigcap_{\Pi_Z^U(z) \in \Sigma(z)} K \cdot [f] \cdot \Pi_Z^U(z)$ is equal to the coset $K \cdot [f]$, and hence is disjoint from the coset $K \cdot [g]$. Thus there is an open neighborhood U of z in Z such that $K \cdot [g]$ is disjoint from $K \cdot [f] \cdot \Pi_Z^U$ so $K \cdot [f] \cdot \Pi_Z^U, K \cdot [g] \cdot \Pi_Z^U$ are disjoint open neighborhoods of $K \cdot [f], K \cdot [g]$ in ${}_K\widehat{Z}$. □

Suppose that (Z, m, z_0) is a pointed metric space. For a subgroup $K \leq \pi_1(Z, z_0)$, we use the lifted pseudo-metric on the universal path space $\widehat{Z}$ (Section 2.7) to define a **lifted pseudo-metric** on the orbit space ${}_K\widehat{Z}$. Given $K \cdot [f], K \cdot [g] \in {}_K\widehat{Z}$, we set:

$$\begin{aligned} d(K \cdot [f], K \cdot [g]) &= \text{glb}\{d([k][f], [k'][g]) : [k], [k'] \in K\} \\ &= \text{glb}\{wt([f]^{-1}[k][g]) : [k] \in K\}. \end{aligned}$$

The pseudo-metric properties hold; the triangle inequality is checked using the multiplicative property of the path class weight function (12) this way:

$$\begin{aligned} d(K \cdot [f], K \cdot [g]) &+ d(K \cdot [g], K \cdot [h]) \\ &= \text{glb}\{wt([f]^{-1}[k'][g]) + wt([g]^{-1}[k''][h]) : [k'], [k''] \in K\} \\ &\geq \text{glb}\{wt([f]^{-1}[k][h]) : [k] \in K\} \\ &= d(K \cdot [f], K \cdot [h]) \end{aligned}$$

Theorem 3.8 *Let (Z, m, z_0) be a pointed metric space and let $K \leq \pi_1(Z, z_0)$.*

(a) The pseudo-metric d determines the orbit space topology on ${}_K\widehat{Z}$.

(b) The orbit space ${}_K\widehat{Z}$ is metrizable if and only if it is Hausdorff, which happens if and only if, for each path class $[f] \in \widehat{Z} = \pi(Z)_{z_0}$, the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, f(1))$, in which case the pseudo-metric d is a metric on ${}_K\widehat{Z}$.

Proof: As in Lemma 2.12(b), given $K \cdot [f] \in {}_K\widehat{Z}$ and $\epsilon > 0$, one checks that

$$B_\epsilon(K \cdot [f]) \subseteq K \cdot [f]\Pi_Z^{B_\epsilon(f(1))} \subseteq B_{2\epsilon}(K \cdot [f])$$

and the result (a) follows. If the orbit space ${}_K\widehat{Z}$ is metrizable, then it is Hausdorff so each groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$, $[f] \in \widehat{Z}$, is totally closed in $\pi_1(Z, f(1))$ by Theorem 3.7. If $K \cdot [f], K \cdot [g] \in {}_K\widehat{Z}$ and $d(K \cdot [f], K \cdot [g]) = 0$, then $f(1) = g(1)$ and for each $\epsilon > 0$ there are path classes $[k_\epsilon] \in K$ and $[\lambda_\epsilon] \in \Pi_Z^{B_\epsilon(f(1))}(f(1))$ such that $[f]^{-1}[k_\epsilon][g] = [\lambda_\epsilon]$. It follows that $[g] \in \bigcap_{\Pi_Z^U(f(1)) \in \Sigma_{f(1)}} K \cdot [f] \cdot \Pi_Z^U(f(1))$. Given that the groupoid conjugate $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, f(1))$, it follows that $\bigcap_{\Pi_Z^U(f(1)) \in \Sigma_{f(1)}} K \cdot [f] \cdot \Pi_Z^U(f(1)) = K \cdot [f]$ so $K \cdot [f] = K \cdot [g]$. This proves (b). □

4 Lifting Properties

The endpoint projections on the universal path space and its orbit spaces are not always covering projections in the traditional sense. Even so, these endpoint projections do admit path liftings. This section is devoted to an investigation of unique path lifting and its consequences. Subsequent sections will examine the availability of unique path lifting, even in the absence of evenly covered neighborhoods.

4.1 Path lifting

Let (Z, z_0) be a pointed space. Given any path $f : (I, 0) \to (Z, z_0)$ and any element $t \in I$, the **truncated path** $f_t : (I, 0) \to (Z, z_0)$ is given by $f_t(s) = f(st)$ for all $s \in I$. We define

$$\widehat{f} : (I, 0, 1) \to (\widehat{Z}, \widehat{z_0}, [f]), \widehat{f}(t) = [f_t], \tag{17}$$

where $\widehat{z_0}$ is the path homotopy class $[z_0^*]$ of the constant path z_0^* at z_0 in Z. Since $p \circ \widehat{f} = f$, we refer to $\widehat{f}$ (17) as the **standard lift** of the path f through the endpoint projection p. Given any subgroup $K \leq \pi_1(Z, z_0)$ of the fundamental group of the base, we also obtain the composite function $q_K \circ \widehat{f} : (I, 0, 1) \to ({}_K\widehat{Z}, K \cdot \widehat{z_0}, K[f])$, called the **standard lift** of the path f through the intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$.

Lemma 4.1 *The standard lift* $\widehat{f} : (I, 0, 1) \to (\widehat{Z}, \widehat{z_0}, [f])$ *is continuous.*

Proof: Let $\widehat{f}(t) = [f_t] \in [g] \cdot \Pi_Z^U$. Then $f(t) \in U$ and $[g] \cdot \Pi_Z^U = [f_t] \cdot \Pi_Z^U$. For an interval neighborhood V of $t \in I$ such that $f(V) \subseteq U$, we have $\widehat{f}(V) \subseteq [f_t] \cdot \Pi_Z^U$. □

Corollary 4.2 *Let* (Z, z_0) *be a pointed space and let* $K \leq \pi_1(Z, z_0)$. *The orbit space* ${}_K\widehat{Z}$ *is connected and locally path connected.*

Proof: Any element of the basic open set $[f] \cdot \Pi_Z^U \subset \widehat{Z}$ has the form $[f][\epsilon]$, for some path $\epsilon : (I, 0) \to (U, f(1))$ that begins at the terminal point $f(1) \in U \subseteq^{op} Z$ of $[f]$. The standard lift $\widehat{\epsilon} : (I, 0, 1) \to (\Pi_Z^U, \widehat{f(1)}, [\epsilon])$ in the universal path space $\pi(Z)_{f(1)}$ at $f(1) \in Z$ and the change of basepoint homeomorphism $T_{[f]} : \pi(Z)_{f(1)} \to \pi(Z)_{z_0}$ compose as a path $T_{[f]} \circ \widehat{\epsilon} : (I, 0, 1) \to (\Pi_Z^U, \widehat{f(1)}, [\epsilon]) \to ([f] \cdot \Pi_Z^U, [f], [f][\epsilon])$ from $[f]$ to $[f][\epsilon]$ in $[f] \cdot \Pi_Z^U$. This shows that the basic open set $[f] \cdot \Pi_Z^U \subseteq^{op} \widehat{Z} = \pi(Z)_{z_0}$ is path connected. Thus the universal path space $\widehat{Z}$ is locally path connected. Since $\widehat{Z} = [z_0^*] \cdot \Pi_Z^Z$, the universal path space is also path connected. Since the orbit map $q_K : \widehat{Z} \to {}_K\widehat{Z}$ is a continuous open surjection, the orbit space ${}_K\widehat{Z}$ is connected and locally path connected. □

Corollary 4.3 $K \subseteq p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0}))$. □

Proof: For each $[k] \in K$, the standard lift $q_K \circ \widehat{k}$ is a loop based at $K \cdot \widehat{z_0}$ in ${}_K\widehat{Z}$. □

Corollary 4.4 *When* $[f] \in \pi(Z)_{z_0} = \widehat{Z}$, *the following subgroups of* $\pi_1(Z, z_0)$ *are identical:*

$$[f] \cdot p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [f])) \cdot [f]^{-1} = p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})).$$

Proof: The standard lift of f through p_K is the path $q_K \circ \widehat{f}$ in ${}_K\widehat{Z}$ from $K \cdot \widehat{z_0}$ to $K \cdot [f]$. Under conjugation by the path class $[q_K \circ \widehat{f}]$ in the fundamental groupoid $\pi({}_K\widehat{Z})$, we have $[q_K \circ \widehat{f}] \cdot \pi_1({}_K\widehat{Z}, K \cdot [f]) \cdot [q_K \circ \widehat{f}]^{-1} = \pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})$. The result follows upon application of the homomorphism $p_{K\#}$. □

4.2 Unique path lifting

Consider any subgroup K of the fundamental group $\pi_1(Z, z_0)$, with its orbit space ${}_K\widehat{Z}$. The intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ has **unique path lifting** provided every continuous lift $F : (I, 0) \to ({}_K\widehat{Z}, K \cdot \widehat{z_0})$ of any path $p_K \circ F = f : (I, 0) \to (Z, z_0)$ has terminal point $F(1) = K \cdot [f]$. In other words, the lift F is required to have the same terminal point as the standard lift $q_K \circ \widehat{f}$ of f. The subgroup K is a **recoverable** subgroup of $\pi_1(Z, z_0)$ if $p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})) = K$. Here's the connection.

Theorem 4.5 *Suppose that (Z, z_0) is a connected and locally path connected pointed space and that $K \leq \pi_1(Z, z_0)$. The intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ has unique path lifting if and only if K is a recoverable subgroup of $\pi_1(Z, z_0)$.*

Proof: Suppose that $p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})) = K$. Let $F : (I, 0) \to ({}_K\widehat{Z}, K \cdot \widehat{z_0})$ be a path such that $p_K \circ F = f$. Set $F(1) = K \cdot [g]$. The standard lift $q_K \circ \widehat{g}$ is a path in the orbit space ${}_K\widehat{Z}$ from $K \cdot \widehat{z_0}$ to $K \cdot [g] = F(1)$. Now $[F]^{-1}[q_K \circ \widehat{g}]$ is a loop in ${}_K\widehat{Z}$ based at $K \cdot [g]$, so applying the endpoint projection p_K, we find that $[f]^{-1}[g] \in p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [g])) = [g]^{-1} \cdot p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})) \cdot [g]$, with the second equality coming from Corollary 4.4. We conclude that $[f]^{-1}[g] \in [g]^{-1} \cdot K \cdot [g]$, so $[g][f]^{-1} \in K$. Thus $F(1) = K \cdot [g] = K \cdot [f]$. We conclude that p_K has unique path lifting.

Now assume that p_K has unique path lifting. Corollary 4.3 provides that K is contained in $p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0}))$. For the reverse containment, a given loop F in ${}_K\widehat{Z}$ based at $\widehat{z_0}$ lies over the loop $f = p_K \circ F$ in Z based at z_0. By unique path lifting, $K \cdot \widehat{z_0} = F(1) = K \cdot [f]$, so $p_{K\#}([F]) = [f] \in K$. □

Our next result shows that for a recoverable subgroup K the intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ enjoys all of the crucial theoretical features of ordinary covering projections, except that the fibers of the endpoint projection need not be discrete. As noted in Corollaries 2.10 and 3.6, the absence of relatively simply connected open sets in the base space Z can produce fibres that are totally disconnected and perfect.

Theorem 4.6 *Let (Z, z_0) be a connected and locally path connected pointed space. Let K be a recoverable subgroup of the fundamental group $\pi_1(Z, z_0)$.*

(a) For each $z \in Z$, the fibre $p_K^{-1}(z)$ is a totally disconnected Hausdorff space.

(b) Let $F : Y \to Z$ be a continuous map on a connected and locally path connected pointed space Y. Given $y_0 \in Y$ and $K \cdot [f] \in p_K^{-1}(F(y_0))$, the following are equivalent.

(i) $F_\#(\pi_1(Y, y_0)) \subseteq p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [f]))$.

(ii) There is a unique continuous map $\widehat{F} : Y \to {}_K\widehat{Z}$ such that $p_K \circ \widehat{F} = F$ and $\widehat{F}(y_0) = K \cdot [f]$.

(c) *The intermediate endpoint projection* $p_K : {}_K\widehat{Z} \to Z$ *is a Hurewicz fibration: For any space* X *and homotopy* $H : X \times I \to Z$*, any partial lifting* $\widehat{H}_0 : X \times \{0\} \to {}_K\widehat{Z}$ *extends to a lift* $\widehat{H} : X \times I \to {}_K\widehat{Z}$ *of the homotopy* H *through* p_K*.*

(d) *The induced homomorphism* $p_{K\#} : \pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0}) \to \pi_1(Z, z_0)$ *of fundamental groups is a monomorphism with image* K*. Thus* $\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0}) \cong K$*.*

(e) *The group* $Aut(p_K)$ *of* p_K*-equivariant self-homeomorphisms of the orbit space* ${}_K\widehat{Z}$ *is canonically isomorphic to the quotient group* $N_{\pi_1(Z,z_0)}(K)/K$*, where* $N_{\pi_1(Z,z_0)}(K)$ *is the normalizer of* K *in* $\pi_1(Z, z_0)$*.*

Proof: Suppose that the fibre $p_K^{-1}(z)$ is not totally disconnected, so that it contains a nontrivial indiscrete component, as in Corollary 3.6. Select an indiscrete doubleton $\{K \cdot [f], K \cdot [g]\} \subseteq p_K^{-1}(z)$ and define $F : (I, 0) \to ({}_K\widehat{Z}, \widehat{z_0})$ to agree with the standard lift $q_K \circ f$ of f on the half-open interval $[0, 1)$ but set $F(1) = K \cdot [g]$. This alternate continuous lift F of f through the intermediate endpoint projection proves that p_K does not have unique path lifting. So the assertion (a) is established.

The implication $(ii) \Rightarrow (i)$ in (b) is trivial. For the implication $(i) \Rightarrow (ii)$, we first observe that the uniqueness of a continuous map $\widehat{F} : Y \to {}_K\widehat{Z}$ such that $p_K \circ \widehat{F} = F$ and $\widehat{F}(y_0) = K \cdot [f]$ follows from the path connectedness of Y and the unique path lifting property of the intermediate endpoint projection p_K. For the existence portion of $(i) \Rightarrow (ii)$, suppose that $F_\#(\pi_1(Y, y_0)) \subseteq p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [f]))$. Given any point $y \in Y$, select a path $\alpha : (I, 0, 1) \to (Y, y_0, y)$ in Y from y_0 to y. The path f in Z ends at $f(1) = p_K([K \cdot [f]]) = F(y_0)$, so the path $f * (F \circ \alpha)$ is defined in Z and begins at z_0. We may therefore define $F : Y \to Z$ by letting

$$\widehat{F}(y) = \widehat{F}(\alpha(1)) = K \cdot [f * (F \circ \alpha)]$$

for each path $\alpha : (I, 0, 1) \to (Y, y_0, y)$. To see that $\widehat{F}$ is well-defined, suppose that $\beta : (I, 0, 1) \to (Y, y_0, y)$. Now $[F \circ \alpha][F \circ \beta]^{-1} \in F_\#(\pi_1(Y, y_0)) \subseteq p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [f]))$, which by Lemma 4.3 is equal to $[f]^{-1} \cdot p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot [z_0^*])) \cdot [f]$. Therefore, by Theorem 4.5, we conclude that $[F \circ \alpha][F \circ \beta]^{-1} \in [f]^{-1} \cdot K \cdot [f]$. This implies that $K \cdot [f * (F \circ \alpha)] = K \cdot [f * (F \circ \beta)]$ so $\widehat{F}$ is well-defined. It is easy to see that $p_K \circ \widehat{F} = F$ and that $\widehat{F}(y_0) = K \cdot [f]$. To show that $\widehat{F}$ is continuous, let $y \in Y$ and let α be a path in Y from y_0 to y. A basic open neighborhood of $\widehat{F}(y) = K \cdot [f * (F \circ \alpha)]$ has the form $K \cdot [f * (F \circ \alpha)] \cdot \Pi_Z^U \subseteq^{op} {}_K\widehat{Z}$ where $F(y) \in U \subseteq^{op} Z$. Let W be a path connected open neighborhood of y contained in $F^{-1}(U)$. We show that $\widehat{F}(W) \subseteq K \cdot [f * (F \circ \alpha)] \cdot \Pi_Z^U$. Let $w \in W$ and let $\epsilon : (I, 0, 1) \to (W, y, w)$ be a path in W from y to w. The path product $\alpha * \epsilon$ is then a path in Y from y_0 to w so

$$\widehat{F}(w) = K \cdot [f * (F \circ (\alpha * \epsilon))] = K \cdot [f * (F \circ \alpha)][F \circ \epsilon] \in K \cdot [f * (F \circ \alpha)] \cdot \Pi_Z^U.$$

This shows that $\widehat{F}$ is continuous and completes the proof of the implication $(i) \Rightarrow (ii)$. The proofs of (c), (d) and (e) proceed along the lines of the analogous results from traditional covering space theory. The implication $(i) \Rightarrow (ii)$ in the lifting result (b) enables a straighforward proof of the homotopy lifting property as in (c). Lifts of path homotopies in Z serve as path homotopies in the orbit space because the fibres of the endpoint projection are totally disconnected by (a). The resulting path homotopy lifting property suffices to prove the monotonicity result (d). The identification of the automorphism group $Aut(p_K)$ in (e) is proved by applying the lifting result (b). The proof relies on the conjugacy result Corollary 4.4 and the fact (Corollary 4.2) that the orbit space is connected and locally path connected. □

In short, unique path lifting assures the essential features of covering space theory even in the absence of evenly covered neighborhoods. We will show that unique path lifting for p_K is intimately connected to the omega-group status of $\pi_1(Z, z_0)$ and its subgroup K.

Corollary 4.7 *The universal endpoint projection $p : \widehat{Z} \to Z$ for a connected and locally path connected space Z has unique path lifting if and only if $\widehat{Z}$ is simply connected, in which case the group $Aut(p)$ of p-equivariant self-homeomorphisms of $\widehat{Z}$ is canonically isomorphic to the fundamental group $\pi_1(Z, z_0)$.*

5 Universal Path Space for a Metric Bouquet

In this section, we analyze the universal path space for the metric bouquet of circles $Z = Z(X, wt)$ that realizes a weighted alphabet (X, wt). This analysis includes the traditional Hawaiian earring that realizes the countable weighted alphabet $\{(x_n^{\pm 1}, wt\frac{1}{n}) : n \geq 1\}$.

5.1 Word intervals, metrics, and assembly

For each reduced non-identity word (ω, x) for the weighted alphabet (X, wt), let $I_{(\omega,x)}$ be a copy of the closed unit interval and let the **word-path** $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$ spell the word (ω, x) by mapping the points of the order-type ω to the basepoint $z_0 \in Z$ and their complementary intervals $i \in I_\omega$ to the 1-spheres $S^1_{x^{\pm 1}}$ in Z according to the labeling function $x : I_\omega \to X$. Continuity in the Euclidean metric topology on $I_{(\omega,x)}$ follows from the weight restriction on the word (ω, x) and the construction of $Z = Z(X, wt)$. But we prefer to give each interval $I_{(\omega,x)}$ its own **word-metric** $m_{(\omega,x)}$:

$$m_{(\omega,x)}(p, q) = m\text{-diam}\, \rho_{(\omega,x)}([p, q]),$$

defined as the metric diameter in (Z, m) of the image of the closed subinterval $[p, q] \subseteq I_{(\omega,x)}$ under the word-path $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$. We have:

Lemma 5.1 *The word-metric $m_{(\omega,x)}$ on $I_{(\omega,x)}$ is topologically equivalent to the Euclidean metric d on I.* □

Proof: For two order-type points $p < q \in \omega$, the image $\rho_{(\omega,x)}([p,q])$ is the one-point union of the circles $S^1_{x(i)}$ of diameter $wt(x(i))$ for the complementary intervals $i \in I_\omega$ contained in $[p,q]$. Because the metric m on Z is inherited from the max metric on the weak product of the Euclidean planes $E^2_{x^{\pm1}}, x^{\pm1} \in X$, the m-diameter of that one-point union, $m_{(\omega,x)}(p,q) = m\text{-diam}\,(\rho_{(\omega,x)}([p,q])$, equals $\max\{wt(x(i)) : i \subset [p,q]\}$, the word-weight $w\text{-}wt(\omega_{pq}, x_{pq})$ of the subword (ω_{pq}, x_{pq}) of (ω, x). And the Euclidean distance $d(p,q)$ is the sum of the Euclidean lengths $l(i)$ for the complementary intervals $i \in I_\omega$ contained in $[p,q]$. Because of the weight restriction, $wt(x(i)) \to 0$ if and only if $l(i) \to 0$, we conclude that $p \to q$ in $(I_{(\omega,x)}, m_{(\omega,x)})$ if and only if $p \to q$ in (I,d). The argument for arbitrary points $p, q \in I_{(\omega,x)}$ builds on these ideas. □

Form the disjoint union $\bigcup\{I_{(\omega,x)} : 1_X \neq [(\omega,x)] \in \Omega(X,wt)\}$, with one word-metric interval $I_{(\omega,x)}$ for each (similarity class of a) reduced non-identity word (ω,x) for the weighted alphabet (X,wt). Whenever (λ,y) is an initial subword of a reduced word (ω,x), say on the subinterval $[0,r]$, there is an isometry that identifies the word-metric interval $(I_{(\lambda,y)}, m_{(\lambda,y)})$ with the initial portion $[0,r]$ of the word-metric interval $(I_{(\omega,x)}, m_{(\omega,x)})$ in a manner compatible with their word-paths to Z.

The **word interval assembly** $\Gamma = \Gamma(X,wt)$ is the quotient set of the union $\cup I_{(\omega,x)}$ obtained by the isometric identification of $(I_{(\lambda,y)}, m_{(\lambda,y)})$ with the initial subinterval of $(I_{(\omega,x)}, m_{(\omega,x)})$ for each initial subword (λ,y) of a reduced word (ω,x) in (X,wt). In the assembly Γ, any two word-metric intervals $I_{(\omega,x)}$ and $I_{(\omega',x')}$ intersect in the word-metric interval $I_{(\lambda,y)}$ that carries the maximum common initial subword (λ,y) for the non-identity reduced order-type words (ω,x) and (ω',x'). The later exists:

Lemma 5.2 *Any two reduced non-identity order-type words have a maximum common initial subword, uniquely determined up to associativity.*

Proof: Given reduced non-identity order-type words (ω,x) and (ω',x'), let p^* be the least upper bound of the set J of all $p \in \omega$ for which the initial subword (ω_p, x_p) is (associated to) an initial subword of (ω',x'). Now $p^* \in J$ and (ω_{p^*}, x_{p^*}) is the maximum common initial subword of (ω,x) and (ω',x'). One can argue this from a nondecreasing sequence $p_n \in J$ that converges to p^*, by stacking side-by-side a sequence of associativity squares $(\omega_{p_n p_{n+1}}, x_{p_n p_{n+1}}) \overset{a}{\sim} (\omega'_{q_n q_{n+1}}, x'_{q_n q_{n+1}})$ for consecutive subwords bounded by the p_n. □

5.2 Omega-group action and word-metric

Each non-identity element of $\Omega = \Omega(X,wt)$ is represented by a reduced order-type word (ω,x), uniquely determined up to association of order-type words. Since an interval $I_{(\lambda,y)}$ is wholely identified only with an initial portion of another interval $I_{(\omega,x)}$ in Γ, the terminal point $1_{(\lambda,y)}$ equals a terminal point $1_{(\omega,x)}$ only if $(\lambda,y) = (\omega,x)$. In this way we may identify the free omega-group Ω with the set of end-points $0, 1_{(\omega,x)} \in I_{(\omega,x)}$ in Γ of the word-metric intervals indexed by the reduced non-identity words (ω, x) for (X, wt).

The action of the group $\Omega(X, wt)$ on itself by left multiplication extends to an action of $\Omega(X, wt)$ on the set $\Gamma = \Gamma(X, wt)$. To see this action, recall that multiplication in $\Omega(X, wt)$ is concatenation of reduced words, followed by reduction. For example, consider any two reduced words (ω, x) and $(\underline{\omega}, \underline{x})$. Let (λ, y) be the maximal common initial subword of the words $(\omega, x)^{-1}$ and $(\underline{\omega}, \underline{x})$, say $(\omega, x) = (\omega', x') \cdot (\lambda, y)^{-1}$ and $(\underline{\omega}, \underline{x}) = (\lambda, y) \cdot (\underline{\omega}', \underline{x}')$, then the product $(\omega, x) \cdot (\underline{\omega}, \underline{x})$ is the reduced word $(\omega', x') \cdot (\underline{\omega}', \underline{x}')$.

Then the two non-trivial reduced words $(\omega, x) = (\omega', x') \cdot (\lambda, y)^{-1}$ and $(\omega', x') \cdot (\underline{\omega}', \underline{x}')$ share the maximal common initial subword (ω', x'). In the assembly Γ, the two word-metric intervals $I_{(\omega,x)} = I_{(\omega',x')\cdot(\lambda,y)^{-1}}$ and $I_{(\omega',x')\cdot(\underline{\omega}',\underline{x}')}$ are identified along their initial subintervals that are isometric copies of the word-metric interval $I_{(\omega',x')}$. They form a topological Y whose arms constitute an isometric copy of the word-metric interval $I_{(\underline{\omega},\underline{x})} = I_{(\lambda,y)\cdot(\underline{\omega}',\underline{x}')}$. Under the *action* of the group $\Omega(X, wt)$ of words for (X, wt) on the word interval assembly Γ, the word (ω, x) carries the word-metric interval $I_{(\underline{\omega},\underline{x})}$ isometrically to its copy $(\omega, x) * I_{(\underline{\omega},\underline{x})}$ comprised of the arms formed by the two word-metric intervals $I_{(\omega,x)} = I_{(\omega',x')\cdot(\lambda,y)^{-1}}$ and $I_{(\omega',x')\cdot(\underline{\omega}',\underline{x}')}$.

Give the assembly $\Gamma = \Gamma(X, wt)$ the **word-metric**, namely, the unique metric m extending the word-metrics $m_{(\omega,x)}$ on the intervals $I_{(\omega,x)}$ and making the action of each reduced word $(\omega, x) \in \Omega(X, wt)$ an isometry on (Γ, m). To express this another way, distance between points of $I_{(\omega,x)} = I_{(\lambda,y)\cdot(\omega',x')}$ and $I_{(\underline{\omega},\underline{x})} = I_{(\lambda,y)\cdot(\underline{\omega}',\underline{x}')}$, which form a topological Y whose arms constitute an isometric copy of the word-metric interval $I_{(\underline{\omega},\underline{x})} = I_{(\lambda,y)\cdot(\underline{\omega}',\underline{x}')}$, are calculated using which ever metric applies: $m_{(\omega,x)}$, $m_{(\underline{\omega},\underline{x})}$, or $m_{(\omega',x')^{-1}\cdot(\underline{\omega}',\underline{x}')}$.

The free omega-group Ω, which has its own word-weight-derived metric, becomes a metric subspace of the assembly $\Gamma = \Gamma(X, wt)$: The distance in (Γ, m) from $(\omega, x) \in \Omega$ to the initial point $0 = 1_X$ equals its *word-weight*:

$$w\text{-}wt((\omega, x)) = max\{wt(x(i)) : i \in I_\omega\},$$

and the word-metric distance in Γ between $(\omega, x), (\underline{\omega}, \underline{x}) \in \Omega$ equals the word-weight of (the free reduction of) their difference:

$$m((\omega, x), (\underline{\omega}, \underline{x})) = w\text{-}wt((\omega, x)^{-1} \cdot (\underline{\omega}, \underline{x})).$$

5.3 Metric Cayley graph

Let 0 denote the initial point shared by all the intervals $I_{(\omega,x)}$ in Γ. The monosyllabic words (τ, x) and (τ, x^{-1}) for the non-identity element pairs $x^{\pm 1} \neq 1_X$ for the weighted alphabet (X, wt) give intervals I_x and $I_{x^{-1}}$ in Γ that initiate at 0. As the elements $x \in X$ can have weights $wt(x) \to 0$, the terminal end-points $1_{(\omega,x)} \in I_{(\omega,x)}$ of these intervals can accumulate in (Γ, m) at the initial point 0. But, these monosyllabic intervals I_x and $I_{x^{-1}}$, hereafter, called $x^{\pm 1}$**-edges** of Γ, are not the only routes out of the initial point z_0, because the monosyllabic words (τ, x) and (τ, x^{-1}) are not the only way to start an order-type word (ω, x) for the weighted alphabet (X, wt). When the weighted alphabet (X, wt) has elements having weights $wt(x) \to 0$, order-types ω in which the initial point 0 is a limit point of ω make possible words (ω, x) having no initial segment equal to a monosyllabic word (τ, x) or (τ, x^{-1}). The interval $I_{(\omega,x)}$ of such a word has no initial segment that is identified with one of the $x^{\pm 1}$-edges $I_{x^{\pm 1}}$.

Because each partition point $p \in \omega$ determines an initial subinterval $[0, p] \subseteq I_{(\omega,x)}$ and because the initial subword (λ, y) of a reduced word (ω, x) supported by that initial subinterval is also reduced, the set Ω is actually identified with the entire set of partition points of the word-metric intervals $I_{(\omega,x)}$ that constitute Γ. View Ω as the set of **vertices** of Γ. Each non-partition point of Γ belongs to a unique translation $(\lambda, y) * I_x$ or $(\lambda, y) * I_{x^{-1}}$ of an x- or x^{-1}-edge by some reduced word (λ, y) for which the product word $(\lambda, y) \cdot (\tau, x)$ or $(\lambda, y) \cdot (\tau, x^{-1})$ is reduced. View these translates of the $x^{\pm 1}$-edges as the **edges** of Γ.

Thus, $\Gamma = \Gamma(X, wt)$, with its word-metric m, can be viewed as the **metric Cayley graph** of the free omega-group $\Omega(X, wt)$ *with respect to the weighted basis* (X, wt). (This is a weighted analog of the usual Cayley graph of a free group with respect to its free basis.) As indicated above, the graph Γ contains a vertex for each group element of $\Omega(X, wt)$, an x-edge initiating and terminating at each vertex, but also involves the limiting behavior of order-types of those edges at each of the vertices, whenever the weighted alphabet (X, wt) contains elements having vanishing weights. By the construction, we have:

Lemma 5.3 *The free omega-group $\Omega(X, wt)$ acts freely and without inversion by isometries on its metric Cayley graph $\Gamma(X, wt)$ with respect to the weighted basis (X, wt).* □

Theorem 5.4 *For any weighted alphabet (X, wt), the metric Cayley graph $\Gamma(X, wt)$ of the free omega-group $\Omega(X, wt)$ with respect to its weighted basis (X, wt) is contractible.*

Proof: We construct a contraction $H : constant \simeq 1_\Gamma : \Gamma \times I \to \Gamma$ of the metric graph (Γ, m) as the assembly of contractions $H_{(\omega,x)} : constant \simeq 1_{I_{(\omega,x)}} : I_{(\omega,x)} \times I \to I_{(\omega,x)}$, one for each word-metric interval $I_{(\omega,x)}$ associated with a non-trivial reduced word (ω, x) for the weighted basis (X, wt). Each contraction $H_{(\omega,x)}$ folds the square $I_{(\omega,x)} \times I$ onto $I_{(\omega,x)}$ by collapsing its left-hand edge $\{0\} \times I$ and its bottom edge $I_{(\omega,x)} \times \{0\}$ to the initial vertex $0 \in I_{(\omega,x)}$. More specifically, each partition point $p \in \omega \subseteq I_{(\omega,x)}$ cuts off an initial subinterval $[0, p]$ that supports an initial subword (ω_p, x_p) of the word (ω, x). The contraction $H_{(\omega,x)}$ maps the vertical slice $\{r\} \times I$ according to the isometry of $I_{(\omega_p,x_p)}$ with the initial subinterval $[0, p]$ of $I_{(\omega,x)}$. Each complementary subinterval $i = (a_i, b_i)$ of the order-type ω supports the monosyllabic subword $a_i < x(i) < b_i$ which is the terminal factor of the initial subword $(\omega_{b_i}, x_{b_i}) = (\omega_{a_i}, x_{a_i}) \cdot x(i)$. There is a retraction $[a_i, b_i] \times I \to \{a_i\} \times I \cup [a_i, b_i] \times \{1\}$ that carries the metrized right-hand side $\{b_i\} \times I_{(\omega_{b_i}, x_{b_i})}$ of $[a_i, b_i] \times I$ isometrically onto the metrized union $\{a_i\} \times I_{(\omega_{a_i}, x_{a_i})} \cup [a_i, b_i] \times \{1\}$ of the left-hand side and top of $[a_i, b_i] \times I$. The contraction $H_{(\omega,x)}$ uses this retraction to extend the isometric embeddings

$$\{a_i\} \times I = I_{(\omega_{a_i}, x_{a_i})} \to [0, a_i] \subseteq I_{(\omega,x)} \text{ and } \{b_i\} \times I = I_{(\omega_{b_i}, x_{b_i})} \to [0, b_i] \subseteq I_{(\omega,x)}$$

over the vertical section $[a_i, b_i] \times I$ to a map $[a_i, b_i] \times I \to [0, b_i] \subseteq I_{(\omega,x)}$. Together the extensions define the contraction $H_{(\omega,x)} : I_{(\omega,x)} \times I \to I_{(\omega,x)}$. Each initial subword (ω_p, x_p) of (ω, x) determines its own contraction $H_{(\omega_p,x_p)} : I_{(\omega_p,x_p)} \times I \to I_{(\omega_p,x_p)}$, which is compatible with $H_{(\omega,x)}$ under the isometric identification of $I_{(\omega_p,x_p)}$ with the initial subinterval of $I_{(\omega,x)}$. So the individual contractions $H_{(\omega,x)}$ assemble into a contraction $H : \Gamma \times I \to \Gamma$. All appropriate metric distances in $\Gamma \times I$ are reduced by H so it is a metric contraction. □

By this theorem, the metric graph $\Gamma = \Gamma(X, wt)$ may be called the **metric Cayley tree** for the free omega-group $\Omega = \Omega(X, wt)$ with respect to its weighted basis (X, wt). Recall that the ordinary free group $F = F(X)$ with unweighted basis X is recognized by the fact that its Cayley graph with respect to X is a tree. So the action of Ω on the metric Cayley tree Γ justifies calling Ω a *free* omega-group. It also has the universal property of freeness in the category of weighted groups ([S97, Theorem 2.4]).

Let $p : \Gamma \to Z$ be the **projection map** induced by the paths $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$ that spell the reduced non-identity words $(\omega, x) \in \Omega(X, wt)$. The projection map is continuous on the metric space (Γ, m), because the metric m is defined to equivariantly extend the word-metrics $m_{(\omega,x)}$ on the intervals $I_{(\omega,x)}$, and each of those is defined using the diameters in Z of images of subintervals in $I_{(\omega,x)}$ under the word-path $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$. The fibre $p^{-1}(z_0) \subseteq \Gamma$ of the projection map $p : \Gamma \to Z$ is the set $\Omega = \Omega(X, wt)$ of vertices of the metric Cayley graph $\Gamma = \Gamma(X, wt)$.

5.4 Universal path space

As in Section 2, let $\widehat{Z}$ denote the universal path space for the metric 1-point union $Z = Z(X, wt)$. Each non-union point of Z has a contractible neighborhood so is a regular point. The union point z_0 of Z is not regular but is nonsingular: The infinitesimal subgroup at z_0 is trivial because the only reduced word (ω, x) that is supported by every open ball neighborhood of z_0 contains only trivial labels so is the trivial element of the free omega-group $\Omega = \Omega(X, wt)$. By Theorem 2.13, $\widehat{Z}$ acquires the lifted metric d and the fiber Ω of the endpoint projection $p : \widehat{Z} \to Z$ is a totally disconnected perfect metric subspace.

Each word-path $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$ has its standard lift $\widehat{\rho_{(\omega,x)}} : I_{(\omega,x)} \to \widehat{Z}$ into the universal path space $\widehat{Z}$ for Z, i.e., $\widehat{\rho_{(\omega,x)}}(t) = [\rho_{(\omega,x)t}] \in \widehat{Z}$. These standard lifts assemble into a function

$$\widehat{\rho} : \Gamma \to \widehat{Z}$$

that factors $p : \Gamma \to Z$, i.e., $p = p \circ \widehat{\rho} : \Gamma \to \widehat{Z} \to Z$. Indeed, for $t \in I_{(\omega,x)}$ one has

$$p \circ \widehat{\rho}(t) = p(\widehat{\rho_{(\omega,x)}}(t)) = p([\rho_{(\omega,x)t}]) = \rho_{(\omega,x)t}(1) = \rho_{(\omega,x)}(t) = p(t).$$

Theorem 5.5 *The function $\widehat{\rho} : \Gamma \to \widehat{Z}$ is an isometry from the word-metric m on the metric Cayley tree Γ to the lifted metric d on the universal path space $\widehat{Z} = \pi(Z)_{z_0}$.*

Proof: A homotopy analysis in Z ([B-S97(1)]) shows that every loop class $[g] \in \pi_1(Z, z_0)$ arises as the class of a word-path $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$ for a unique reduced word (ω, x) for $\Omega(X, wt)$. Also every non-loop $g : (I, 0) \to (Z, z_0)$ can be extended (by a completion of the circuit of the circle on which it terminates) to a loop that represents a path-class $[\rho_{(\omega,x)}] \in \pi_1(Z, z_0)$ having an initial portion $\rho_{(\omega,x)t} : I \to Z$ path homotopic to g. So the function $\widehat{\rho} : \Gamma \to \widehat{Z}$ is surjective.

The function $\widehat{\rho} : \Gamma \to \widehat{Z}$ is also injective: A coincidence of $\widehat{\rho_{(\omega,x)}}(t) = [\rho_{(\omega,x)t}]$ and $\widehat{\rho_{(\omega',x')}}(t') = [\rho_{(\omega',x')t'}]$ means that there is a path-homotopy $\rho_{(\omega,x)t} \simeq \rho_{(\omega',x')t'}$, in particular, these paths share the terminal point $\rho_{(\omega,x)}(t) = \rho_{(\omega',x')}(t')$ in Z. When this terminal point is the join point in Z, then $t \in \omega$ and $t' \in \omega'$ and the path-homotopy implies that the subword of (ω, x) on $[0, t]$ is associated to the subword of (ω', x') on $[0, t']$. In other words, the two words (ω, x) and (ω, x) have a common initial subword (λ, y), so that the subintervals $[0, t] \subseteq I_{(\omega,x)}$ and $[0, t'] \subseteq I_{(\omega',x')}$, are isometrically identified with the word-metric interval $I_{(\lambda,y)}$ in the assembly Γ. In particular, $t \in I_{(\omega,x)}$ and $t' \in I_{(\omega',x')}$ are identified in Γ. When the terminal point $\rho_{(\omega,x)}(t) = \rho_{(\omega',x')}(t')$ is not the join point in Z, the path-homotopy $\rho_{(\omega,x)t} \simeq \rho_{(\omega',x')t'}$ implies that the two paths $\rho_{(\omega,x)}$ and $\rho_{(\omega',x')}$ are wrapping around the same 1-sphere S^1_x in the same direction at the same point at times t and t', respectively. (Else, the path-homotopy provides a null-homotopy of the loop $\rho_{(\omega,x)t} \cdot (\rho_{(\omega',x')t'})^{-1}$ which spells a reduced non-identity word.) So the points t and t' belong to subintervals $i \in \mathcal{I}_\omega$ and $i' \in \mathcal{I}_{\omega'}$ having the same label $x(i) = x = x'(i')$, and the given path-homotopy extends over the remainder of these subintervals. So the initial subwords of (ω, x) and (ω', x') that are determined by the first partition points $t < \bar{t} \in \omega$ and $t' < \bar{t}' \in \omega'$ are associated. This reduces the coincidence question to the previous case.

To show that $\widehat{\rho} : (\Gamma, m) \to (\widehat{Z}, d)$ is an isometry, consider two points $r < s$ in $I_{(\omega,x)}$. Their distance in Γ is given by the word-metric: $m_{(\omega,x)}(r, s) = \text{diam } \rho_{(\omega,x)}([r, s])$. Their images under $\widehat{\rho}$ have distance $d([\rho_{(\omega,x)r}], [\rho_{(\omega,x)s}])$, which is evaluated as the greatest lower bound of the diameters diam $h([0, 1])$ for paths h representing the path class difference

$$[\rho_{(\omega,x)r}]^{-1} \cdot [\rho_{(\omega,x)s}] = [\rho_{(\omega,x)[r,s]}].$$

Because the word (ω, x) is reduced, this distance is achieved by diam $\rho_{(\omega,x)}([r, s])$, i.e., the distance $m_{(\omega,x)}(r, s)$. So $\widehat{\rho} : \Gamma \to \widehat{Z}$ is an isometry on each interval $(I_{(\omega,x)}, m_{(\omega,x)})$. Since $\widehat{\rho}$ is also equivariant with respect to the actions of $\Omega(X, wt)$ on Γ and $\widehat{Z}$, and since Γ is given the unique metric extending the word-metrics $m_{(\omega,x)}$ and making the action isometric, it follows that $\widehat{\rho} : (\Gamma, m) \to (\widehat{Z}, d)$ is an isometry. □

Because the universal path space $\widehat{Z}$ of the metric bouquet $Z = Z(X, wt)$ is the contractible metric tree $\Gamma = \Gamma(X, wt)$ by Theorems 5.4 and 5.5, then Theorems 4.5 and 4.6 yield:

Corollary 5.6 *The endpoint projection $p : \widehat{Z} \to Z$ for the metric bouquet $Z = Z(X, wt)$ has unique path lifting, so the free omega-group $\Omega(X, wt)$ acts as the full group $Aut(p)$ of p-equivariant cellular isometries of the Cayley graph.* □

The endpoint projection for the metric bouquet exemplifies a connection, established in Theorem 7.2, between simple connectivity of the universal path space $\widehat{Z}$ and the omega-group structure of the fundamental group $\pi_1(Z, z_0)$ of the base space Z.

6 Orbit Spaces of Wild Metric Trees

The universal path space $\widehat{Z}$ of the metric bouquet $Z = Z(X, wt)$ that models a weighted alphabet (X, wt) has been identified as the metric Cayley tree $\Gamma = \Gamma(X, wt)$ of the free omega-group $\Omega = \Omega(X, wt)$. In this section we examine orbit graphs ${}_K\widehat{Z} \equiv {}_K\Gamma$ corresponding to subgroups K of the word-weighted fundamental group $\Omega \cong \pi_1(Z, z_0)$.

6.1 Totally closed omega-subgroups of Ω

Fix a subgroup $K \leq \Omega \cong \pi_1(Z, z_0)$. Although the endpoint projection $p : \widehat{Z} \to Z$ for the metric bouquet $Z = Z(X, wt)$ has unique path lifting by Corollary 5.6, the intermediate endpoint projections $p_K : {}_K\widehat{Z} \to Z$ need not have unique path lifting and the orbit space ${}_K\widehat{Z}$ need not be metrizable. We begin with the metrizability question.

Theorem 6.1 *The orbit graph ${}_K\widehat{Z} \equiv {}_K\Gamma$ for a subgroup $K \leq \Omega$ is metrizable if and only if the subgroup K is totally closed in Ω.*

Proof: Theorem 3.7 shows that K is totally closed in Ω when ${}_K\widehat{Z}$ is metrizable. For the converse, we use Theorem 3.8. For this, consider a path class $[f] \in \pi(Z)_{z_0} = \widehat{Z}$ and the associated groupoid conjugate $[f]^{-1} \cdot K \cdot [f] \leq \pi_1(Z, f(1))$. If $f(1) \neq z_0$, then $f(1)$ is a regular point of Z, so that $[f]^{-1} \cdot K \cdot [f]$ is a totally closed subgroup of the discrete group $\pi_1(Z, f(1))$. When $f(1) = z_0$ and K is totally closed, each coset $K \cdot [f][g]$, $[g] \in \Omega$, is closed in Ω. It follows that $([f]^{-1} \cdot K \cdot [f]) \cdot [g]$ is closed in Ω since left translation by $[f]^{-1}$ is a self-homeomorphism of Ω. This shows that $[f]^{-1} \cdot K \cdot [f]$ is totally closed in $\pi_1(Z, f(1))$, so the orbit space ${}_K\widehat{Z}$ is metrizable. □

The unique path lifting problem is much more subtle. Consider liftings through the intermediate endpoint projection $p_K : {}_K\widehat{Z} \to Z$ of the word-path $\rho_{(\omega,x)} : I_{(\omega,x)} \to Z$ associated to a reduced order-type word (ω, x) for the weighted alphabet (X, wt). The restriction of the standard lift $q_K \circ \widehat{\rho_{(\omega,x)}} : (I_{(\omega,x)}, 0) \to ({}_K\widehat{Z}, K \cdot 1_\Omega)$ to the order-type $\omega \subseteq I_{(\omega,x)}$ is a continuous function

$$h : \omega \to K\backslash\Omega = p_K^{-1}(z_0)$$

satisfying the initial condition $h(0) = K \cdot 1_\Omega$ and the equations

$$h(a_i) \cdot x(i) = h(b_i)$$

in the coset space $K\backslash\Omega$, for all $i = (a_i, b_i) \in \mathcal{I}_\omega$. This is due to the fact that $h(p) = K \cdot (\omega_p, x_p)$ for each $p \in \omega$ and the word product calculation

$$(\omega_{a_i}, x_{a_i}) \cdot x(i) = (\omega_{b_i}, x_{b_i})$$

that holds for each complementary interval $i = (a_i, b_i)$ of the order-type ω. For a finite word (ω, x), the system of equations is a recursive finite sequence of equations that is solved trivially and uniquely. But for a word based on an arbitrary order-type ω, it is a serious challenge to establish uniqueness of word-path lifting to the coset space ${}_K\widehat{Z}$. The next lemma shows how to construct possible alternative lifts of word-paths.

Lemma 6.2 *These paths exist in the metric bouquet Z and the orbit graph ${}_K\widehat{Z}$:*

(a) For any $u \in \Omega \cong \pi_1(Z, z_0)$, there is a word path $\rho_{(\omega,x)} : (I, 0, 1) \to (Z, z_0, z_0)$ with $u = [\rho_{(\omega,x)}]$ and m-diam $(\rho_{(\omega,x)})(I)) = w\text{-}wt(u)$.

(b) Given an order-type word (ω, g) for the weighted set $(\Omega(X, wt), w\text{-}wt)$, a continuous function $h : \omega \to K\backslash\Omega = p_K^{-1}(z_0)$ such that $h(0) = K \cdot 1$ and $h(b_i) = h(a_i) \cdot g(i)$ for each $i \in \mathcal{I}_\omega$, and continuous loops $\rho(i) : (I, 0, 1) \to (Z, z_0, z_0)$, $i \in \mathcal{I}_\omega$ such that $g(i) = [\rho(i)]$ and $\text{diam}\,(\rho(i)(I)) \to 0$ as $l(i) \to 0$, then there is a continuous path $\rho : (I, 0) \to ({}_K\widehat{Z}, K \cdot 1)$ such that $\rho|_\omega = h$ and the loop $p_K \circ \rho : (I, 0, 1) \to (Z, z_0, z_0)$ restricts to each $i = (a_i, b_i) \in \mathcal{I}_\omega$ as a reparametrized version of the loop $\rho(i)$.

Proof: Each element $u \in \Omega$ is represented by a reduced order-type word (ω, x) in (X, wt), and the word-path $\rho = \rho_{(\omega,x)}$ is a homotopy representative for $[\rho_{(\omega,x)}] = u$ whose image $\rho(I)$ is the one-point union of the circles $S^1_{x(i)}$ of diameter $wt(x(i))$ for the complementary intervals $i \in I_\omega$. Because the metric m on Z is inherited from the max metric on the weak product of the Euclidean planes $E^2_{x^{\pm 1}}, x^{\pm 1} \in X$, the m-diameter of that one-point union equals the word-weight $w\text{-}wt(\omega, x) = \max\{wt(x(i)) : i \in \mathcal{I}_\omega\}$, which proves (a).

Given the word (ω, g) and paths $\rho(i)$ as in (b), select elements $u_p \in \Omega$, $p \in \omega$, representing the cosets $K \cdot u_p = h(p) \in K\backslash\Omega$. For each $i = (a_i, b_i) \in \mathcal{I}_\omega$, the action of $u_{a_i} \in \Omega$ on $\widehat{Z}$ transports the standard lift of $\widehat{\rho(i)} : (I, 0) \to ({}_K\widehat{Z}, [z_0^*])$ to produce the path

$$\alpha(i) = q_K \circ u_{a_i} \circ \widehat{\rho(i)} : (I, 0, 1) \to ({}_K\widehat{Z}, K \cdot u_{a_i}, K \cdot u_{a_i} \cdot g(i)).$$

We now define $\rho : (I, 0) \to ({}_K\widehat{Z}, K \cdot 1)$ so that $\rho|_\omega = h$ so that ρ restricts to the interval $[a_i, b_i]$ to be a reparametrized version of the path $\alpha(i)$. The function ρ is well-defined since $K \cdot u_{a_i} = h(a_i)$ and $K \cdot u_{a_i} \cdot g(i) = h(a_i) \cdot g(i) = h(b_i)$. The function ρ is continuous since the diameters $\text{diam}\,(\alpha(i)(I)) \leq \text{diam}\,(\rho(i)(I))$ tend to zero as the lengths $l(i) = b_i - a_i$

tend to zero. The restriction of $p_K \circ \rho$ to the interval $[a_i, b_i]$ is the reparametrized loop $\rho(i)$ since $p = p_K \circ q_K$ and the action of Ω on $\widehat{Z}$ is p-equivariant. $\square$

The following result gives the first concrete indication of the essential role of omega-group structures in universal path space theory, using this terminology. A subgroup K of an omega-group (G, wt) is an **omega-subgroup** of (G, wt) if K contains the evaluation $\omega\langle k\rangle$ of each order-type word for the weighted set (K, wt). This connection is made precise in Theorems 6.12 and 7.2 below.

Theorem 6.3 *For a subgroup $K \leq \Omega$ and its orbit graph ${}_K\widehat{Z} \equiv {}_K\Gamma$, the image subgroup $p_{K\#}(\pi_1({}_K\widehat{Z}, K\cdot 1)) \leq \pi_1(Z, z_0)$ contains the evaluation $\omega\langle k\rangle$ of each order-type word (ω, k) for the weighted set $(K, w\text{-}wt)$. Therefore, when K is recoverable, it is a totally closed omega-subgroup of the word-weighted fundamental group $(\Omega, w\text{-}wt)$.*

Proof: Given a word (ω, k) in $(K, w\text{-}wt)$, Lemma 6.2(a) shows that we can select loops $\rho(i)$ in Z representing $[\rho(i)] = k(i) \in K$ and with diameters $\operatorname{diam}(\rho(i)(I))$ tending to zero as the lengths $l(i)$, $i \in \mathcal{I}_\omega$, tend to zero. Lemma 6.2(b) then shows that the constant function $h : \omega \to K\backslash\Omega$, $h(p) = K\cdot 1$, determines a loop $\rho : (I, 0, 1) \to ({}_K\widehat{Z}, K\cdot 1, K\cdot 1)$ with image $\omega\langle k\rangle = p_{K\#}([\rho]) \in p_{K\#}(\pi_1({}_K\widehat{Z}, K\cdot 1))$. Theorem 4.5 thus implies that if K is recoverable, then $p_{K\#}(\pi_1({}_K\widehat{Z}, K\cdot 1)) = K$ is an omega-subgroup of $(\Omega, w\text{-}wt)$. In addition, when K is recoverable, the fibre $p^{-1}(z_0)$ is totally disconnected (Theorem 4.6) so that K is totally closed in Ω by Corollary 3.6. $\square$

6.2 Distance calculations in orbit graphs

Let a subgroup $K \leq \Omega$ be fixed. Here we examine pseudo-metric distances $m(K\cdot u, K\cdot u')$, $u, u' \in \Omega$, between vertices in the orbit graph ${}_K\Gamma$. The results derived apply principally to non-normal subgroups $K \leq \Omega$. The situation for normal subgroups is examined in more detail in Section 6.5. Given any reduced word (ω, x), we define $k(\omega, x)$ to be the least upper bound of the set of all order-type points $p \in \omega$ for which the initial subword (ω_p, x_p) is associated to an initial subword of some reduced word in K. Then $k(\omega, x) \in \omega$ and we let $\sigma_K(\omega, x)$ be the initial subword of (ω, x) that is determined by $k(\omega, x) \in \omega$:

$$\sigma_K(\omega, x) = (\omega_{k(\omega,x)}, x_{k(\omega,x)}).$$

In case $k(\omega, x) = 0$, we set $\sigma_K(\omega, x) = (\tau, 1_X)$, the reduced identity word based on the trivial order-type $\tau = \{0, 1\}$ in which the single complementary interval $(0, 1)$ is labeled by the identity 1_X. If (ω, x) and (ω', x') are associated reduced words, then the reduced initial subwords $\sigma_K(\omega, x)$ and $\sigma_K(\omega', x')$ are also associated. We can therefore view σ_K as a function $\sigma_K : \Omega \to \Omega$. Given any element $u \in \Omega$ and a reduced representative word (ω, x), the element $\sigma_K(u) \in \Omega$ has reduced representative word $\sigma_K(\omega, x)$.

Given any reduced word $u = (\omega, x) \in \Omega$, the function σ_K determines a reduced K**-initial factorization** of u:

$$u = \sigma_K(u) \cdot \underline{u}. \tag{18}$$

Explicitly, if $k = k(\omega, x)$, then $\sigma_K(u)$ is the reduced initial subword (ω_k, x_k) and $\underline{u}$ is the reduced terminal subword (ω_{k1}, x_{k1}). (If one interprets non-identity elements $u \in \Omega$ interchangeably as vertices or reduced edge paths from 1_X in the metric tree Γ, then $\sigma_K(u)$ is the upper bound of the vertices visited by the path u that separate 1_X from a member of K and $u = \sigma_K(u) \cdot \underline{u}$ is a factorization of the path u at that vertex.) The K-initial factorizations facilitate computation of pseudo-metric distances in the vertex set $K\backslash\Omega$ of the orbit graph ${}_K\Gamma$.

Lemma 6.4 *Given $u, u' \in \Omega$ with K-initial factorizations $u = \sigma \cdot \underline{u}$ and $u' = \sigma' \cdot \underline{u'}$, there are these distance calculations for the vertices $K \cdot u$ and $K \cdot u'$ in the orbit graph ${}_K\Gamma$:*

(a) if $K \cdot \sigma = K \cdot \sigma'$, then $m(K \cdot u, K \cdot u') = m(\underline{u}, \underline{u'})$;

(b) if $K \cdot \sigma \neq K \cdot \sigma'$, then $m(K \cdot u, K \cdot u') = max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), m(K \cdot \sigma, K \cdot \sigma')\}$.

Proof: Distance between vertices in the orbit graph is given by the formula:

$$m(K \cdot u, K \cdot u') = glb\{w\text{-}wt(u^{-1} \cdot k \cdot u') : k \in K\}.$$

Suppose that we are given a reduced word $k \in K$ and a similarity square $S : u^{-1} \cdot k \cdot u' = \underline{u}^{-1} \cdot \sigma^{-1} \cdot k \cdot \sigma' \cdot \underline{u'} \sim v$ where $v \in \Omega$ is a reduced word. We first observe that no arc of S cancels a complementary interval of $\underline{u}^{-1}$ or $\underline{u'}$ with a complementary interval of $\sigma^{-1} \cdot k \cdot \sigma'$. No arc of S cancels a complementary interval of $\underline{u}^{-1}$ with a complementary interval of σ^{-1} since $u = \sigma \cdot \underline{u}$ is reduced. No arc of S cancels a complementary interval of $\underline{u}^{-1}$ with a complementary interval of $k \cdot \sigma'$ since no initial subword of u that properly contains $\sigma = \sigma_K(u)$ can arise as an initial subword of a reduced word in K. It follows that there are just these two possibilities for the similarity square S.

Associating Case: Each arc of S that touches a complementary interval of $\underline{u}^{-1}$ or $\underline{u'}$ is an associating arc.

Cancelling Case: Some arc of S cancels a complementary interval of $\underline{u}^{-1}$ with a complementary interval of $\underline{u'}$.

In the associating case, we have

$$w\text{-}wt(u^{-1} \cdot k \cdot u') = max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')\}.$$

In the cancelling case, the similarity square S provides cancelling arcs to show that $\sigma^{-1} \cdot k \cdot \sigma'$ is similar to an identity word, which implies that $K \cdot \sigma = K \cdot \sigma'$ and

$$w\text{-}wt(u^{-1} \cdot k \cdot u') = w\text{-}wt(\underline{u}^{-1} \cdot \underline{u'}) = m(\underline{u}, \underline{u'}).$$

In the associating case, we further note that

$$w\text{-}wt(u^{-1} \cdot k \cdot u') \geq max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'})\} \geq w\text{-}wt(\underline{u}^{-1} \cdot \underline{u'}) = m(\underline{u}, \underline{u'}).$$

Thus $w\text{-}wt(u^{-1} \cdot k \cdot u') \geq m(\underline{u}, \underline{u'})$ for all $k \in K$, which shows that $m(K \cdot u, K \cdot u') \geq m(\underline{u}, \underline{u'})$.

Now, if $K \cdot \sigma = K \cdot \sigma'$, as in (a), then there is an element $k \in K$ such that $\sigma^{-1} \cdot k \cdot \sigma' = 1$. This implies that $m(K \cdot u, K \cdot u') \leq w\text{-}wt(u^{-1} \cdot k \cdot u') = w\text{-}wt(\underline{u}^{-1} \cdot \underline{u'}) = m(\underline{u}, \underline{u'})$, which completes the proof of (a).

Suppose now that $K \cdot \sigma \neq K \cdot \sigma'$, as in (b). For each element $k \in K$, we have

$$w\text{-}wt(u^{-1} \cdot k \cdot u') = max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')\}$$

since the cancelling case does not occur. Set $M = max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), m(K \cdot \sigma, K \cdot \sigma')\}$. For each $k \in K$ we have

$$w\text{-}wt(\underline{u}) \leq max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')\} = w\text{-}wt(u^{-1} \cdot k \cdot u').$$

This implies that $w\text{-}wt(\underline{u}) \leq m(K \cdot u, K \cdot u')$, and we similarly conclude that $w\text{-}wt(\underline{u'}) \leq m(K \cdot u, K \cdot u')$. For each $k \in K$,

$$\begin{aligned} m(K \cdot \sigma, K \cdot \sigma') \leq w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma') &\leq max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')\} \\ &= w\text{-}wt(u^{-1} \cdot k \cdot u'). \end{aligned}$$

This implies that $m(K \cdot \sigma, K \cdot \sigma') \leq m(K \cdot u, K \cdot u')$. These observations show that $M \leq m(K \cdot u, K \cdot u')$. In order to prove the reverse inequality, we may as well assume that

$$m(K \cdot u, K \cdot u') > max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'})\}.$$

With this assumption, for each $k \in K$ it follows that

$$\begin{aligned} max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'}), w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')\} &= w\text{-}wt(u^{-1} \cdot k \cdot u') \\ &\geq m(K \cdot u, K \cdot u') \\ &> max\{w\text{-}wt(\underline{u}), w\text{-}wt(\underline{u'})\}. \end{aligned}$$

This implies that $w\text{-}wt(u^{-1} \cdot k \cdot u') = w\text{-}wt(\sigma^{-1} \cdot k \cdot \sigma')$ for each $k \in K$, which in turn shows that $m(K \cdot u, K \cdot u') = m(K \cdot \sigma, K \cdot \sigma') \leq M$. In any event, we conclude that $m(K \cdot u, K \cdot u') = M$ and the proof of (b) is complete. □

These calculations reduce computation of the pseudo-metric distance between vertices of the orbit graph ${}_K\Gamma$ to that of the distance between vertices of the form $K \cdot \sigma_K(u)$, $u \in \Omega$. In addition, Lemma 6.4 implies that many metric subtrees of the Cayley tree Γ are mapped isometrically onto metric subtrees of the orbit graph ${}_K\Gamma$. For example, the union in Γ of all word-metric intervals $I_u \subseteq \Gamma$ for which $\sigma_K(u) = 1$ is a metric subtree T of Γ that is mapped isometrically onto a metric subtree of ${}_K\Gamma$. The metric tree T is *non-separating* in Γ in the sense that if $u, u' \in \Omega$ are vertices in Γ for which $u' \preceq u$ and $u' \in T$, then $u \in T$. Our next goal is to show how to prune such isometrically embedded non-separating subtrees from the orbit graph ${}_K\Gamma$.

6.3 Pruning subtrees from orbit graphs

Here we use the K-initial factorizations $u = \sigma_K(u) \cdot \underline{u}$ (18) of elements $u \in \Omega$ to describe a certain strong deformation retract of the orbit graph ${}_K\Gamma$. Recall that the metric Cayley tree Γ is the union of all word-metric intervals $I_u = I_{(\omega,x)}$ for reduced non-identity words $u = (\omega, x)$ in (X, wt). The word-metric intervals corresponding to two reduced non-identity words intersect in the word-metric interval corresponding to their maximum common initial subword.

Let $\Sigma(K)$ denote the set of K-**initial factors** of elements of Ω:

$$\Sigma(K) = \{u \in \Omega : \sigma_K(u) = u\}. \tag{19}$$

The K-**initial spine** of the orbit graph ${}_K\Gamma$ is defined to be the image under the orbit map $q_K : \Gamma \to {}_K\Gamma$ of the union of those word-metric intervals in Γ that correspond to reduced non-identity words in $\Sigma(K)$:

$$sp({}_K\Gamma) = q_K\left(\bigcup\{I_u : 1 \neq u \in \Sigma(K)\}\right). \tag{20}$$

It turns out that the orbit graph ${}_K\Gamma$ consists of its K-initial spine (20) together with a collection of non-separating metric subtrees that can be pruned away. Here are the details.

Given elements $u, u' \in \Omega$, we write $u' \preceq u$ if either $u' = 1$ or there are reduced representative words (ω, x) for u and (ω', x') for u' and an order type element $p \in \omega$ such that (ω', x') is associated to the initial subword (ω_p, x_p) of (ω, x). This is independent of the choice of reduced representatives for u and u'. The relation $\preceq$ is a partial ordering on Ω, and in particular is anti-symmetric: if $u \preceq u'$ and $u' \preceq u$, then $u = u'$ in Ω. (Topologically speaking, this relation $\preceq$ on Ω is a restriction of the cut-point ordering with respect to the vertex 1_X in the metric tree Γ.)

Given reduced non-identity representative words $(\omega, x), (\omega', x')$ for elements $u, u' \in \Omega$, Lemma 5.2 provides a maximum common initial subword (λ, y) of (ω, x) and (ω', x'). The reduced word (λ, y) represents an element of Ω that we denote by $u \cap u'$, as it is independent of the choice of reduced representatives of u and u'. We set $u \cap 1 = 1 \cap u = 1$.

Lemma 6.5 *The function $\sigma_K : \Omega \to \Omega$ has these properties for $u, u' \in \Omega$ and $k \in K$:*

(a) $\sigma_K(u) \preceq u$;

(b) if $u \preceq \sigma_K(u')$, *then* $\sigma_K(u) = u$;

(c) $\sigma_K(u \cap u') = \sigma_K(u) \cap \sigma_K(u')$;

(d) $\sigma_K(k \cdot u) = k \cdot \sigma_K(u)$.

Proof: Verification of the properties $(a) - (c)$ is left to the reader. For the proof of (d), select reduced representative words (ω, x) for u and (λ, y) for k. A reduced representative word for the product $k \cdot u$ is obtained by cancelling the maximum common initial subword of $(\lambda, y)^{-1}$ and (ω, x). Thus there are order-type elements $r \in \lambda$ and $m \in \omega$ such that the element $k \cdot u$ has reduced representative given by $k \cdot u = (\lambda_r, y_r) \cdot (\omega_{m1}, x_{m1})$.

Let $p = k(\omega, x) \in \omega$. Since the initial subword (ω_m, x_m) is associated to an initial subword of the reduced word $(\lambda, y)^{-1} \in K$, we have that $m \leq p$. The product word $k \cdot \sigma_K(u)$ is therefore represented by the reduced initial subword word $(\lambda_r, y_r) \cdot (\omega_{mp}, x_{mp})$ of the reduced representative $(\lambda_r, y_r) \cdot (\omega_{m1}, x_{m1})$ for $k \cdot u$.

Suppose that $q \in \omega$ is such that the reduced word $(\lambda_r, y_r) \cdot (\omega_{mq}, x_{mq})$ is associated to an initial subword (λ'_s, y'_s), $s \in \lambda'$, of a reduced word $k' = (\lambda', y') \in K$. Then (ω_q, x_q) is an initial subword of the reduced representative $(\omega_q, x_q) \cdot (\lambda'_{s1}, y'_{s1})$ for $k^{-1} \cdot k' \in K$, so $q \leq k(\omega, x) = p$. This implies that $\sigma_K(k \cdot u) \preceq k \cdot \sigma_K(u)$.

Now let $q \in \omega$ be such that (ω_q, x_q) is an initial subword of a reduced word $k' = (\lambda', y')$ in K, say (ω_q, x_q) is associated to the initial subword (λ'_s, y'_s), $s \in \lambda'$. Then $(\lambda_r, y_r) \cdot (\omega_{mq}, x_{mq})$ is an initial subword of the reduced representative $(\lambda_r, y_r) \cdot (\omega_{mq}, x_{mq}) \cdot (\lambda'_{s1}, y'_{s1})$ for $k \cdot k' \in K$. This implies that $k \cdot \sigma_K(u) \preceq \sigma_K(k \cdot u)$, and the proof of (d) is complete. □

Theorem 6.6 *For any subgroup $K \leq \Omega$, the orbit graph ${}_K\Gamma$ contains its K-initial spine $sp({}_K\Gamma)$ as a strong deformation retract.*

Proof: A compatible family of contractions $H_{(\omega,x)}$ of word-metric intervals corresponding to reduced non-identity words (ω, x) was constructed in the proof of Theorem 5.4. We use these contractions and the Ω-action on Γ to define a deformation of Γ as follows.

Given any reduced non-identity word $u = (\omega, x) \in \Omega$, the K-initial factorization $u = \sigma_K(u) \cdot \underline{u}$ of u shows how to decompose the word-metric interval $I_u = I_{(\omega,x)}$ as the one-point union of the word-metric interval $I_{\sigma_K(u)}$ and the translated word-metric interval $\sigma_K(u) \cdot I_{\underline{u}}$; these sub-intervals intersect in the vertex $\sigma_K(u)$. There is a deformation D_u of I_u that fixes the initial sub-interval $I_{\sigma_K(u)}$ identically and which uses a translated version of the contraction $H_{\underline{u}}$ to contract the terminal sub-interval $\sigma_K(u) \cdot I_{\underline{u}}$ to the vertex $\sigma_K(u)$. Specifically, given $(s, t) \in I_u \times I$, the deformation $D_u : I_u \times I \to I_u$ satisfies $D_u(s, t) = s$ if $s \in I_{\sigma_K(u)}$ and $D_u(s, t) = \sigma_K(u) \cdot H_{\underline{u}}(\sigma_K(u)^{-1} \cdot s, t)$ if $s \in \sigma_K(u) \cdot I_{\underline{u}}$.

The deformations D_u assemble to form a K-equivariant deformation $D : \Gamma \times I \to \Gamma$ of Γ. To see that D is well-defined, consider two word-metric intervals I_u and $I_{u'}$ for reduced non-identity words $u, u' \in \Omega$ with K-initial factorizations $u = \sigma \cdot \underline{u}$ and $u' = \sigma' \cdot \underline{u'}$. We must show that the deformations D_u and $D_{u'}$ agree on the intersection $(I_u \times I) \cap (I_{u'} \times I) = I_{u \cap u'} \times I$. If either $u \cap u' \preceq \sigma$ or $u \cap u' \preceq \sigma'$, then using Lemma 6.5 (b) and (c), we have $u \cap u' = \sigma_K(u \cap u') = \sigma \cap \sigma'$. This implies that both D_u and $D_{u'}$ restrict to the identity deformation of the intersection $I_{u \cap u'}$. Since $u \cap u'$ and σ (resp. σ') are both initial subwords of u (resp. u'), we may therefore suppose that $\sigma \preceq u \cap u'$ and $\sigma' \preceq u \cap u'$. Since $\sigma \preceq u \cap u'$, we have $\sigma = \sigma \cap (u \cap u')$ and Lemma 6.5 provides that $\sigma = \sigma \cap (\sigma \cap \sigma') = \sigma \cap \sigma'$. From this it follows that $\sigma \preceq \sigma'$ and in a similar manner we obtain $\sigma' \preceq \sigma$. Thus $\sigma = \sigma'$. Now let $(s,t) \in (I_u \times I) \cap (I_{u'} \times I) = I_{u \cap u'} \times I$. If $s \in I_\sigma = I_{\sigma'}$, then $D_u(s,t) = D_{u'}(s,t) = s$. Otherwise, $\sigma^{-1} \cdot s = \sigma'^{-1} \cdot s \in I_{\underline{u}} \cap I_{\underline{u'}}$. Since the contractions $H_{\underline{u}}$ and $H_{\underline{u'}}$ are compatible, we obtain the desired agreement of D_u and $D_{u'}$ at the point (s,t):

$$D_u(s,t) = \sigma \cdot H_{\underline{u}}(\sigma^{-1} \cdot s, t) = \sigma' \cdot H_{\underline{u'}}(\sigma'^{-1} \cdot s, t) = D_{u'}(s,t).$$

The function $D : \Gamma \times I \to \Gamma$ is continuous and defines a strong deformation retraction of the assembly Γ onto the subassembly of word-metric intervals $\bigcup\{I_u : 1 \neq u \in \Sigma(K)\} \subseteq \Gamma$. The deformation D is also K-equivariant. To see this, let $(s,t) \in I_u \subseteq \Gamma$ and a reduced word $k \in K$ be given. Suppose first that $s \in I_{\sigma_K(u)}$, in which case the element $k \cdot s$ lies in the topological Y that is formed as the union of the word-metric intervals I_k and $I_{k \cdot \sigma_K(u)} = I_{\sigma_K(k \cdot u)}$, both of which are fixed identically by D. Thus $D(k \cdot s, t) = k \cdot s = k \cdot D(s,t)$. Otherwise, we may suppose that $s \in \sigma_K(u) \cdot I_{\underline{u}}$, where $u = \sigma_K(u) \cdot \underline{u}$ is the K-initial factorization of u. The K-initial factorization of the element $k \cdot u = k \cdot \sigma_K(u) \cdot \underline{u} \in \Omega$ is given by $k \cdot u = \sigma_K(k \cdot u) \cdot \underline{u}$; and since $k \cdot s \in \sigma_K(k \cdot u) \cdot I_{\underline{u}}$, we compute:

$$\begin{aligned} D(k \cdot s, t) &= \sigma_K(k \cdot u) \cdot D_{\underline{u}}(\sigma_K(k \cdot u)^{-1} \cdot (k \cdot s), t) \\ &= k \cdot (\sigma_K(u) \cdot D_{\underline{u}}(\sigma_K(u)^{-1} \cdot s, t)) \\ &= k \cdot D(s,t). \end{aligned}$$

It follows that D determines a strong deformation retraction of the quotient orbit graph ${}_K\Gamma$ onto its K-initial spine $sp({}_K\Gamma)$. □

6.4 Examples: F, C, and Ω_2

Fix a weighted alphabet (X, wt) with associated word group $\Omega = \Omega(X, wt)$ and metric Cayley tree $\Gamma = \Gamma(X, wt)$. To make things interesting, we assume that X contains non-identity elements $x \in X$ with vanishing weights. By examining orbit graphs of certain non-normal subgroups of Ω, we exhibit: (1) a non-metric orbit graph of the metric Cayley tree Γ, (2) an omega-subgroup of Γ that is not totally closed, (3) a totally closed but non-recoverable subgroup of Ω, and finally (4) a recoverable subgroup of Ω.

Given $K \leq \Omega$, examination of the orbit graph ${}_K\Gamma$ proceeds by first identifying the set $\Sigma(K)$ of K-initial factors. Then the (pseudo-)metric structure of the K-initial spine $sp({}_K\Gamma)$ is determined by computing the distance $m(K \cdot \sigma, K \cdot \sigma')$, $\sigma, \sigma' \in \Sigma(K)$, between vertices of the orbit graph. The following general result aids the first of these tasks.

Lemma 6.7 *For any subgroup $K \leq \Omega$, the set $\Sigma(K)$ of K-initial factors contains the closure $\overline{K}$ of K in Ω.*

Proof: Given any reduced word $u = (\omega, x) \in \overline{K}$ and an order-type element $p \in \omega \cap [0, 1)$, the terminal subword $u_{p1} = (\omega_{p1}, x_{p1})$ has positive word-weight, so there is a reduced word $k \in K$, a reduced word $v \in \Omega$ such that $m(k, u) = w\text{-}wt(v) < w\text{-}wt(u_{p1})$, and a similarity square $S : u \sim k \cdot v$. The complementary interval $i = (q, r) \in \mathcal{I}_{\omega_{p1}}$ whose label $x(i)$ has weight equal to the word-weight of u_{p1} is associated to a complementary interval of the reduced word $k \in K$. Then the initial subword $u_r = (\omega_r, x_r)$ of u is associated to an initial subword of the reduced word $k \in K$, so that $k(\omega, x) \geq r > p$. This shows that $k(\omega, x) = 1$ so $u = \sigma_K(u) \in \Sigma(K)$. □

Example 6.8 *The ordinary free subgroup $F = F(X)$ consisting of all reduced order-type words (ω, x) involving finite order-types ω is not totally closed.*

The subgroup $F \leq \Omega$ is the ordinary free group of finite reduced words for the alphabet X. This is not an omega-subgroup of Ω as F fails to contain the evaluation of any infinite reduced word for $(X, wt) \subseteq (F, w\text{-}wt)$. Theorem 6.3 shows that $p_{F\#}(\pi_1({}_F\Gamma, F \cdot 1)) = \Omega \neq F$, so that F is not a recoverable subgroup of Ω.

The subgroup F is neither totally closed nor closed in Ω. In fact, its closure $\overline{F} = \Sigma(F)$ consists of F together with all reduced order-type words $(\overline{\eta}, y)$ based on the *reverse harmonic* order-type $\overline{\eta} : 0 < \frac{1}{2} < \ldots < \frac{n}{n+1} < \ldots < 1$. For if $u = (\omega, x)$ is a reduced word in $\Sigma(F)$ based on an infinite order-type ω, then every proper initial order-type ω_p, $p \in \omega \cap [0, 1)$, is finite so there is an order-type isomorphism from ω to $\overline{\eta}$ carrying any $p \in \omega \cap [0, 1)$ to $\frac{n-1}{n} \in \overline{\eta}$, where n is the number of elements in $\omega \cap [0, p]$. Further, each reduced word $h = (\overline{\eta}, y)$ lies in the closure of F: for each positive integer n, the initial subword $h_{\frac{n}{n+1}}$ of h lies in F and the word-weights $w\text{-}wt(h^{-1}_{\frac{n}{n+1}} \cdot h)$ tend to zero as n tends to infinity. Similarly, examination of the vertices of the F-initial spine $sp({}_F\Gamma)$ shows that all pseudo-metric distances $m(F \cdot \sigma, F \cdot \sigma')$ for $\sigma, \sigma' \in \Sigma(F)$ are zero: suitable initial subwords $f, f' \in F$ for $\sigma, \sigma' \in \Sigma(F)$, make the word weight $w\text{-}wt(\sigma^{-1} \cdot f \cdot f'^{-1} \cdot \sigma') \leq max\{w\text{-}wt(\sigma^{-1} \cdot f), w\text{-}wt(f'^{-1} \cdot \sigma')\}$ arbitrarily small.

Since F is not (totally) closed, the orbit graph ${}_F\Gamma$ is neither totally disconnected nor metrizable; its vertex set $F\backslash\Omega$ contains the infinite indiscrete subset $\Sigma = q_F(\Sigma(F))$ consisting of all cosets of the form $F \cdot \sigma$, $\sigma \in \Sigma(F) = \overline{F}$.

The orbit graph ${}_F\Gamma$ contains an isometric copy Z of the metric bouquet $Z = Z(X, wt)$, and the intermediate endpoint projection $p_F : {}_F\Gamma \to Z$ is a retraction. The embedded bouquet $Z \subseteq {}_F\Gamma$ arises as the image under the orbit map $q_F : \Gamma \to {}_F\Gamma$ of the union of the word-metric intervals $I_{(\tau,x)} \subseteq \Gamma$ for the monosyllabic words (τ, x), $x \in X$. The orbit graph ${}_F\Gamma$ consists of the F-initial spine $sp({}_F\Gamma)$ together with non-separating metric trees appended. The spine $sp({}_F\Gamma)$ is the one-point union of the metric bouquet Z and the closed indiscrete subspace Σ consisting of all vertices of the form $F \cdot \sigma$, $\sigma \in \Sigma(F)$; the intersection $Z \cap \Sigma$ consists of the vertex $F \cdot 1$. The orbit graph is displayed in Figure 2.

The spine $sp({}_F\Gamma)$ in turn contains the metric bouquet Z as a strong deformation retract: since the indiscrete subspace $\Sigma = q_F(\Sigma(F))$ is closed, a strong deformation retraction

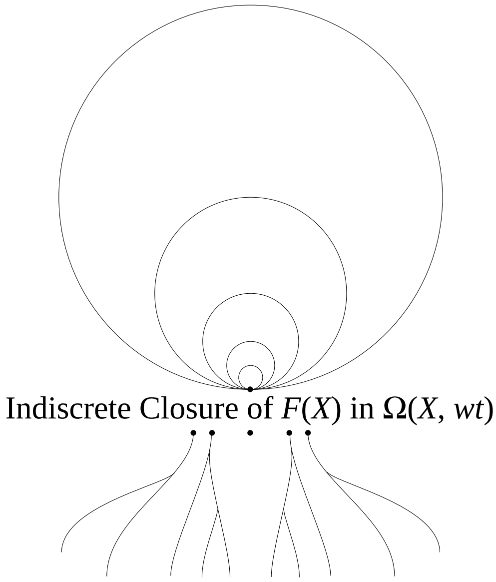

Figure 2: Orbit graph ${}_F\Gamma$

$sp({}_F\Gamma) \times I \to sp({}_F\Gamma)$ of the spine onto the bouquet Z can be defined to fix the spine identically on $sp({}_F\Gamma) \times [0,1)$ and map each point $(F \cdot \sigma, 1)$, $\sigma \in \Sigma(F)$ to the vertex $F \cdot 1$ in the embedded bouquet $Z \subseteq sp({}_F\Gamma)$. Consequently, $\pi_1({}_F\Gamma, F \cdot 1) \cong \pi_1(Z, z_0) \cong \Omega$.

Example 6.9 *For any reduced reverse-harmonic word $h = (\bar{\eta}, y) \in \Omega$, the conjugate subgroup $F' = h \cdot F \cdot h^{-1}$ is an omega-subgroup of Ω that is not totally closed.*

Since the word-weights of non-identity elements in the word-weighted group $(F', w\text{-}wt)$ are uniformly bounded below by $w\text{-}wt(h) > 0$, each reduced order-type word for the weighted set $(F', w\text{-}wt)$ is finite, so F' is an omega-subgroup of Ω in a trivial way. The subgroup F' is not totally closed since its conjugate F is not totally closed.

Example 6.10 *The subgroup C consisting of all order-type words (ω, x) based on countable order-types ω is totally closed but not recoverable.*

The key point is that the set of C-initial factors in Ω is just the subgroup C: $\Sigma(C) = C$. If $u = (\omega, x) \in \Sigma(C)$ then $\sigma_C(u) = u$ and it follows that for each positive integer n, there is an order-type element $p \in \omega \cap (\frac{n}{n+1}, 1]$ such that $\omega \cap [0, p]$ is countable. This implies that $\omega \cap [0, \frac{n}{n+1}]$ is countable for each positive integer n, so ω is countable. It follows from Lemma 6.7 that C is closed in Ω. In fact, distance calculations in the orbit graph ${}_C\Gamma$ reveal that C is totally closed. For suppose that we are given vertices $C \cdot u$ and $C \cdot u'$ corresponding to elements $u, u' \in \Omega$ for which $m(C \cdot u, C \cdot u') = 0$. Using the C-initial factorizations $u = \sigma \cdot \underline{u}$ and $u' = \sigma' \cdot \underline{u'}$, since $C \cdot \sigma = C = C \cdot \sigma'$, Lemma 6.4 shows that $0 = m(C \cdot u, C \cdot u') = m(\underline{u}, \underline{u'})$, so that $\underline{u} = \underline{u'}$. Thus $C \cdot u = C \cdot \underline{u} = C \cdot \underline{u'} = C \cdot u'$. This shows that the vertex set $C\backslash\Omega$ in the orbit graph ${}_C\Gamma$ is metrizable, and hence Hausdorff. That C is totally closed in Ω follows from Theorem 3.4.

Just as in the case of the subgroup F of finite words, the orbit graph ${}_C\Gamma$ contains an isometric copy of the metric bouquet $Z = Z(X, wt)$ that is projected isometrically onto Z under the intermediate endpoint projection $p_C : {}_C\Gamma \to Z$. In fact this metric bouquet is precisely the C-initial spine $sp({}_C\Gamma)$ of the orbit graph. It follows that $\pi_1({}_C\Gamma, C \cdot 1) \cong p_{C\#}(\pi_1({}_C\Gamma, C \cdot 1)) = \Omega \neq C$, so C is not recoverable. The concomitant failure of unique path lifting for $p_K : {}_K\widehat{Z} \to Z$ is demonstrated by any word path $p_{(\omega,x)}$ based on an order-type ω having no countable initial section; it has at least two liftings, one of which remains in and one of which exits immediately the C-initial spine $sp({}_C\Gamma)$.

Example 6.11 *The free omega-subgroup Ω_2 of words of squares is recoverable.*

The subgroup $\Omega_2 \leq \Omega$ consists of all reduced order-type words of squares $x \cdot x$ of members $x \in X$. This is an omega-subgroup of Ω. A reduced word (ω, x) lies in Ω_2 precisely when for each non-identity element $1_X \neq x_0 \in X$, the finite set of complementary intervals $i \in \mathcal{I}_\omega$ with label $x(i) = x_0$ decomposes into adjacent pairs $(p, q) < (q, r)$ of identically labeled intervals that share a common endpoint $q \in \omega$.

The set of Ω_2-initial factors is $\Sigma(\Omega_2) = \Omega_2 \cdot X$. For the containment $\Omega_2 \cdot X \subseteq \Sigma(\Omega_2)$, consider a reduced word $h \in \Omega_2$ and an element $x \in X$. If $x = 1_X$, then $h \cdot x = h \in \Omega_2 \subseteq \Sigma(\Omega_2)$. Suppose that $x \neq 1_X$. If the product word $h \cdot x$ is reduced, then $h \cdot x$ is an initial subword of the reduced word $h \cdot x^2 \in \Omega_2$. Otherwise, a reduced form of $h \cdot x$ is obtained by cancelling the second factor x with the last letter of h so a reduced form of $h \cdot x$ is associated to an initial subword of $h \in \Omega_2$. This shows that $\Omega_2 \cdot X \subseteq \Sigma(\Omega_2)$. For the reverse containment, let $u = (\omega, x)$ be a reduced word in $\Sigma(\Omega_2)$. We consider two cases:

The non-limiting case: Suppose first that the terminal point $1 \in \omega$ is an endpoint of a complementary interval $i = (p, 1) \in \mathcal{I}_\omega$ of ω. Then u itself is associated to an initial subword of a reduced word in Ω_2. Thus either $u \in \Omega_2 = \Omega_2 \cdot 1_X \subseteq \Omega_2 \cdot X$ or else $u_p \in \Omega_2$ and $u = u_p \cdot x(i) \in \Omega_2 \cdot x(i) \subseteq \Omega_2 \cdot X$.

The limiting case: Suppose, then, that the terminal point $1 \in \omega$ is not an endpoint of any complementary interval of ω. In this case, we argue that $u \in \Omega_2$. Consider a non-identity element $x_0 \in X$. Among the finite set of complementary intervals $i \in \mathcal{I}_\omega$ with label $x(i) = x_0$, let $i_0 = (p, q) \in \mathcal{I}_\omega$ be the maximum such interval with respect to the natural ordering on $\mathcal{I}_\omega$. Since the terminal point $1 \in \omega$ is a limit point of the order-type ω, there is complementary interval $j_0 = (r, s) \in \mathcal{I}_\omega$ such that $s > q$. As the initial subword u_s belongs to $\Sigma(\Omega_2)$, the non-limiting case shows that u_s lies in $\Omega_2 \cdot X$. Since the non-trivial reduced word u_{qs} has no complementary intervals with label x_0, the x_0-labeled complementary intervals of u_s, and hence of u, fall into adjacent pairs. Since this holds for arbitrary $x_0 \in X$, we conclude that $u \in \Omega_2$.

It follows that the vertex set of the Ω_2-initial spine of the orbit graph ${}_{\Omega_2}\Gamma$ consists of all cosets of the form $\Omega_2 \cdot x$, $x \in X$. Given $x, x' \in X$, we have

$$m(\Omega_2 \cdot x, \Omega_2 \cdot x') = glb\{w\text{-}wt(x^{-1} \cdot h \cdot x' : h \in \Omega_2\} \le w\text{-}wt(x^{-1} \cdot 1 \cdot x') \le max\{wt(x), wt(x')\}.$$

Without loss of generality we may suppose that $wt(x) \ge wt(x')$. Consider a similarity square $S : x^{-1} \cdot h \cdot x' \sim v$ where $h \in \Omega_2$ and $v \in \Omega$ are reduced words. If $w\text{-}wt(x^{-1} \cdot h \cdot x') = w\text{-}wt(v) < wt(x)$, then there is a cancelling arc of S that cancels the first factor x^{-1} of $x^{-1} \cdot h \cdot x'$ with either the last factor x' or else with a letter of the reduced word h. If x^{-1} cancels with x' then $x = x'$ and $h = 1$. If x^{-1} cancels with a letter of h, then it must cancel with the first letter of h, since h is reduced. In this case it follows that $h \in \Omega_2$ has a reduced factorization $h = x \cdot x \cdot h'$ and the second occurrence of x in h must cancel in S with the last factor x' of $x^{-1} \cdot h \cdot x'$, so, in this case, we find that $x' = x^{-1}$ and $h = x \cdot x$. These considerations show that either $x' = x^{\pm 1}$, in which case $m(\Omega_2 \cdot x, \Omega_2 \cdot x') = 0$, or else $x' \ne x^{\pm 1}$ and $m(\Omega_2 \cdot x, \Omega_2 \cdot x') = max\{wt(x), wt(x')\}$.

These distance calculations enable us to show that Ω_2 is totally closed in Ω. Consider elements $u, u' \in \Omega$ with Ω_2-initial factorizations $u = \sigma \cdot \underline{u}$ and $u' = \sigma' \cdot \underline{u'}$. There are elements $h, h' \in \Omega_2$ and $x, x' \in X$ such that $\sigma = h \cdot x$ and $\sigma' = h' \cdot x'$. Thus $\Omega_2 \cdot \sigma = \Omega_2 \cdot x$ and $\Omega_2 \cdot \sigma' = \Omega_2 \cdot x'$. If $\Omega_2 \cdot \sigma \ne \Omega_2 \cdot \sigma'$, then $x' \ne x^{\pm 1}$ and Lemma 6.4 shows that

$$m(\Omega_2 \cdot u, \Omega_2 \cdot u') \ge m(\Omega_2 \cdot x, \Omega_2 \cdot x') = max\{wt(x), wt(x')\} > 0.$$

Therefore, if we suppose that $m(\Omega_2 \cdot u, \Omega_2 \cdot u') = 0$, then it follows that $\Omega_2 \cdot x = \Omega_2 \cdot \sigma = \Omega \cdot \sigma' = \Omega_2 \cdot x'$. By Lemma 6.4, $0 = m(\Omega_2 \cdot u, \Omega_2 \cdot u') = m(\underline{u}, \underline{u'})$, which implies that $\underline{u} = \underline{u'}$, so $\Omega_2 \cdot u = \Omega \cdot x \cdot \underline{u} = \Omega_2 \cdot x' \cdot \underline{u'} = \Omega_2 \cdot u'$. Thus the coset space $\Omega_2 \backslash \Omega$ is metrizable, so Ω_2 is totally closed in Ω.

Because the set of Ω_2-initial factors is $\Sigma(\Omega_2) = \Omega_2 \cdot X$, the Ω_2-initial spine $sp({}_{\Omega_2}\Gamma)$ equals the image Z_2 under the orbit map $q : \Gamma \to {}_{\Omega_2}\Gamma$ of the union of all word-metric intervals $I_{x \cdot x}$ for the squares $x \cdot x$ of non-identity elements $1_X \ne x \in X$. This spine Z_2 can be viewed as the metric realization $Z(X \cdot X, wt)$ of the weighted set $(X \cdot X, wt)$ of squares $x \cdot x$, with the image of $I_{x \cdot x}$ a copy of the circle $S^1_{(x \cdot x)^{\pm 1}}$ of Z that has been subdivided to include a new vertex $\Omega_2 \cdot x = \Omega_2 \cdot x^{-1}$ at distance $wt(x)$ from, and antipodal to, the join point $\Omega_2 \cdot 1$. The intermediate endpoint projection $p_{\Omega_2} : {}_{\Omega_2}\Gamma \to Z$ maps each subdivided circle $S^1_{(x \cdot x)^{\pm 1}}$ onto $S^1_{x^{\pm 1}} \subseteq Z$ by a two-fold wrap. The entire orbit graph ${}_{\Omega_2}\Gamma$, with strong deformation retract Z_2, is depicted in Figure 3.

By [B-S97(1), Theorem 1.4], the fundamental group $\pi_1({}_{\Omega_2}\Gamma, \Omega_2 \cdot 1) \cong \pi_1(Z_2, \Omega_2 \cdot 1)$ is the free omega-group $\Omega(X \cdot X, wt)$ on the alphabet of squares $x \cdot x$ of elements of X. On fundamental groups, the intermediate endpoint projection $p_{\Omega_2} : {}_{\Omega_2}\Gamma \to Z$ maps the omega-generator $x \cdot x \in \Omega(X \cdot X, wt)$ to the square $x \cdot x \in \Omega = \Omega(X, wt)$. This shows that the image of $p_{\Omega_2 \#} : \pi_1({}_{\Omega_2}\Gamma, \Omega_2 \cdot 1) \to \pi_1(Z, z_0)$ is the omega-subgroup Ω_2 of Ω. So Ω_2 is a recoverable subgroup of $\Omega = \pi_1(Z, z_0)$, equivalently by Theorem 4.5, the intermediate endpoint projection $p_{\Omega_2} : {}_{\Omega_2}\Gamma \to Z$ has unique path lifting.

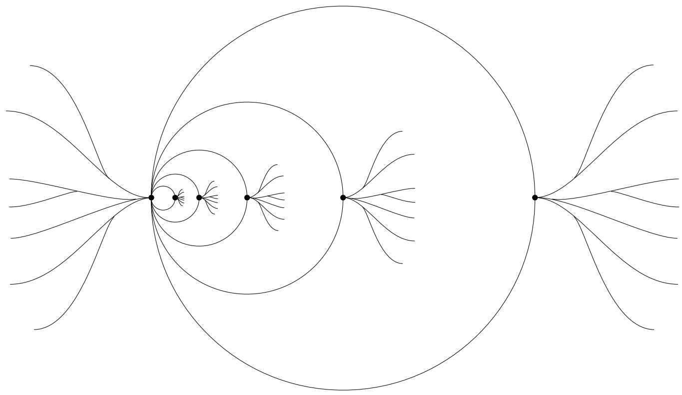

Figure 3: Orbit graph ${}_{\Omega_2}\Gamma$

6.5 Recoverable normal subgroups of Ω

We next characterize the recoverability of normal subgroups K of the free omega-group Ω, using the following terminology from [S97]. Let K be a subgroup of an omega-group (G, wt). Given order-type words (ω, g) and (ω, k) for (G, wt) and (K, wt) with the same underlying order-type ω, there is the order-type word $(\omega, g \cdot k)$ for (G, wt) whose label on each complementary interval $i \in \mathcal{I}_\omega$ is the product $g(i) \cdot k(i)$ of the corresponding labels for the order-type words (ω, g) and (ω, k). Since (G, wt) is an the omega-group (G, wt), the product $\omega\langle g \cdot k\rangle(\omega\langle g\rangle)^{-1}$ exists in G and is called a **weighted consequence** of (K, wt) in (G, wt). The subgroup K is a **normal omega-subgroup** of (G, wt) provided it contains each weighted consequence $\omega\langle g \cdot k\rangle \cdot (\omega\langle g\rangle)^{-1}$ of (K, wt) in (G, wt). A normal omega-subgroup K is an omega-subgroup of (G, wt) (take (ω, g) to be the identity word $(\omega, 1_G)$) and is normal in G (take ω to be the trivial order-type $\tau = \{0, 1\}$).

When K is any normal subgroup of an omega-group (G, wt), there is a real-valued function $wt : G/K \to \mathbf{R}$ on the quotient group G/K given by $wt(gK) = \text{glb}\{wt(gk) : k \in K\}$, $g \in G$. This is a weight function on the quotient group if and only if K is totally closed in G. When K is totally closed, the weighted group $(G/K, wt)$ may not be an omega-group.

Theorem 6.12 *Let $Z = Z(X, wt)$ be the metric bouquet of circles with join point z_0 that realizes the weighted alphabet (X, wt). The following are four equivalent conditions on a normal subgroup $K \leq \Omega = \Omega(X, wt)$ of the word-weighted free omega-group $\Omega \cong \pi_1(Z, z_0)$:*

(a) The subgroup K is a normal omega-subgroup of the free omega-group $(\Omega, w\text{-}wt)$.

(b) The quotient group $(\Omega/K, w\text{-}wt)$ is an omega-group.

(c) The subgroup K is a recoverable subgroup of the fundamental group $\Omega \cong \pi_1(Z, z_0)$.

(d) The endpoint projection $p : {}_K\widehat{Z} \to Z$ has unique path lifting.

Proof: Statements (a) and (b) are equivalent by [S97, Lemma 1.3, Theorem 1.4]. Statements (c) and (d) are equivalent by Theorem 4.5.

To prove that $(d) \Rightarrow (a)$, consider order-type words (ω, g) and (ω, k) for $(\Omega, w\text{-}wt)$ and $(K, w\text{-}wt)$ with the same underlying order-type ω. Via their initial subwords, (ω, g) and $(\omega, g \cdot k)$ determine continuous functions $h : \omega \to K\backslash\Omega$ given by $h(p) = K \cdot \omega_p\langle g_p\rangle$ and $h' : \omega \to K\backslash\Omega$ with $h'(p) = \omega_p\langle (g \cdot k)_p\rangle$, $p \in \omega$.

For each $i = (a_i, b_i) \in \mathcal{I}_\omega$, we have $h(b_i) = h(a_i) \cdot g(i)$ in $K\backslash\Omega$. Since K is normal in Ω, we likewise have these calculations in $K\backslash\Omega$:

$$h'(b_i) = h'(a_i) \cdot g(i) \cdot k(i) = h'(a_i) \cdot g(i).$$

Selecting loops $\rho(i) : (I, 0, 1) \to (Z, z_0, z_0)$ representing $[\rho(i)] = g(i)$, $i \in \mathcal{I}_\omega$, and with diameters tending to zero as the lengths $l(i)$ tend to zero, Lemma 6.2 provides paths $\rho, \rho' : (I, 0) \to ({}_K\widehat{Z}, K \cdot 1)$ with $\rho|_\omega = h$, $\rho'|_\omega = h'$ and $p_K \circ \rho = p_K \circ \rho'$. By unique path lifting for p_K, we conclude that $\rho = \rho'$, and in particular that $K \cdot \omega\langle g\rangle = h(1) = h'(1) = K \cdot \omega\langle g \cdot k\rangle$. This shows that $\omega\langle g \cdot k\rangle \cdot (\omega\langle g\rangle)^{-1} \in K$, so K is a normal-omega subgroup of Ω, as in (a).

To prove that $(b) \Rightarrow (c)$, it suffices to show $p_{K\#}(\pi_1({}_K\widehat{Z}, K \cdot \widehat{z_0})) \subseteq K$, as the converse holds by Corollary 4.3. We examine path homotopy classes of based loops in the orbit graph ${}_K\widehat{Z}$. The metric Cayley tree $\widehat{Z} = \Gamma(X, wt)$ has 0-skeleton $\Omega = p^{-1}(z_0)$ and open 1-cells $e^1_x(u)$ with diameter $wt(x)$ and endpoints $u, u \cdot x \in \Omega$ corresponding to vertices $u \in \Omega$ and non-identity elements $x \in X$. The orbit map $q_K : \widehat{Z} \to {}_K\widehat{Z}$ carries the open 1-cell $e^1_x(u)$ homeomorphically onto an open 1-cell $e^1_x(K \cdot u)$, with diameter $wt(x)$ in the orbit graph, having endpoints $K \cdot u$ and $K \cdot u \cdot x$ in the 0-skeleton $p_K^{-1}(z_0) = K\backslash\Omega$ of ${}_K\widehat{Z}$.

Consider any loop $f : I \to {}_K\widehat{Z}$ based at the vertex $K \cdot 1$ in the orbit graph. We must show that $p_{K\#}([f]) = [p_K \circ f] \in K$. We first normalize the loop f. As in the discussion preceding [B-S97(1), Lemma 1.2], there is a continuous closed surjection $q : I \to I$ that collapses the components of the closed set $(p_K \circ f)^{-1}(z_0) \subseteq I$ onto individual points of an order-type $\omega \subseteq I$. The loop f factors through the quotient map q to give a loop $f' : I \to {}_K\widehat{Z}$ based at $K \cdot 1$ for which $(p_K \circ f')^{-1}(z_0) = \omega$ and $[f] = [f \circ 1_I] = [f \circ q] = [f'] \in \pi_1({}_K\widehat{Z}, K \cdot 1)$.

The map $f' : I \to {}_K\widehat{Z}$ carries each complementary interval $i \in \mathcal{I}_\omega$ into some open 1-cell $e^1_x(K \cdot u)$. Since the omega-group $\Omega/K = K\backslash\Omega$ is a weighted group, the subgroup K is totally closed in Ω and so the orbit graph is metrizable and Hausdorff. This implies that the boundary of each open 1-cell $e^1_x(K \cdot u)$ consists of just its endpoints $K \cdot u, K \cdot u \cdot x \in K\backslash\Omega$. Thus the restriction of f' to the closure of the complementary interval $i = (a_i, b_i) \in \mathcal{I}_\omega$, when re-parametrized to a path $I \to {}_K\widehat{Z}$, is path-homotopic either to a constant loop at one of the endpoints $K \cdot u$ or $K \cdot u \cdot x$, or to a positively- or negatively-oriented essential traverse of the closed 1-cell $\bar{e}^1_x(K \cdot u) = e^1_x(K \cdot u) \cup \{K \cdot u, K \cdot u \cdot x\}$. This uniquely determines a labeling function $x : \mathcal{I}_\omega \to X$ for an order-type word (ω, x) in $(X, wt) \subseteq (\Omega, w\text{-}wt)$ and a composite labeling function $g : \mathcal{I}_\omega \to X \subseteq \Omega \to \Omega/K$ for an order-type word (ω, g) in $(\Omega/K, w\text{-}wt)$ for which $\omega\langle g\rangle = \omega\langle x\rangle \cdot K$. In addition, the restriction of f' to the order-type ω is a continuous function $h : \omega \to \Omega/K$ satisfying $h(0) = 1_{\Omega/K}$ and $h(b_i) = h(a_i) \cdot g(i)$ in Ω/K for each $i \in \mathcal{I}_\omega$.

By the Uniqueness Property for the omega-group $(\Omega/K, w\text{-}wt)$ (Section 1.1), h is determined by the initial subwords of the word (ω, g), and in particular $K \cdot 1 = h(1) = \omega\langle g\rangle = \omega\langle x\rangle \cdot K = K \cdot \omega\langle x\rangle$. Thus $[p_K \circ f] = [p_K \circ f'] = \omega\langle x\rangle \in K$, which completes the proof. □

7 Universal Path Spaces of Metric 2-Complexes

In this section we investigate the universal path spaces of wild metric 2-complexes. We present two extreme possibilities. The fundamental group can be an omega-group (see [S97] and Section 1.1), in which case, the universal path space is a simply-connected metric 2-complex whose 0-skeleton is a totally disconnected perfect copy of the fundamental group. The fundamental group can equal its infinitesimal subgroup, in which case, the universal path space is a pseudo-metric 2-complex whose 0-skeleton is an indiscrete copy of the fundamental group.

7.1 Universal Path Spaces for a Pair

To prepare for the investigation of the universal path space of a wild metric 2-complex, we consider the kernel subgroup that arises from a topological inclusion $j : Z \subseteq W$ of a pair of spaces. Let

$$K = \ker\, j_\# : \pi_1(Z, z_0) \to \pi_1(W, z_0).$$

By the functorality of the universal path space (Theorem 2.6), the inclusion $j : Z \subseteq W$ lifts through the endpoint projections $p : \widehat{Z} \to Z$ and $p : \widehat{W} \to W$ to a continuous function $\widehat{j} : (\widehat{Z}, \widehat{z_0}) \to (\widehat{W}, \widehat{z_0})$.

Lemma 7.1 *(a) When every path in W from z_0 to a point of Z is path homotopic in W to a path in Z, the image of $\widehat{j} : \widehat{Z} \to \widehat{W}$ is the pre-image $p^{-1}(Z)$ under $p : \widehat{W} \to W$;*

(b) for all $[f], [g] \in \widehat{Z}$, $\widehat{j}([f]) = \widehat{j}([g])$ in $\widehat{W}$ if and only if $[f][g]^{-1}$ belongs to $K = \ker\, j_\#$;

(c) when W has a base of open sets U such that all paths in U having endpoints in Z are path homotopic in U to a path in $U \cap Z$, then $\widehat{j} : \widehat{Z} \to p^{-1}(Z) \subseteq \widehat{W}$ factors as the quotient orbit map $q_K : \widehat{Z} \to {}_K\widehat{Z}$ and an embedding $\kappa : {}_K\widehat{Z} \to p^{-1}(Z) \subseteq \widehat{W}$, $\kappa(K \cdot [f]) = j_\#([f])$.

Proof: Under the hypothesis in (a), each $[h] \in \pi(W)_{z_0} = \widehat{W}$ having endpoint $p([h]) = h(1) \in Z$ is the image $j_\#([f]) = [h]$ of some $[f] \in \pi(Z)_{z_0} = \widehat{Z}$.

For (b), recall that $\widehat{j} : \widehat{Z} \to \widehat{W}$ is just a restriction of the groupoid homomorphism $j_\# : \pi(Z) \to \pi(W)$. So $\widehat{j}([f]) = \widehat{j}([g])$ if and only if $[f][g]^{-1}$ belongs to $K = \ker j_\# : \pi_1(Z, z_0) \to \pi_1(W, z_0)$.

The hypothesis in (c), shows that the base set $K \cdot [f] \cdot \Pi_Z^{U \cap Z}$ for ${}_K\widehat{Z}$ arising from the subspace base set $U \cap Z$ has image under $\widehat{j} : \widehat{Z} \to p^{-1}(Z) \subseteq \widehat{W}$ equal to the base set $([j \circ f] \cdot \Pi_W^U) \cap p^{-1}(Z)$ for the subspace $p^{-1}(Z) \subseteq \widehat{W}$. □

As noted in Section 1.2, it is shown in [B-S97(1)] that each weighted presentation

$$\mathcal{P} =< (X, wt) : (R, r\text{-}wt) >$$

has a **realization** by a metric 2-complex $Z(\mathcal{P})$ whose 1-skeleton is the metric bouquet of circles $Z = Z(X, wt)$ with 0-cell z_0 and whose 2-cells are attached and metrized in accordance with the weighted relator set $(R, r\text{-}wt)$. In [B-S97(1), Theorem 2.5] it is shown that the weighted presentation $\mathcal{P}$ presents the fundamental group $\Pi(\mathcal{P}) \cong \pi_1(Z(\mathcal{P}), z_0)$.

The metric 2-complex $Z(\mathcal{P})$ has closed skeleta, the k-cells are open in the k-skeleton for $k = 1, 2$, and the usual technique of radially expanding central portions of the 2-cells defines a deforming homotopy $H : Z(\mathcal{P}) \times I \to Z(\mathcal{P})$ of the identity map $1_{Z(\mathcal{P})}$ relative the 1-skeleton. Moreover this deforming homotopy $H : Z(\mathcal{P}) \times I \to Z(\mathcal{P})$ respects neighborhood bases in the sense that each point z in the 1-skeleton has a local basis of neighborhoods $N_\epsilon(z)$ such that $H(N_\epsilon(z) \times I) \subseteq N_\epsilon(z)$. We call such a space a **wild metric 2-complex**. As the 1-cells and 2-cells may limit on 0-cells, a wild metric 2-complex can support essential loops of arbitrarily small metric diameter.

Now the hypotheses of Lemma 7.1 are available whenever W is a wild metric 2-complex with 1-skeleton Z, and we can use Theorem 6.12 to determine when the universal path space $\widehat{W}$ is simply connected. This makes possible the following

Theorem 7.2 *When W is a wild metric 2-complex whose 1-skeleton $Z = Z(X, wt)$ is a metric bouquet of circles with join point z_0 and $K = ker(j_\# : \pi_1(Z, z_0) \to \pi_1(W, z_0))$ is the kernel subgroup, the following are equivalent:*

(a) the universal path space $\widehat{W}$ is simply connected;

(b) the endpoint projection $p : \widehat{W} \to W$ has unique path lifting;

(c) the endpoint projection $p_K : {}_K\widehat{Z} \to Z$ has unique path lifting;

(d) the kernel K is a recoverable subgroup of $\pi_1(Z, z_0) \cong \Omega(X, wt)$; and

(e) the word-weighted fundamental group $(\pi_1(W, z_0), wt) \cong (\Omega(X, wt)/K, w\text{-}wt)$ is an omega-group.

Proof: Statements (a) and (b) are equivalent by Theorem 4.7. Statements (c), (d), and (e) are equivalent by Theorem 6.12. By Lemma 7.1, the endpoint projection $p_K : {}_K\widehat{Z} \to Z$ is a restriction of the endpoint projection $p : \widehat{W} \to W$. So statement (b) implies (c).

We conclude by proving $(d) \Rightarrow (b)$. Theorem 4.5 shows that it suffices to prove that the homomorphism $p_{\#} : \pi_1(\widehat{W}, \widehat{z_0}) \to \pi_1(W, z_0)$ is trivial. Let $F : I \to \widehat{W}$ be a loop based at $\widehat{z_0} \in \widehat{W}$. In the wild metric 2-complex (W, Z), the complement $W \backslash Z$ is a union of open 2-cells with the weak topology. Each one is a relatively simply connected neighborhood in W, over which the endpoint projection $p : \widehat{W} \to W$ is a (covering) projection of isometric open 2-cells. The path F can first be deformed off of central portions of the 2-cells of $\widehat{W}$, which are the components of the pre-image of central portions of the 2-cells of W. Then, a deforming homotopy $H : W \times I \to W$ lifts to a deforming homotopy $\widehat{H} : \widehat{W} \times I \to \widehat{W}$, which can be used to slide the loop F off the 2-cells of $\widehat{W}$ to become a loop $F : I \to {}_K\widehat{Z}$ based at $\widehat{z_0}$. Now $[p_K \circ F] \in p_{K\#}(\pi_1({}_K\widehat{Z}, \widehat{z_0})) = K = ker j_{\#}$ since K is recoverable in $\pi_1(Z, z_0)$ by (d). Then $p \circ F = j \circ p_K \circ F$ so $[p \circ F] = 1$ in $\pi_1(W, z_0)$ as desired. □

7.2 Universal path spaces for metric models

Here we complete the discussion begun in Section 2.8 by examining the unique path lifting status for the endpoint projections on the universal path spaces of the wild metric 2-complexes introduced in Section 1.2.

The nontrivial fundamental group of the harmonic archipelago HA (Example 1.1, Figure 1) modeled on the weighted presentation $\mathcal{P}_{HA}$ (1) is not an omega-group, nor even a weighted group since the fundamental group is a non-trivial infinitesimal group:

$$1 \neq \pi_1(HA, z_0) = \Pi_{z_0} = \bigcap \{\Pi^U_{HA}(z_0) : z_0 \in U \subseteq^{op} HA\}.$$

The endpoint projection $p_{HA} : \widehat{HA} \to HA$ does not have unique path lifting and by Theorem 7.2, the universal path space $\widehat{HA}$ is a non-simply-connected pseudo-metric 2-complex. Indeed, its 1-skeleton is the orbit graph ${}_K\Gamma$ of the metric Cayley tree $\Gamma = \Gamma(X, wt)$ modulo the action of $K = N_{\Omega(X,wt)}(R)$. The vertex set of the orbit graph is the fundamental group $\pi_1(HA, z_0) = K \backslash \Omega$ of traces $K \cdot (\omega, x)$ of reduced words $(\omega, x) \in \Omega$, with the indiscrete topology. The complement HA^* of the union point z_0 in HA consists of regular points. Over HA^*, the universal path space $\widehat{HA}$ is a topological disjoint union of copies of HA^*, indexed by $\pi_1(HA, z_0)$, each of which projects homeomorphically to HA^*. In $\widehat{HA}$ the component indexed by the trace $K \cdot (\omega, x)$ has its x_n-edges join the vertex $K \cdot (\omega, x)$ to the single vertex $K \cdot (\omega, x) \cdot x_n$, all of which coincide since they are the same trace. All of these components are adjoined in a swirl to the indiscrete vertex set $\pi_1(HA, z_0)$, as depicted in Figure 4.

The fundamental group of the harmonic projective plane HP (Example 1.2) modeled on the weighted presentation $\mathcal{P}_{HP}$ (3) is the non-trivial word-weighted quotient group $(\Omega(X, wt)/N(XX, wt), w\text{-}wt)$ of the free omega-group $\Omega(X, wt)$ on the weighted generator

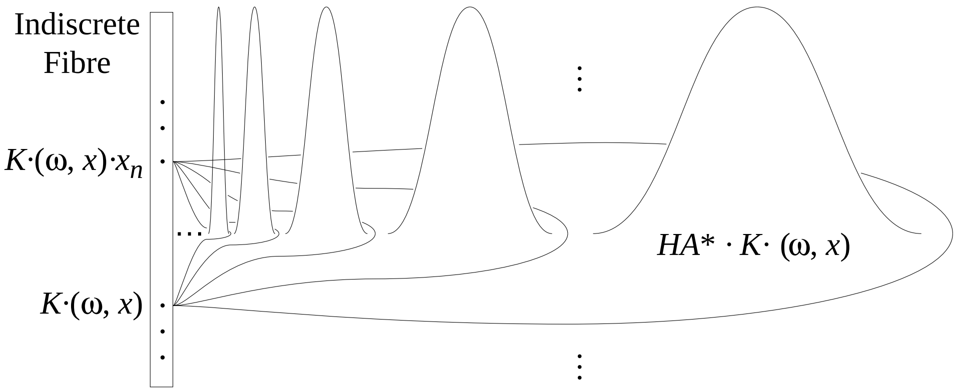

Figure 4: Universal Path Space of Harmonic Archipelago

alphabet (X, wt), modulo the weighted normal closure $N(XX, wt)$ in $\Omega(X, wt)$ of the weighted relator set (XX, wt) of squares xx of members $x \in X$. This word-weighted fundamental group is shown to be an omega-group in [B-S97(2)].

By Theorem 7.2, the endpoint projection $p_{HP} : \widehat{HP} \to HP$ has unique path lifting and the universal path space of HP is a simply-connected wild metric 2-complex whose 1-skeleton is the orbit graph ${}_K\Gamma$ of the metric Cayley tree $\Gamma = \Gamma(X, wt)$ modulo the action of the normal omega-subgroup $K = N(XX, wt)$. The realization HP is the metric 1-point union of a sequence of copies P_n of the real projective plane whose diameters have limit zero. The inclusion of each projective plane P_n into the harmonic projective plane HP induces an injection on fundamental groups. This implies that over P_n, the universal path space $\widehat{HP}$ is a topological disjoint union of copies of the 2-sphere $\widehat{P}_n$, indexed by $\pi_1(HP)$, each of which double covers P_n. In $\widehat{HP}$, the component of $p^{-1}(P_n)$ indexed by the coset $K \cdot (\omega, x)$ has an x_n-edge connecting the vertex $K \cdot (\omega, x)$ to the vertex $K \cdot (\omega, x) \cdot x_n$, which are distinct members of the fundamental group, and a second x_n-edge connecting the same vertices in opposite order.

The result is a metric union of 2-spheres meeting at their north and south poles that is invariant under antipodal involutions centered on each sphere and scaling by a half centered on each pole (assuming dyadic weights $1/2^n, n \geq 1$ instead). Some impression of this space is conveyed by Figure 5, but the lifted metric on $\widehat{HP}$ is not the Euclidean metric of the figure, as each vertex is a union point of 2-spheres of all diameters $1/2^n, n \geq 1$.

The fundamental group of the projective telescope PT (Example 1.3) modeled on the weighted presentation $\mathcal{P}_{HP}$ (3) is shown to be an omega-group in [B-S97(2)]. By Theorem 7.2, the endpoint projection $p_{PT} : \widehat{PT} \to PT$ has unique path lifting and the universal path space of PT is a simply-connected wild metric 2-complex.

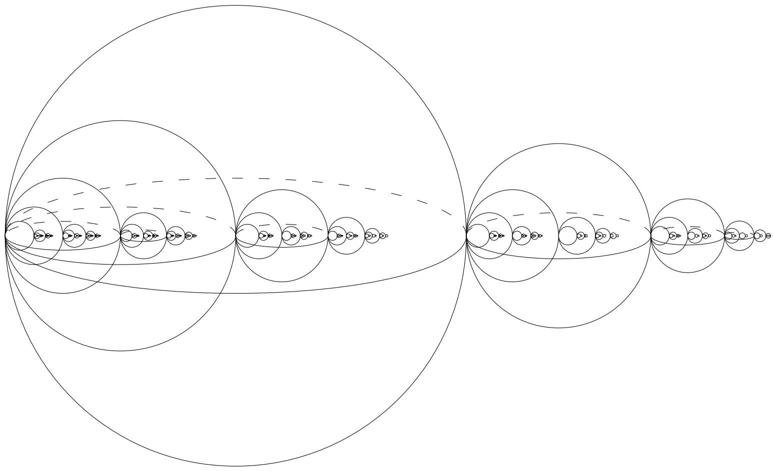

Figure 5: Universal Path Space of Harmonic Projective Plane

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