1. Noncommutative approximation theory

Let $G$ be a compact topological group that acts continuously and faithfully on a compact metric space $(X, m)$. If $A \subseteq G$ is a finite subset containing at least two points, we describe two results that establish the existence of a nonidentity element in the difference set

$$\{ab^{-1} : a \in A, b \in A, \text{ and } a \neq b\}$$

that moves the points of $X$ minimally. We consider explicit results of this sort in the case of the unitary group $G = U(N)$ of $N \times N$ unitary matrices acting on the complex surface

$$X = \{x \in \mathbb{C}^N : |x|^2 = 1\}.$$ 

We define a function $\varphi : U(N) \to [0, \infty)$ by

$$\varphi(A) = \sup \{|Ax - x|_2 : x \in X\},$$

so that $\varphi(A)$ measures the maximum distance that $A$ moves a point $x$ in $X$.

**Theorem 1.** Let $A \subseteq U(N)$ be a finite subset of cardinality $|A| \geq 2$. If

$$\delta(A) = \min \{\varphi(AB^{-1}) : A \in A, B \in A, \text{ and } A \neq B\},$$

then we have

$$\delta(A) \leq 2\pi|A|^{-1/N^2}.$$ 

As an application of Theorem 1, we obtain the following noncommutative approximation theorem.

**Theorem 2.** Let $A$ and $B$ be matrices in the unitary group $U(N)$, and let $J$ and $K$ be positive integers. If

$$\delta_{J,K}(A, B) = \min \{\varphi(A^jb^k) : |j| \leq J, |k| \leq K, \text{ and } (j, k) \neq (0, 0)\},$$

then

$$\delta_{J,K}(A, B) \leq 2\pi(J + 1)^{-1/N^2}(K + 1)^{-1/N^2}.$$ 

This is joint work with Clay Petsche.