MTH 341 - Homework 5 (due Friday, October 13)

1) Let 
\[ v_1 = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix}. \]

Set \( A = (v_1 \ v_2 \ v_3) \), and let \( p = \begin{pmatrix} -6 \\ 1 \\ 17 \end{pmatrix} \). Is \( p \) in the column space of \( A \)?

2) Determine if the collection of vectors 
\[ \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} \]

is a basis for \( \mathbb{R}^3 \).

3) Determine if the collection of vectors 
\[ \begin{pmatrix} 3 \\ -8 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix} \]

is a basis for \( \mathbb{R}^3 \).

4) State whether the following are true or false. Explain your answer.

(a) If \( B \) is an echelon form of the matrix \( A \), then the columns of \( B \) containing leading entries form a basis for the column space of \( A \).

(b) The columns of an invertible \( n \times n \) matrix form a basis for \( \mathbb{R}^n \).

5) In parts (a) and (b) below, find a basis for \( \text{Col} \ A \) (the column space of \( A \)) and \( \text{Nul} \ A \) (the null space of \( A \)). State the dimension of these subspaces.

(a) 
\[ A = \begin{pmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix}, \]

(b) 
\[ A = \begin{pmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{pmatrix}. \]
6) Let 
\[ v_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 9 \end{pmatrix}. \]
The vector \( x \) is in a subspace \( H \) which has basis \( B = \{v_1, v_2\} \). (You don’t need to show this.) Find the representation of \( x \) with respect to \( B \).

7) State whether the following are true or false. Explain your answer.

(a) The dimensions of \( \text{Col} \ A \) and \( \text{Nul} \ A \) add up to the number of columns in \( A \).

(b) The dimension of \( \text{Nul} \ A \) is the number of variables in the matrix equation \( A x = 0 \).

8) Construct a \( 3 \times 4 \) matrix with rank 1.

9) Let \( A \) be an \( n \times p \) matrix whose column space is \( p \)-dimensional. Explain why the columns of \( A \) must be linearly independent.

10) Find a basis for the subspace spanned by the vectors
\[
\begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -3 \\ -1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 4 \\ -7 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -7 \\ 6 \\ -9 \end{pmatrix}.
\]

11) Find a basis for solution set of the linear system
\[
x_1 + x_2 + 5x_3 = 0 \\
2x_1 + 2x_2 + 10x_3 = 0 \\
-4x_1 - 4x_2 - 20x_3 = -0.
\]