SAMPLE FINAL QUESTION
MTH 420

1. CURVATURE OF A CURVE

Let \( C \) be a curve in (Euclidean) \( \mathbb{R}^3 \). Either using the heuristic argument

\[
ds^2 = dx^2 + dy^2 + dz^2 \quad \implies \quad ds = \frac{dx}{ds} \, dx + \frac{dy}{ds} \, dy + \frac{dz}{ds} \, dz
\]

where \( s \) is arc length, or by noting that the unit tangent vector \( \vec{T} \) to \( C \) satisfies

\[
\vec{T} \cdot d\vec{r} = \vec{T} \cdot \vec{T} \, ds = ds
\]

we see that it is natural to define \( T = ds \) to be the unit 1-form tangent to the curve \( C \). In practice, it is common to parameterize the curve using an arbitrary parameter \( t \), not necessarily arc length. In this case, the \( s \) derivatives are replaced using chain rule, and we have in general

\[
T = \frac{dx}{dt} \, dx + \frac{dy}{dt} \, dy + \frac{dz}{dt} \, dz
\]

\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
\]

FACT: If \( dT \neq 0 \), there is a unique positive function \( \kappa \) and a unique unit 1-form \( N \) such that \( N \) is orthogonal to \( T \) and

\[
dT = \kappa \, T \wedge N
\]

(If \( dT = 0 \) we define \( \kappa = 0 \) and \( N \) is not defined.) We call \( \kappa \) the curvature of \( C \).

(a) Find the curvature of a circle of radius \( R \).

(b) Find the curvature of the \( x \)-axis.

(c) Find the curvature of any other curve of your choice. You are encouraged to discuss your choice with me; overly simple curves may not receive full credit.