1. (16 pts) In this problem the curve $C$ is the quarter of the circle of radius 2, centered at the origin, which lies in the first quadrant, oriented from $(2, 0)$ to $(0, 2)$.

(a) Compute $\int_C f\, ds$ where $f(x, y) = \frac{xy}{x^2 + y^2}$.

The first thing which is needed is a parametrization of the curve. While it is possible to use a representation as a graph, the most convenient is $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$. In terms of the parametrization, $\int_C f\, ds = \int_0^{\pi/2} f(x(t), y(t))|\vec{r}'(t)|\, dt$. An easy computation gives $|\vec{r}'(t)| = 2$, so the integral becomes

$$\int_0^{\pi/2} 2 \cos t \sin t\, dt = \int_0^{\pi/2} 2 \sin t \cos t\, dt = \sin^2 t \bigg|_0^{\pi/2} = 1$$

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = y\vec{i} - x\vec{j}$.

This is a line integral given in terms of the parametrization by $\int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$ which evaluates to

$$\int_0^{\pi/2} (2 \sin t\vec{i} - 2 \cos t\vec{j}) \cdot (-2 \sin t\vec{i} + 2 \cos t\vec{j})\, dt = \int_0^{\pi/2} -4\, dt = -2\pi$$

2. (16 pts) In this problem $C$ is the half of the circle $x^2 + y^2 = 9$ in the right half plane ($x \geq 0$) together with the segment on the $y$-axis from $(0, 3)$ to $(0, -3)$ oriented counterclockwise, and $\vec{F} = -x^2\vec{i} + xy^2\vec{j}$.

(a) Compute $\int_C \vec{F} \cdot d\vec{r}$.

This curve is a closed curve, bounding the half disk of radius three in the right half plane. Thus Green’s theorem is applicable. Using Green’s theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{\text{Half disk}} \left( \frac{\partial}{\partial x} (xy^2) - \frac{\partial}{\partial y} (-x^2y) \right) \, dA$$

$$= \iint y^2 + x^2 \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^2 \, r \, dr \, d\theta$$

$$= \pi \frac{1}{4} r^4 \bigg|_0^3 = \frac{81\pi}{4}$$
(b) Compute \( \int_C \vec{F} \cdot \vec{N} ds \), where \( \vec{N} \) is the outer unit normal.

Using the variation of Green’s theorem for this flux integral (integral of the normal component of \( \vec{F} \)) gives
\[
\int_C \vec{F} \cdot \vec{N} ds = \iint \text{div} \vec{F} dA = \iint 0 dA = 0
\]

3. (9 pts) Compute \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F} = \text{grad} \left((x + 2y + 3z)e^{xz}\right) \) and \( C \) is the piecewise smooth curve made of the straight line segment from \((0, 1, 1)\) to \((1, 2, 1)\) followed by the straight line segment from \((1, 2, 1)\) to \((2, 1, 0)\).

Since this is the line integral of a gradient, the fundamental theorem for line integrals says that it is equal the difference of the function evaluated at the terminal point less the value at the initial point. Substituting in the function, the difference is \(4 - 5 = -1\)

4. (9 pts) Let \( \vec{F} = (3y^2 - 3x^2 + y) \vec{i} + (6xy + x + z) \vec{j} + y \vec{k} \)

(a) Compute \( \text{curl} \vec{F} = 0 \)
(b) Compute \( \text{div} \vec{F} = 0 \)
(c) Is \( \vec{F} \) conservative on \( \mathbb{R}^3 \)? Justify. Yes, since \( \mathbb{R}^3 \) is simply connected, and the curl of \( \vec{F} \) is zero.

5. (a) (4 pts) Let \( u = 2x + y \) and \( v = -x + 4y \). Compute the Jacobian \( \frac{\partial(x, y)}{\partial(u, v)} \)

The easiest method is to compute \( \frac{\partial(u, v)}{\partial(x, y)} \) which is
\[
\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = 9
\]

and then use that \( \frac{\partial(x, y)}{\partial(u, v)} \) is the reciprocal, which is \( \frac{1}{9} \).

(b) (6 pts) Compute \( \iint_D \sin(2x + y) dA \) where \( D \) is the region bounded by the lines
\(-x + 4y = 4, -x + 4y = 12, 2x + y = 0 \) and \( 2x + y = \pi \).

Using the change of variables from above
\[
\iint_D \sin(2x + y) dA = \int_4^{12} \int_0^\pi \sin(u) \frac{\partial(x, y)}{\partial(u, v)} du dv
\]
\[
= \int_4^{12} \int_0^\pi \frac{1}{9} \sin(u) du dv = 8 * 2 * \frac{1}{9} = \frac{16}{9}
\]