1. Each part of this problem concerns the parametric curve \( \vec{r}(t) = (t^2 - 2) \hat{i} + (2t - 1) \hat{j} + \ln(t) \hat{k}, \quad t > 0. \)

(a) Find the velocity, \( \vec{v}, \) the speed, \( \nu, \) and the acceleration, \( \vec{a}, \) at every \( t > 0. \)

The velocity is \( \vec{v}(t) = 2t \hat{i} + 2 \hat{j} + \frac{1}{t} \hat{k}. \) The acceleration is \( \vec{a}(t) = 2 \hat{i} - \frac{1}{t^2} \hat{k}. \) The speed is \( \sqrt{4t^2 + 4 + \frac{1}{t^2}} = 2t + \frac{1}{t}. \)

(b) Find the length of the parametric curve from \((-1, 1, 0, \ln(3))\) to \((7, 5, \ln(3)).\)

The length is the integral from 1 to 3 of the speed. It is equal to \( 8 + \ln 3. \)

(c) Find the unit tangent vector \( \vec{T} \) and the tangential component of acceleration, \( a_T, \) at the point \((-1, 1, 0), \) that is, when \( t = 1. \)

The unit tangent at the given point is \( \vec{T} = \frac{1}{3}(2 \hat{i} + 2 \hat{j} + \hat{k}). \) The tangential component of acceleration is the dot product of acceleration with the unit tangent. It is equal to \( \frac{1}{3}. \)

(d) Find the curvature \( \kappa, \) normal component of acceleration, \( a_N, \) and the principle unit normal vector \( \vec{N} \) at \((-1, 1, 0), \) that is when \( t = 1. \)

\( a_N = 2, \quad \vec{N} = \frac{1}{5}(8 \hat{i} - \hat{j} + 4 \hat{k}), \quad \kappa = \frac{2}{5} \)

2. In this problem the function \( f \) is given by \( f(x, y, z) = x^3y - 3x^2yz^4. \)

(a) Compute \( \nabla f(x, y, z) \) and find the directional derivative of \( f \) at the point \((1, 1, 1)\) in the direction towards the origin.

\[ \nabla f(x, y, z) = (3x^2y - 6xyz^4) \hat{i} + (x^3 - 3x^2z^4) \hat{j} - 12x^2yz^3 \hat{k} \]

The directional derivative is the dot product of the gradient at \((1, 1, 1)\) with the unit vector pointing from \((1, 1, 1)\) to the origin. This is \( \frac{17}{\sqrt{3}}. \)

(b) In what direction does \( f \) increase most rapidly at the point \((1, 1, 1)\) and what is the rate of increase of \( f \) in that direction?

In the direction of the gradient and the most rapid rate of increase is the magnitude of the gradient. Computing: \( \hat{u} = \frac{1}{\sqrt{157}}(-3 \hat{i} - 2 \hat{j} - 12 \hat{k}) \) and the magnitude is \( \sqrt{157}. \)

(c) Consider the level surface \( f(x, y, z) = -2, \) where \( f \) is as above. Find an equation of the tangent plane and of the normal line to this surface at the point \((1, 1, 1).\)

Equation of tangent plane: \( 3x + 2y + 12z = 17, \) equation of normal line \( \vec{r}(t) = (1, 1, 1) + t(-3, -2, 12). \)

3. In this problem \( f(x, y) = 3xy - x^2y - xy^2. \)
(a) Find all critical points of $f$ and determine whether $f$ has a local maximum, local minimum, or a saddle point at each critical point. (There are 4 distinct critical points.)

The four critical points are: $(0, 0)$, $(3, 0)$, and $(0, 3)$ all of which are saddles since $D = -9$, and $(1, 1)$ which is a local maximum since $D = 3$ while $f_{xx}(1, 1) = -2$ is negative.

(b) Find the maximum and minimum values of the function $f(x, y)$ on (and inside) the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(3, 6)$.

Restrict $f$ to the boundary and use one variable calculus to find the extreme values on the boundary, and compare with the local maximum which was in the interior. The maximum of 1 occurs at the point $(1, 1)$, while the minimum is $-108$ at the vertex $(3, 6)$.

4. In this problem, the vector field $\vec{F}$ is given by $\vec{F} = (2xz + \sin(y)) \hat{i} + x\cos(y) \hat{j} + x^2 \hat{k}$.

(a) Show that $\vec{F}$ is conservative.

Compute $\text{curl} \vec{F} = 0$.

(b) Compute the line integral \( \int_C \vec{F} \cdot \vec{n} \, ds \) where $C$ is the curve $\vec{r}(t) = \cos(t) \hat{i} + \sin(t) \hat{j} + t \hat{k}$, with $0 \leq t \leq \pi/2$.

Either use path independence and find a new path where the computation is easier, or find $f$ such that $\nabla f = \vec{F}$ and use the fundamental theorem of line integrals. The result is 0. If you used the second method, $f(x, y, z) = x^2z + x \sin y$ would work.

5. In this problem $\vec{F}$ is the vector field in the plane given by $\vec{F}(x, y) = \frac{x}{x^2 + y^2} \hat{i} + \frac{y}{x^2 + y^2} \hat{j}$.

(a) Compute $\text{div} \vec{F}$.

The divergence is equal to 0.

(b) Compute $\int_C \vec{F} \cdot \vec{n} \, ds$ for each of the three curves: $C_1 = \{(x, y) : x^2 + y^2 = 4\}$, $C_2 = \{(x, y) : (x - 1)^2 + (y - 1)^2 = 1\}$, and $C_3 = \{(x, y) : x^2 + y^2/2 = 1\}$. All curves are oriented counterclockwise.

The divergence form of Green’s theorem does not apply to the first curve because the components of $\vec{F}$ are not differentiable at $(0, 0)$. Direct computation gives that the integral over the first curve is equal to $2\pi$. The divergence form of Green’s theorem does apply to the second curve, since the singular point lies outside of the region bounded by the curve. Since the divergence is zero, the integral is 0. For the integral over the third curve, direct computation is hard, but the divergence form of Green’s theorem applies to the region lying between the first and third curves, which shows that the integrals over these two curves are equal. Thus the third integral must also be equal to $2\pi$. 

6. A surface $S$ in the shape of a helicoid is given by the parametric equations
\[ \vec{r}(u,v) = u \cos(v) \vec{i} + u \sin(v) \vec{j} + v \vec{k}, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi. \]

Suppose that the density of the surface at the point $(x,y,z)$ is given by $\rho(x,y,z) = [1 + x^2 + y^2]^{1/2}$. Find the total mass of the surface, that is, compute the surface integral
\[ \iint_{S} \rho(x,y,z) \, dS. \]

There is no alternative to direct computation. Using the parametrization
\[ \iint \rho \, dS = \int_{0}^{2\pi} \int_{0}^{1} \sqrt{1 + u^2 \sqrt{1 + u^2}} \, du \, dv = \frac{8\pi}{3}. \]

In the integral, the first square root factor comes from evaluation of $\rho(\vec{r}(u,v))$ while the second is $|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|$.

7. Let $\vec{F} = xy \vec{i} + yz \vec{j} + zx \vec{k}$ and let $P$ be the triangular region in space with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let $C$ be the boundary of $P$ with the positive orientation being counterclockwise as viewed from above. Compute the line integral $\int_{C} \vec{F} \cdot d\vec{r}$.

Using Stokes' theorem gives that the line integral is equal to the flux of the curl of the vector field across the triangle. Compute the curl, and the flux integral. The result is $-\frac{1}{2}$.

8. Let $\vec{F} = x^2 \vec{i} + xy \vec{j} + z \vec{k}$ and let $E$ be the three dimensional solid region that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$. Let $S$ be the boundary of $E$. Compute the total flux of the vector field $\vec{F}$ out of $E$ across the surface $S$.

The divergence theorem applies, and the flux integral is equal to the integral of the divergence of the region bounded above by the plane and below by the paraboloid. Converting the integral to cylindrical coordinates and integrating yields $8\pi$. 