Teaching Calculus Coherently

Tevian Dray

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian
co-he-rent:
logically or aesthetically ordered

cal-cu-lus:
a method of computation *in a special notation*

differential calculus:
a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*
Mathematics vs. Physics

Mathematics

Physics
Mathematics vs. Physics

[Diagram showing a divide between Mathematics and Physics]
Suppose the temperature on a rectangular slab of metal is given by

\[ T(x, y) = k(x^2 + y^2) \]

where \( k \) is a constant. What is \( T(r, \theta) \)?

**A:** \( T(r, \theta) = kr^2 \)

**B:** \( T(r, \theta) = k(r^2 + \theta^2) \)
What are Functions?

**MATH**

\[ T = f(x, y) = k(x^2 + y^2) \]
\[ T = g(r, \theta) = kr^2 \]

**PHYSICS**

\[ T = T(x, y) = k(x^2 + y^2) \]
\[ T = T(r, \theta) = kr^2 \]

Two disciplines separated by a common language...
Mathematics vs. Physics

- Physics is about things.
- Physicists can’t change the problem.

- Mathematicians do algebra.
- Physicists do geometry.
Projection:

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \]

\[ \vec{u} \cdot \vec{v} = u_x v_x + u_y v_y \]

Law of Cosines:

\[ (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2 \vec{u} \cdot \vec{v} \]

\[ |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}| |\vec{v}| \cos \theta \]

Addition Formulas:

\[ \vec{u} = \cos \alpha \hat{i} + \sin \alpha \hat{j} \]

\[ \vec{v} = \cos \beta \hat{i} + \sin \beta \hat{j} \]

\[ \vec{u} \cdot \vec{v} = \cos(\alpha - \beta) \]
Flux is the total amount of electric field through a given area.

\[ \Phi = \int \vec{E} \cdot d\vec{a} \]

\[ \sum \text{over all rectangles} \]
Differentials

\[ d(u + cv) = du + c \, dv \]
\[ d(uv) = u \, dv + v \, du \]
\[ d(u^n) = nu^{n-1} \, du \]
\[ d(e^u) = e^u \, du \]
\[ d(\sin u) = \cos u \, du \]
\[ d(\cos u) = -\sin u \, du \]
\[ d(\ln u) = \frac{1}{u} \, du \]
Derivatives:

\[
\frac{d}{du} \sin u = \frac{d}{du} \sin u = \cos u
\]

Chain rule:

\[
\frac{d}{dx} \sin u = \frac{d}{dx} \sin u = \frac{d}{du} \sin u \frac{du}{dx} = \cos u \frac{du}{dx}
\]

Inverse functions:

\[
\frac{d}{du} \ln u = \frac{d}{du} q = \frac{dq}{du} = \frac{1}{du/dq} = \frac{1}{de^q/dq} = \frac{1}{e^q} = \frac{1}{u}
\]
Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

“Zap equations with $d$”
The central idea in calculus is not the limit.
The central idea of derivatives is not slope.
The central idea of integrals is not area.
The central idea of curves and surfaces is not parameterization.
The central representation of a function is not its graph.

The central idea in calculus is the differential.
The central idea of derivatives is rate of change.
The central idea of integrals is total amount.
The central idea of curves and surfaces is “use what you know”.
The central representation of a function is data attached to the domain.
The Vector Calculus Bridge Project

- **Differentials** (*Use what you know!*)
- **Multiple representations**
- **Symmetry** (*adapted bases, coordinates*)
- **Geometry** (*vectors, div, grad, curl*)

- Small group activities
- Instructor’s guide (in preparation)

http://www.math.oregonstate.edu/bridge
Vector Differentials

\[ d\vec{r} = dx \hat{i} + dy \hat{j} \]

\[ d\vec{r} = dr \hat{r} + r d\phi \hat{\phi} \]

<table>
<thead>
<tr>
<th>Expression</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ds)</td>
<td>(</td>
</tr>
<tr>
<td>(d\vec{A})</td>
<td>(d\vec{r}_1 \times d\vec{r}_2)</td>
</tr>
<tr>
<td>(dA)</td>
<td>(</td>
</tr>
<tr>
<td>(dV)</td>
<td>((d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3)</td>
</tr>
<tr>
<td>(df)</td>
<td>(\vec{\nabla}f \cdot d\vec{r})</td>
</tr>
</tbody>
</table>
Roles

**Task Master:** The task master ensures that the group completes all of the parts of the work.
“Part 1 says that we must ... . How shall we do it?”
“What you had for lunch doesn’t seem relevant. Can we get back to the main question?”

**Cynic:** The cynic questions everything the group does and ensures that everyone in the group understands what is going on.
“Why are we doing it this way?”
“Wouldn’t it be better if we did it this other way?”
“I don’t understand this part ..., let’s go over it again.”

**Recorder:** The recorder records the group’s answers.
“Do we agree that the answer to Part 3 is ... ?”
“I have written ... for Part 2. Is that what we want to say?”

**Reporter:** The reporter reports to the class.
Small Group Activities
Concept Tests
Examples

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ConcepTests

- conceptual multiple-choice questions
- Eric Mazur
- [http://math.arizona.edu/~lomen/conceptests.html](http://math.arizona.edu/~lomen/conceptests.html)

- Focus on a single concept
- Can’t be solved using equations
- Have good multiple-choice answers
- Are clearly worded
- Are of intermediate difficulty
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if there is no such thing as too many pizzas and too much cola?

Hughes Hallett et al (2005)
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if there is such a thing as too many pizzas or too much cola?

Hughes Hallett et al (2005)
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if *there is such a thing as too much cola but no such thing as too many pizzas*?

![Graph A](image1.png)

![Graph B](image2.png)

![Graph C](image3.png)

![Graph D](image4.png)

Hughes Hallett et al (2005)
Which of the graphs below could represent the derivative of the function graphed at the right?

A B C D

Hughes Hallett et al (2005)
Which of the graphs below could represent the function whose derivative is graphed at the right?

A  B  C  D

Hughes Hallett et al (2005)
Calculus Working Group

- Weekly meetings
- Faculty, Instructors, GTAs
- Coordinate schedule
- Plan and discuss labs
- Discuss pedagogy
Calculus Concept Inventory

- pretest/posttest
- measures conceptual understanding
- Jerome Epstein
- modeled on Force Concept Inventory

Example
(Deleted)
Normalized Gain

\[
\text{normalized gain} = \frac{\text{gain}}{\text{possible gain}}
\]

- Traditional lectures: 15–20%
- Active engagement: 30%

OSU:
- 9 sections under 20%
- 2 sections @ 30%
- Made heavy use of ConcepTests
- Wasn’t mine...
SUMMARY

Active engagement is essential.

Concepts matter.

Coherence is nice.

I took this class a year ago, and I still remember all of it...