Teaching Calculus Coherently

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co-he-rent:

logically or aesthetically ordered

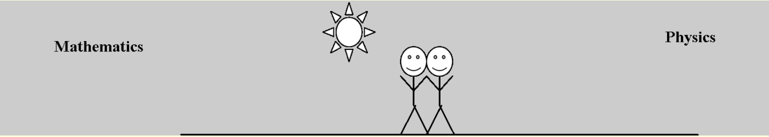
cal-cu-lus:

a method of computation *in a special notation*

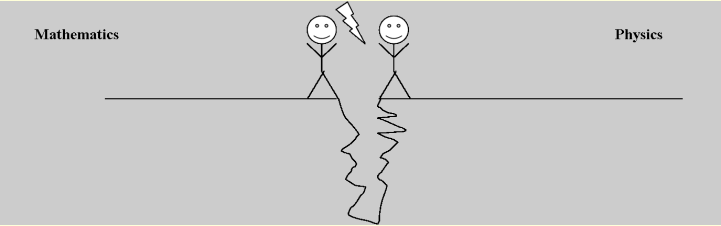
differential calculus:

a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*

Mathematics vs. Physics



Mathematics vs. Physics

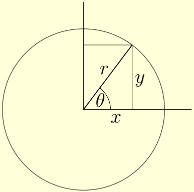


What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

$$T(x, y) = k(x^2 + y^2)$$

where k is a constant. What is $T(r, \theta)$?



A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$

What are Functions?

MATH

$$\begin{aligned} T &= f(x,y) = k(x^2+y^2) \\ T &= g(r,\theta) = kr^2 \end{aligned}$$

PHYSICS

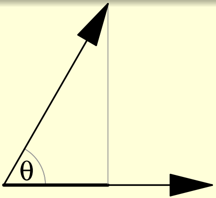
$$\begin{aligned} T &= T(x,y) = k(x^2+y^2) \\ T &= T(r,\theta) = kr^2 \end{aligned}$$

Two disciplines separated by a common language...

Mathematics vs. Physics

- **Physics is about things.**
- **Physicists can't change the problem.**

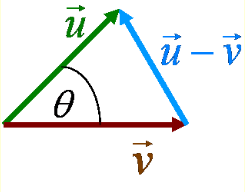
- **Mathematicians do algebra.**
- **Physicists do geometry.**



Projection:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$

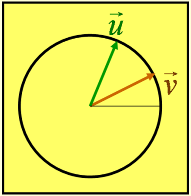
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_x v_x + u_y v_y$$



Law of Cosines:

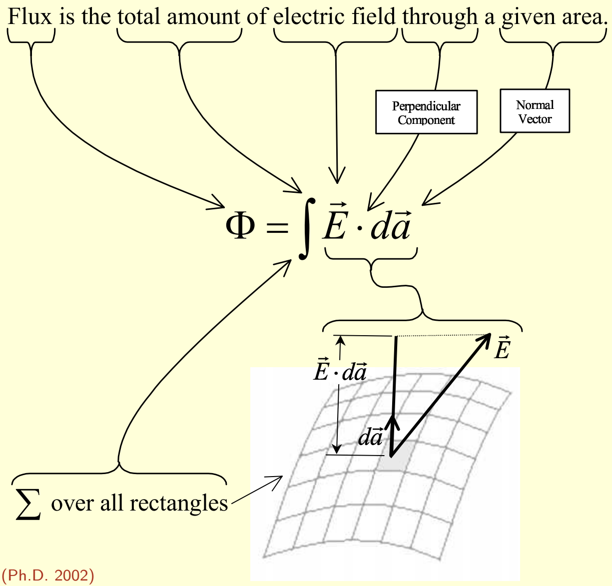
$$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} - \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - 2\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$$

$$|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2 = |\vec{\mathbf{u}}|^2 + |\vec{\mathbf{v}}|^2 - 2|\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$$



Addition Formulas:

$$\vec{\mathbf{u}} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$
$$\vec{\mathbf{v}} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \cos(\alpha - \beta)$$



Kerry Browne (Ph.D. 2002)

Differentials

$$d(u + cv) = du + c\,dv$$
$$d(uv) = u\,dv + v\,du$$

$$d\left(u^n\right) = nu^{n-1}\,du$$
$$d\left(e^u\right) = e^u\,du$$
$$d(\sin u) = \cos u\,du$$
$$d(\cos u) = -\sin u\,du$$
$$d(\ln u) = \frac{1}{u}\,du$$

Derivatives

Derivatives:

$$\frac{d}{du}\sin u = \frac{d\sin u}{du} = \cos u$$

Chain rule:

$$\frac{d}{dx}\sin u = \frac{d\sin u}{dx} = \frac{d\sin u}{du}\,\frac{du}{dx} = \cos u\,\frac{du}{dx}$$

Inverse functions:

$$\frac{d}{du}\ln u = \frac{d}{du}\,q = \frac{dq}{du} = \frac{1}{du/dq} = \frac{1}{de^q/dq} = \frac{1}{e^q} = \frac{1}{u}$$

Derivatives

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with d"

A Radical View of Calculus

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.

- The central idea in calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is "use what you know".
- The central representation of a function is data attached to the domain.

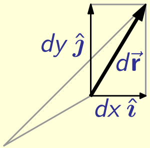
The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

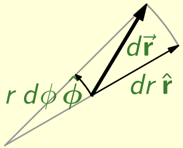
- Small group activities
- Instructor's guide (in preparation)

`http://www.math.oregonstate.edu/bridge`

Vector Differentials



$d\vec{\mathbf{r}} = dx\,\hat{\imath} + dy\,\hat{\jmath}$



$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\phi}$

$$
\begin{aligned}
ds &= |d\vec{\mathbf{r}}| \\
d\vec{\mathbf{A}} &= d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2 \\
dA &= |d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2| \\
dV &= (d\vec{\mathbf{r}}_1 \times d\vec{\mathbf{r}}_2) \cdot d\vec{\mathbf{r}}_3 \\
df &= \vec{\nabla} f \cdot d\vec{\mathbf{r}}
\end{aligned}
$$

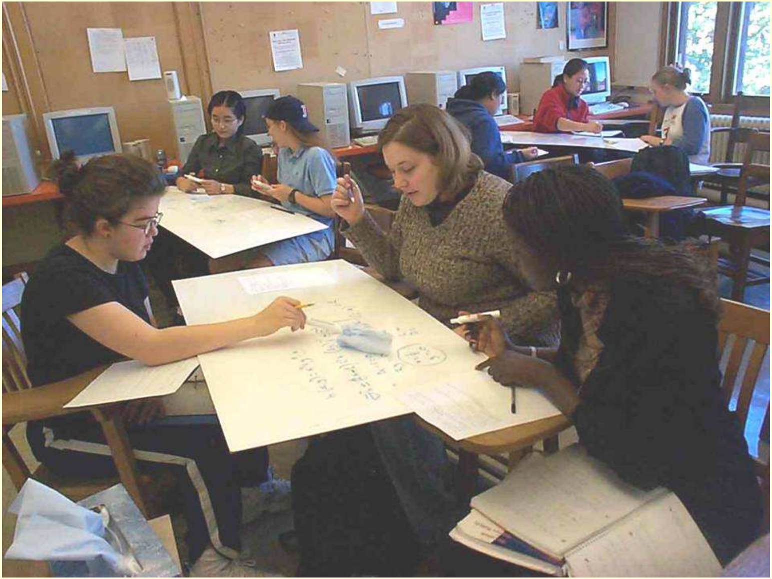
Roles

Task Master: *The task master ensures that the group completes all of the parts of the work.*
"Part 1 says that we must ... . How shall we do it?"
"What you had for lunch doesn't seem relevant. Can we get back to the main question?"

Cynic: *The cynic questions everything the group does and ensures that everyone in the group understands what is going on.*
"Why are we doing it this way?"
"Wouldn't it be better if we did it this other way?"
"I don't understand this part ..., let's go over it again."

Recorder: *The recorder records the group's answers.*
"Do we agree that the answer to Part 3 is ... ?"
"I have written ... for Part 2. Is that what we want to say?"

Reporter: *The reporter reports to the class.*

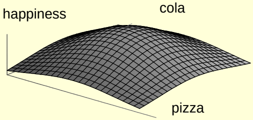


ConcepTests

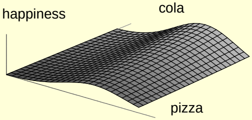
- conceptual multiple-choice questions
- Eric Mazur
- http://math.arizona.edu/~lomen/conceptests.html

- Focus on a single concept
- Can't be solved using equations
- Have good multiple-choice answers
- Are clearly worded
- Are of intermediate difficulty

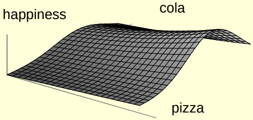
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if *there is no such thing as too many pizzas and too much cola*?



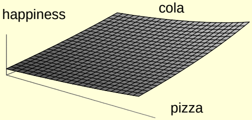
A



B



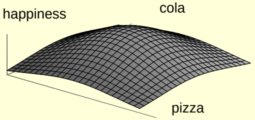
C



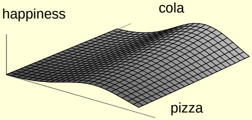
D

Hughes Hallett et al (2005)

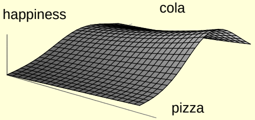
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if *there is such a thing as too many pizzas or too much cola*?



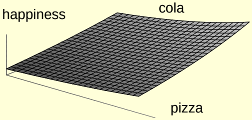
A



B



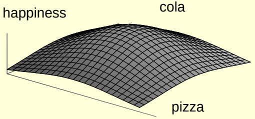
C



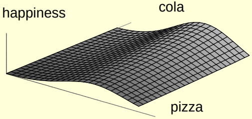
D

Hughes Hallett et al (2005)

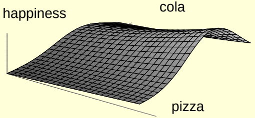
You like pizza and you like cola. Which of the graphs below represents your happiness as a function of how many pizzas and how much cola you have if *there is such a thing as too much cola but no such thing as too many pizzas*?



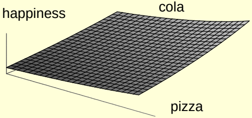
A



B



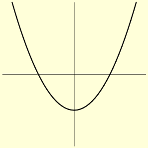
C

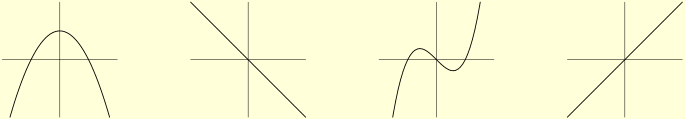


D

Hughes Hallett et al (2005)

Which of the graphs below could represent the derivative of the function graphed at the right?

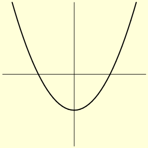


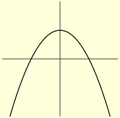


A B C D

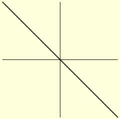
Hughes Hallett et al (2005)

Which of the graphs below could represent the function *whose derivative* is graphed at the right?

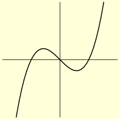




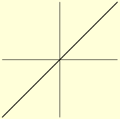
A



B



C



D

Hughes Hallett et al (2005)

Calculus Working Group

- Weekly meetings
- Faculty, Instructors, GTAs
- Coordinate schedule
- Plan and discuss labs
- Discuss pedagogy

Calculus Concept Inventory

- pretest/posttest
- measures conceptual understanding
- Jerome Epstein
- modeled on Force Concept Inventory

Example

(Deleted)

Normalized Gain

$$\text{normalized gain} = \frac{\text{gain}}{\text{possible gain}}$$

- Traditional lectures: 15–20%
- Active engagement: 30%

OSU:

- 9 sections under 20%
- 2 sections @ 30%
- Made heavy use of ConcepTests
- Wasn't mine...

SUMMARY

Active engagement is essential.

Concepts matter.

Coherence is nice.

I took this class a year ago, and I still remember all of it...