TEACHING CALCULUS COHERENTLY

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I: Coherent Calculus
II: Vector Calculus Bridge Project
III: Concept Tests
IV: Calculus Concept Inventory
Coherent Calculus

co-her-ent:
logically or aesthetically ordered

cal-cu-lus:
a method of computation in a special notation
A Radical View of Calculus

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.
A Radical View of Calculus

• The central idea in calculus is the differential.
• The central idea of derivatives is rate of change.
• The central idea of integrals is total amount.
• The central idea of curves and surfaces is “use what you know”.
• The central representation of a function is data attached to the domain.
Coherent Calculus

**coherent:**
logically or aesthetically ordered

**calculus:**
a method of computation in a special notation

**differential calculus:**
a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives and differentials
Differentials

\[ df = \frac{df}{dx} \, dx \]

- Shorthand for limit argument
- Nonstandard analysis (hyperreal numbers)
- Smooth infinitesimal analysis
- Differential forms

“Differentials of variables”
not
“Differentials of functions”!
Differentials

\[ d(u + cv) = du + c \, dv \]
\[ d(uv) = u \, dv + v \, du \]
\[ d\left( u^n \right) = nu^{n-1} \, du \]
\[ d\left( e^u \right) = e^u \, du \]
\[ d(\sin u) = \cos u \, du \]
\[ d(\cos u) = -\sin u \, du \]
\[ d(\ln u) = \frac{1}{u} \, du \]
Derivatives:

\[ \frac{d}{du} \sin u = \frac{d}{du} \sin u = \cos u \]

Chain rule:

\[ \frac{d}{dx} \sin u = \frac{d}{dx} \sin u = \frac{d}{du} \sin u \cdot \frac{du}{dx} = \cos u \frac{du}{dx} \]

Inverse functions:

\[ \frac{d}{du} \ln u = \frac{d}{du} q = \frac{dq}{du} = \frac{1}{du/dq} = \frac{1}{de^q/dq} = \frac{1}{e^q} = \frac{1}{u} \]
Derivatives

Instead of:

• chain rule
• related rates
• implicit differentiation
• derivatives of inverse functions
• difficulties of interpretation (units!)

One coherent idea:

“Zap equations with $d$”
Vector Calculus Bridge Project

- **Differentials** (*Use what you know!*)
- **Multiple representations**
- **Symmetry** (*adapted bases, coordinates*)
- **Geometry** (*vectors, div, grad, curl*)

- Small group activities
- Instructor’s guide (in preparation)

http://www.math.oregonstate.edu/bridge
Vector Differentials \((d\vec{r})\)

\[
\begin{align*}
d\vec{r} &= dx \hat{i} + dy \hat{j} \\
d\vec{r} &= dr \hat{r} + r d\theta \hat{\theta}
\end{align*}
\]

- \(ds = |d\vec{r}|\)
- \(d\vec{S} = d\vec{r}_1 \times d\vec{r}_2\)
- \(dS = |d\vec{r}_1 \times d\vec{r}_2|\)
- \(dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3\)
- \(df = \vec{\nabla} f \cdot d\vec{r}\)
ConcepTests

- conceptual multiple-choice questions
- Eric Mazur
- http://math.arizona.edu/~lomen/conceptests.html
- Focus on a single concept
- Can’t be solved using equations
- Have good multiple-choice answers
- Are clearly worded
- Are of intermediate difficulty
ConcepTest Example

Which of the graphs below could represent the derivative of the function graphed at the right?

A  B  C  D
Calculus Concept Inventory

• pretest/posttest
• measures conceptual understanding
• Jerome Epstein
• modeled on Force Concept Inventory

Example:
If a number very close to zero is divided by another number very close (but not equal) to zero, the result
(a) must be a number very close to zero
(b) must be a number close to one
(c) could be any number
(d) might not be a number at all.
Normalized Gain

\[
\text{normalized gain} = \frac{\text{gain}}{\text{possible gain}}
\]

- Traditional lectures: 15–20%
- Active engagement: 30%

**OSU:**
- 7 sections under 20%
- 1 section @ 30%
- Made heavy use of ConcepTests
- Wasn’t mine...
SUMMARY

Active engagement is essential.

Concepts matter.

Coherence is nice.