Bridging the Gap between Mathematics and Physics

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Mathematics vs. Physics

Mathematics

Physics
Bridging the Gap between Mathematics and Physics
What are Functions?

Suppose the temperature on a rectangular slab of metal is given by

\[ T(x, y) = k(x^2 + y^2) \]

where \( k \) is a constant. What is \( T(r, \theta) \)?

A: \( T(r, \theta) = kr^2 \)

B: \( T(r, \theta) = k(r^2 + \theta^2) \)
What are Functions?

**MATH**

\[
T = f(x, y) = k(x^2 + y^2)
\]

\[
T = g(r, \theta) = kr^2
\]

**PHYSICS**

\[
T = T(x, y) = k(x^2 + y^2)
\]

\[
T = T(r, \theta) = kr^2
\]

Two disciplines separated by a common language...
Differential Geometry!

\[ T(x, y) \iff T \circ (x, y)^{-1} \]

\[ T(r, \theta) \iff T \circ (r, \theta)^{-1} \]

Two disciplines separated by a common language...

physical quantities ≠ functions
Mathematics vs. Physics

- Physics is about things.
- Physicists can’t change the problem.

- Mathematicians do algebra.
- Physicists do geometry.
Write down something that you know about the dot product.

Geometry:
\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \]

Algebra:
\[ \vec{u} \cdot \vec{v} = u_x v_x + u_y v_y \]
Projection:
\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \]

\[ \vec{u} \cdot \vec{v} = u_x v_x + u_y v_y \]

Law of Cosines:
\[ (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2 \vec{u} \cdot \vec{v} \]

\[ |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2 |\vec{u}| |\vec{v}| \cos \theta \]

Addition Formulas:
\[ \vec{u} = \cos \alpha \hat{i} + \sin \alpha \hat{j} \]
\[ \vec{v} = \cos \beta \hat{i} + \sin \beta \hat{j} \]
\[ \vec{u} \cdot \vec{v} = \cos(\alpha - \beta) \]
Find the angle between the diagonal of a cube and the diagonal of one of its faces.

**Algebra:**

\[ \vec{u} = \hat{i} + \hat{j} + \hat{k} \]
\[ \vec{v} = \hat{i} + \hat{k} \]

\[ \implies \vec{u} \cdot \vec{v} = 2 \]

**Geometry:**

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = \sqrt{3} \sqrt{2} \cos \theta \]

Need both!
Flux is the total amount of electric field through a given area.

\[ \Phi = \int \vec{E} \cdot d\vec{a} \]

\[ \sum \text{ over all rectangles} \]
CUPM

MAA Committee on the Undergraduate Program in Mathematics
Curriculum Guide

CRAFTY

Subcommittee on Curriculum Renewal Across the First Two Years
Voices of the Partner Disciplines
http://www.maa.org/cupm/crafty
The Vector Calculus Bridge Project

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

- Small group activities
- Instructor’s guide
- Online text *(http://www.math.oregonstate.edu/BridgeBook)*

`http://www.math.oregonstate.edu/bridge`
The Vector Calculus Bridge Project

Bridge Project homepage hits in 2009
Mathematicians’ Line Integrals

- Start with Theory

\[
\int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \hat{T} \, ds
\]

\[
= \int \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| \, dt
\]

\[
= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \ldots = \int P \, dx + Q \, dy + R \, dz
\]

- Do examples starting from next-to-last line

Need parameterization \( \vec{r} = \vec{r}(t) \)
Physicists’ Line Integrals

- Theory
  - Chop up curve into little pieces $d\vec{r}$.
  - Add up components of $\vec{F}$ parallel to curve (times length of $d\vec{r}$)
- Do examples directly from $\vec{F} \cdot d\vec{r}$

Need $d\vec{r}$ along curve
\[ \vec{F}(x, y) = \frac{-y \hat{i} + x \hat{j}}{x^2 + y^2} \quad \vec{r} = x \hat{i} + y \hat{j} \]

\[ x = 2 \cos \theta \]
\[ y = 2 \sin \theta \]

\[ \int \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(x(\theta), y(\theta)) \cdot \vec{r}'(x(\theta), y(\theta)) \, d\theta \]
\[ = \int_0^{\pi/2} \frac{1}{2} (-\sin \theta \hat{i} + \cos \theta \hat{j}) \cdot 2(-\sin \theta \hat{i} + \cos \theta \hat{j}) \, d\theta \]
\[ = \ldots = \frac{\pi}{2} \]
\[ \vec{F} = \frac{\hat{\phi}}{r} \]

\[ d\vec{r} = r \, d\phi \, \hat{\phi} \]

**I:** \( |\vec{F}| = \text{const}; \vec{F} \parallel d\vec{r} \implies \int \vec{F} \cdot d\vec{r} = \frac{1}{2}(2 \, \frac{\pi}{2}) \)

**II:** Do the dot product

\[ \int \vec{F} \cdot d\vec{r} = \int_{0}^{\frac{\pi}{2}} \frac{\phi}{2} \cdot 2 \, d\phi \, \hat{\phi} = \int_{0}^{\frac{\pi}{2}} d\phi = \frac{\pi}{2} \]
Vector Differentials

\[ d\mathbf{r} = dx \hat{i} + dy \hat{j} \]

\[ d\mathbf{r} = dr \hat{r} + r d\phi \hat{\phi} \]

\[
\begin{align*}
  ds &= |d\mathbf{r}| \\
  d\mathbf{A} &= d\mathbf{r}_1 \times d\mathbf{r}_2 \\
  dA &= |d\mathbf{r}_1 \times d\mathbf{r}_2| \\
  dV &= (d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3
\end{align*}
\]
Coherent Calculus

co-he-re-nt:
logically or aesthetically ordered

cal-cu-lus:
a method of computation *in a special notation*

differential calculus:
a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives *and differentials*
Derivatives

**Instead of:**
- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

**One coherent idea:**

“Zap equations with $d$”

Tevian Dray & Corinne A. Manogue,
*Putting Differentials Back into Calculus*,
The central idea in calculus is not the limit.
The central idea of derivatives is not slope.
The central idea of integrals is not area.
The central idea of curves and surfaces is not parameterization.
The central representation of a function is not its graph.

The central idea in calculus is the differential.
The central idea of derivatives is rate of change.
The central idea of integrals is total amount.
The central idea of curves and surfaces is “use what you know”.
The central representation of a function is data attached to the domain.
I took this class a year ago, and I still remember all of it...