Interacting with Partial Derivatives

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(& Corinne Manogue, Liz Gire, OSUPER)
The Paradigms in Physics Project

- Complete redesign of physics major – 20 new courses
- Junior-year “paradigms” designed around common themes.
- Senior-year “capstones” finish traditional disciplinary content.
- 24 years of continuous NSF funding.
- Living curriculum: *Monthly curriculum meetings for 25 years!*
- Paradigms 2.0 implemented in 2017.
- Active engagement: *(300+ documented activities!)*

http://physics.oregonstate.edu/portfolioswiki
Suppose the temperature on a rectangular slab of metal is given by

\[ T(x, y) = k(x^2 + y^2) \]

where \( k \) is a constant. What is \( T(r, \theta) \)?

A: \[ T(r, \theta) = kr^2 \]

B: \[ T(r, \theta) = k(r^2 + \theta^2) \]

Mathematics and Physics are two disciplines separated by a common language!
Which vector field is conservative?
Which vector field has nonzero curl?
Which vector field has nonzero divergence?

Which vector field could represent an electric field? a magnetic field?
Suppose you are standing on a hill. You have a topographic map, which uses rectangular coordinates \((x, y)\) measured in miles. Your global positioning system says your present location is at one of the points shown. Your guidebook tells you that the height \(h\) of the hill in feet above sea level is given by

\[
h = a - bx^2 - cy^2
\]

where \(a = 5000 \text{ ft}\), \(b = 30 \frac{\text{ft}}{\text{mi}^2}\), and \(c = 10 \frac{\text{ft}}{\text{mi}^2}\).
Kinesthetic Activity

Stand up and close your eyes. Hold out your right arm in the direction of the gradient where you are standing.
Each surface is dry-erasable, as are the matching contour maps.

*Raising Calculus to the Surface* (Aaron Wangberg)
*Raising Physics to the Surface* (+ Liz Gire, Robyn Wangberg)

http://raisingcalculus.winona.edu
State Variables:

\[ T = \text{temperature} \]
\[ S = \text{entropy} \]
\[ p = \text{pressure} \]
\[ V = \text{volume} \]

First Law:
\[ dU = T \, dS - p \, dV \]
\((U = \text{internal energy})\)

- Compressibility = \(-\frac{1}{V} \frac{\partial V}{\partial p}\)
- Design an experiment to measure compressibility.
What is this material? 

Isothermal: \[-\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T \]

Isoentropic: \[-\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_S \]

What are the independent variables? 

Can not hold "everything else" fixed!

\[ \left( \frac{\partial p}{\partial T} \right)_V \neq \left( \frac{\partial p}{\partial T} \right)_S. \]
- Developed for junior-level thermodynamics course
- Two positions, $x_i$, two string tensions (masses), $F_i$.
- “Find $\frac{\partial x}{\partial F}$.”
- Idea: Measure $\Delta x$, $\Delta F$; divide.
- Mathematicians: “That’s not a derivative!”

Roundy et al., *Experts’ Understanding of Partial Derivatives Using the Partial Derivative Machine*, PERC 2014
Math: ∃ “bright line” between average rate of change and instantaneous rate of change.
(Such averages are used to approximate derivatives.)

Physics: “Average” refers to secant lines, not (good) approximations to tangent lines.

Move the bright line!

Thick Derivatives!

(Derivatives are fundamentally ratios of small changes, not limits.)

[Dray, AMS Blog on Education, 5/31/16]
Learning Progression

Learning Progression for Partial Derivatives

- Successively more sophisticated ways of thinking about a topic.
- Sequences supported by research on learner’s ideas and skills.
- Lower anchor grounded in students’ prior ideas and skills.
- Upper anchor grounded in knowledge and practices of experts.

Duschle et al., NRC, 2007; Plummer, 2012; Sikorski et al., 2009, 2010
Manogue, Dray, Emigh, Gire, & Roundy, PERC 2017
Flux is the total amount of electric field through a given area.

\[ \Phi = \int \vec{E} \cdot d\vec{a} \]

\[ \sum \text{over all rectangles} \]

Kerry Browne (Ph.D. 2002)
### Extended Theoretical Framework for Concept of Derivative

<table>
<thead>
<tr>
<th>Process-object layer</th>
<th>Graphical</th>
<th>Verbal</th>
<th>Symbolic</th>
<th>Numerical</th>
<th>Physical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Slope</td>
<td>Rate of Change</td>
<td>Difference Quotient</td>
<td>Ratio of Changes</td>
<td>Measurement</td>
</tr>
<tr>
<td><strong>Ratio</strong></td>
<td><img src="#" alt="Graph" /></td>
<td>“avg. rate of change”</td>
<td>[ \frac{f(x + \Delta x) - f(x)}{\Delta x} ]</td>
<td>numerically</td>
<td><img src="#" alt="Measurement" /></td>
</tr>
<tr>
<td><strong>Limit</strong></td>
<td><img src="#" alt="Graph" /></td>
<td>“inst. rate of change”</td>
<td>[ \lim_{\Delta x \to 0} \ldots ]</td>
<td>...with ( \Delta x ) small</td>
<td><img src="#" alt="Measurement" /></td>
</tr>
<tr>
<td><strong>Function</strong></td>
<td><img src="#" alt="Graph" /></td>
<td>“…at any point/time”</td>
<td>[ f'(x) = \ldots ]</td>
<td>depends on ( x )</td>
<td>tedious repetition</td>
</tr>
</tbody>
</table>

No entry for symbolic differentiation!!

Differentials

Does $\frac{df}{dx}$ mean “$f'(x)$” or “$df$ over $dx$”? 

$$d(u^2) = 2u \, du$$
$$d(\sin u) = \cos u \, du$$

Instead of:

- chain rule
- related rates
- implicit differentiation
- derivatives of inverse functions
- difficulties of interpretation (units!)

One coherent idea:

"Zap equations with $d$"

(infinitesimal reasoning)

Dray & Manogue, CMJ 34, 283–290 (2003); CMJ 41, 90–100 (2010).
Vector Calculus

**Vector calculus is about one coherent concept:**

**Infinitesimal Displacement**

\[ d\mathbf{r} = dx \mathbf{\hat{x}} + dy \mathbf{\hat{y}} \]

\[ d\mathbf{r} = dr \mathbf{\hat{r}} + r d\phi \mathbf{\hat{\phi}} \]

\[
\begin{align*}
    ds &= |d\mathbf{r}|
    \\
    d\mathbf{A} &= d\mathbf{r}_1 \times d\mathbf{r}_2 \\
    dA &= |d\mathbf{r}_1 \times d\mathbf{r}_2|
    \\
    dV &= (d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3
\end{align*}
\]
SUMMARY

- Use multiple representations, including geometry, measurement, numerical data;
- Always ask both “With respect to what,” and “With what held constant.”

[Links]
- http://math.oregonstate.edu/bridge
- http://math.oregonstate.edu/BridgeBook
- http://physics.oregonstate.edu/portfolioswiki
- http://osuper.science.oregonstate.edu
- http://raisingcalculus.winona.edu