MTH 675 – ASSIGNMENT 2

Due: 10/24/2008

1. Surfaces of revolution: Let $c : [0,1] \longrightarrow \mathbb{R}^2$ be a plane curve parametrized by arc length. Set $c(u) = (r(u), z(u))$ and assume that $r > 0$. If the image of c is a submanifold of $\mathbb{R}^2$, then the surface S obtained by rotation of c around the z-axis is a submanifold of $\mathbb{R}^3$.

(a) Find local coordinate charts of S.

(b) Equip S with the metric g induced by the Euclidean metric of $\mathbb{R}^3$, write this metric in the local coordinate system of (a) and compute the length of a parallel $(u = const)$.

(c) Which conditions on the curve c do you need in order to obtain a sphere, i.e. $S = S^2$? What is the metric in the local coordinates of (a) of the sphere?

2. Let $C \subset \mathbb{R}^3$ be the catenoid: C is the surface of revolution generated by rotation around the z-axis of the curve $x = \cosh(z)$. Let H be the helicoid: H is generated by the straight lines which are parallel to the xy-plane and meet both the z-axis and the helix $t \mapsto (cost, sint, t)$.

(a) Show that H and C are submanifolds of $\mathbb{R}^3$, and give a "natural" parametrization for both.

(b) If g is the Euclidean metric of $\mathbb{R}^3$, give the expressions of $g|H$ and $g|C$ in the parametrizaions defined in (a), and show that C and H are locally isometric. Are they globally isometric?

Note that it is not possible to guess from the embeddings that C and H are locally the "same" from a Riemannian geometry point of view. For example, C does not contain any straight line, not even a segment of a line: the local isometries between C and H do not come from isometries of the ambient space.

3. (a) Show that the curves with "constant tangent" (i. e. such that the segment of the tangent line between the point of tangency and some line in the plane, which does not meet the curve, is constantly equal to one) are obtained by horizontal translation of the curve $t \mapsto (t - \tanh t, \frac{1}{\cosh t})$.

(b) The *pseudo sphere* is the surface generated by rotation around the x-axis of the curve defined in (a). Show that the pseudo sphere, with the circle generated by the cusps taken away, is locally isometric to $(\mathbb{R}^2, \frac{dx^2+dy^2}{y^2})$.