Tomography and Sampling Theory

Adel Faridani

Department of Mathematics
Oregon State University
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The Divergent Beam X-Ray Transform

\[ D_z f(\omega) = \int_0^\infty f(z + t\omega) \, dt, \quad z \in \mathbb{R}^n, \ \omega \in S^{n-1}. \]

Parametrization in 2D:
\[ D_z f(\omega) = D f(\beta, \alpha), \quad z = (r \cos \beta, r \sin \beta) \]
The Divergent Beam X-Ray Transform

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Symmetry:

\[ Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha). \]
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The goal of tomography

The goal is to reconstruct \( f(x) \) from finitely many measurements of line integrals \( D f(\beta, \alpha) \).
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We will use Shannon Sampling Theory to address this question. The answer depends on the frequency content of the data function $Df$, that is both on the size and the shape of the support of the Fourier transform of $Df$. 
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Change of variables:

$$g(s, t) = Df(\beta, \alpha), \quad (s, t) \in [0, 1)^2 = \mathbb{T}^2.$$
The essential support of $\hat{g}$

A-priori information used: $f$ has support in unit disk and ’essential bandwidth’ $b = 100$. 
Sampling lattices

The data is a function $g$ on some group $G$, here $G = T^2$. We measure $g$ on a discrete subgroup (lattice) $L$. 
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Characterization of sampling lattices

$P = \text{number of equidistant source positions}$

$Q = \text{number of equidistant rays measured for each source position}$

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$L = \{(s_j, t_{jl}) : s_j = j \Delta s, \Delta s = 1/P,\]

\[ t_{jl} = l \Delta t + \delta_j, \Delta t = 1/Q, \delta_j = jN/(PQ) \]

\[ j = 0, \ldots, P - 1, l = 0, \ldots, Q - 1 \}.$
Let \( g(x) \) be such that \( \hat{g}(\xi) = 0 \) for \( \xi \notin K = [-b, b) \).
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If \( 0 < h \leq \pi/b \) then

\[
g(x) = \frac{hb}{\pi} \sum_{l=-\infty}^{\infty} g(hl) \text{sinc}(b(x - hl)),
\]

where \( \text{sinc}(x) = (\sin x)/x \).
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- $g(x)$ is sampled on a subgroup $L = h\mathbb{Z}$ of $\mathbb{R}$. 

Classical Sampling Theorem on $\mathbb{R}$
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Note:

- $g(x)$ is sampled on a subgroup $L = h\mathbb{Z}$ of $\mathbb{R}$.

- $2b \text{sinc}(bx)$ is the inverse Fourier transform of the indicator function $\chi_K(\xi)$ of $K$.

$\chi_K(\xi) = 1$ for $\xi \in K$ and zero otherwise.
Motivation of the Sampling Theorem

Assume $\hat{g}$ is very small outside a set $K$. 
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Discrete Fourier Transform of measured data:

\[
\text{DFT}(g) = c \sum_{y \in L} g(y) e^{-2\pi i \langle y, \xi \rangle}
\]

\[
= \hat{g}(\xi) + \sum_{0 \neq \eta \in \mathbb{L}^\perp} \hat{g}(\xi + \eta)
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Hence \( \text{DFT}(g) \approx \hat{g}(\xi) \) if \( \xi \in K \) and \( \xi + \eta \notin K \), that is, if the translates \( K + \eta, \eta \in L^\perp \) are disjoint.
Interpolation

\[ Sg(x) = \text{IFT} [\chi_K \text{DFT}(g)] \]

\[ = c \sum_{y \in L} \tilde{\chi}_K(x - y) g(y) \]

\[ \chi_K = \text{indicator fct. of } K \]

\[ \tilde{\chi}_K = \text{IFT} [\chi_K] \]
Theorem 1 \( L = L(N, P, Q) \) a sampling lattice such that \( K + \eta, \eta \in L^\perp \) are disjoint. For \( z \in \mathbb{T}^2 \) define

\[
Sg(z) = \frac{1}{PQ} \sum_{y \in L} \tilde{\chi}_K(z - y)g(y).
\]

Then

\[
|g(z) - Sg(z)| \leq 2 \int_{\mathbb{Z}^2 \setminus K} |\hat{g}(\zeta)| \, d\zeta.
\]
Achieving Efficiency

1. Find lattices $L$ as sparse as possible so that the translates $K + \eta, \eta \in L^\perp$ are disjoint.

2. Exploit the symmetry relation

$$Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha).$$

This requires sampling theorems for sampling sets which are not lattices.
Choice of lattices

Find lattices $L$ as sparse as possible so that the translates $K + \eta$, $\eta \in L^\perp$ are disjoint.

Example: Standard lattice ($N=0$).

$$L_S(P, Q) = \{(j/P, l/Q) : j = 0, \ldots, P - 1 \atop l = 0, \ldots, Q - 1\}$$

$$L_S^\perp = \{(Pk, Qm) : k, m \in \mathbb{Z}\}$$
Translates \( K + \eta, \eta \in L_S^{\perp} \)

\[
N = 0, \quad P = 156, \quad Q = 600, \quad |L_S| = PQ = 93,600
\]
More efficient lattice

\[ N = 110, \quad P = 330, \quad Q = 200 \quad |L| = PQ = 66,000 \]
Reconstruction

Two basic strategies for reconstructing the function $f$ from samples of $Df$.

1. **Direct.** Reconstruct directly from the sampled data. (This is the only possibility for local tomography.) Need for error analysis of reconstruction algorithm.

2. **Interpolated.** First use sampling theorem to interpolate data onto a denser grid. Then reconstruct from the interpolated data.
Direct local reconstruction of $\Lambda f$


Parallel-beam. (F. & Ritman, 2000)
Direct vs. interpolated for fan-beam

\[ f(x) = \left(1 - 100 |x - x_0|^2\right)^3, \quad x_0 = (0.4, 0.7) \]

Theoretical explanation: Izen (2005)
Artifacts from undersampling

Left: Standard    Right: Efficient.
Top: $P$ too small
Bottom: $Q$ too large (!)
Undersampling by sampling more data

\[ N = 110, \quad P = 330, \quad Q = 240 > 200. \]
Extensions of the Sampling Theorem

Goal: Extend to sampling sets which are not subgroups but still retain group structure.

Periodic sampling: $S = \bigcup_{n=1}^{m} (x_n + \mathbb{L})$
Sampling set is invariant with respect to shifts by elements of $\mathbb{L}$. (Well understood; see, e.g., Kohlenberg (1954), F. (1994), Izen (2005) and many other authors.)
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- Non-periodic sampling: $S = \bigcup_{n=1}^{m} (x_n + L_n)$
  (More recent development. See, e.g., preprint by Behmard, F. & Walnut, 2005).
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- Of course, these are not the only extensions! (See, e.g., Aldroubi & Unser, Feichtinger & Gröchenig, Zayed, . . .)
Periodic sampling

If the translates $K + \eta$, $\eta \in L^\perp$ are not disjoint, decompose $K$ as

$$K = \bigcup_{l=1}^{L} K_l$$

where each $K_l$ consists of the points $\xi \in K$ overlapped by the same translates, i.e.,

$$\xi \in K_l \Leftrightarrow \xi \in K + \eta_j^{(l)}, \quad j = 0, \ldots, m_l - 1.$$ 

Let $m = \max_{l=1,\ldots,L} m_l$ and use $m$ shifted copies of $L$ as sampling set,

$$S = \bigcup_{n=0}^{m} (x_n + L).$$
Overlapping translates of $K$

$P = 156, \quad N = 0, \quad Q = 312$
Decomposition $K = \bigcup K_l$
Approximate $\hat{g}(\xi)$ for $\xi \in K_l$ by

$$\hat{g}(\xi) \simeq c \sum_{n=0}^{m-1} \beta_n^{(l)} \sum_{y \in L} g(x_n + y) e^{-2\pi i <y+x_n,\xi>}.$$ 

Determine the coefficients $\beta_n^{(l)}$ such that the approximation is exact if $\hat{g}$ vanishes outside $K$. This requires

$$\sum_{n=0}^{m-1} \beta_n^{(l)} = 1$$

$$\sum_{n=0}^{m-1} \beta_n^{(l)} e^{2\pi i <x_n,\eta_j^{(l)}} = 0, \quad j = 1, \ldots, m_l - 1$$
Reconstruction Formula

\[ Sg(x) = c \sum_{n=0}^{m-1} \sum_{y \in \mathbf{L}} g(x_n + y) k_n(x - x_n - y) \]

\[ k_n(z) = \sum_{l=1}^{L} \beta_n^{(l)} \tilde{\chi}_{K_l}(z) \]

\( \tilde{\chi}_{K_l}(z) \) is the inverse Fourier transform of the indicator function of \( K_l \).
Estimate for the Aliasing Error

\[ |Sg(z) - g(z)| \leq \gamma \int_{\mathbb{Z}^2 \setminus K} |\hat{g}(\zeta)| \, d\zeta \]

\[ \gamma = 1 + m \max_l \sum_{n=0}^{m-1} |\beta_n^{(l)}| \]
Applications of periodic sampling in CT

- Additional efficient 2D sampling schemes
- "Preferred pitch" in 3D helical CT
- Higher resolution in 2D fan-beam CT

Applications of non-periodic sampling

- Higher resolution in 2D fan-beam CT
Exploiting Symmetry

\[ Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha). \]

Standard lattice with constant detector shift:

\[ \mathbf{L}_S = \{ (\beta_j, \alpha_l) : \beta_j = \frac{2\pi j}{P}, \quad \alpha_l = \frac{\pi(l + \delta)}{Q}, \]

\[ j = 0, \ldots, P - 1, \quad l = -Q/2, \ldots, Q/2 - 1, \quad \delta \geq 0 \} \]

‘Reflected lattice’

\[ \mathbf{L}_R = \{ (\beta_j + 2\alpha_l + \pi, -\alpha_l) : (\beta_j, \alpha_l) \in \mathbf{L}_S \}. \]

\( \mathbf{L}_S \) and \( \mathbf{L}_R \) are cosets of two different subgroups.
Key observation by Izen et al. (2005)

$\mathbf{L}_S \cup \mathbf{L}_R$ is a union of $Q / \gcd(P, Q/2)$ shifted copies of the smaller lattice

$$L_P = \{(2\pi j / P, \pi l / \gcd(P, Q/2)),$$

$$j = 0, \ldots, P - 1, \ |l| \le \gcd(P, Q/2) \}$$

So the periodic sampling theorem can be applied!

(Izen, Rohler & Sastry (2005) used an alternative reconstruction method.)

Thus effective bandwidth $b$ can be doubled by only having to double $P$ but not $Q$. 

Use of periodic sampling

Periodic Sampling with 2 cosets. $l_2\text{err} = 0.024841$

$P=156, q=78, \delta=0.25, R=3, b=100, \rho=1, \tau=0.95, k=156, n_{ray}=0$
Standard (Direct) Reconstruction

Direct standard reconstruction

b=200, P=312, Q/2=q=300, alphamax=\pi/2
High-Resolution with periodic sampling

Periodic Sampling with 2 cosets

Mitchell (2005)
Gratton (2005) found an ingenious way to show that a non-periodic sampling theorem can be applied to the sampling set $L_S \cup L_R$ to exploit the symmetry.
Standard Rec. of Calibration Object

Real data.
High-Resolution Reconstruction

Challenge: Increased noise.
Possible remedy: Edge preserving denoising with TV-based algorithm (R. Hass, 2005).
Denoised Standard vs. Denoised High-R.
Standard vs. Denoised High-Resolution
Three dimensions

- Rich in potential sampling geometries. (Family of lines has 4 parameters. Need only a 3 parameter subfamily).

- Investigation of sampling issues has begun (see, e.g, Desbat and Grangeat (2004), Gratton (2005)) but very much remains to be done.
Conclusions

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- Both size and shape of the bandregion (spectrum) of $g$ matter.

- Awareness and proper use of a-priori information is important. Here we only assumed that $f$ has support in the unit disk and ’essential bandwidth’ $b$. 
Conclusions, continued

Reconstruction can be done directly from the measured data or after interpolating data on a denser grid with the sampling theorem. The latter is often better.
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- Theory strictly applies only to sufficiently smooth functions, but turns out to give valuable guidance for other functions as well.

- Extensions of the classical sampling theorem allow to exploit the symmetry relation and also to construct additional efficient sampling schemes.

- Noise is a challenge for efficient sampling. Post-processing with edge preserving denoisers is promising.