1. Conway, chapter 3, section 3, problem 8 If $T \ z = \frac{az + b}{cz + d}$ show that $T(R_\infty) = R_\infty$ if and only if $a, b, c, d$ can be chosen to be real numbers.

It is clear that if $a, b, c, d$ are real, then $T$ maps the extended real axis to the extended real axis. The qualification “if $a, b, c, d$ can be chosen to be real numbers” is used, since the transformation $T$ is unchanged if each of the coefficients is multiplied by a non-zero complex number.

To prove the converse, we suppose that $T$ maps the extended real axis to the extended real axis. We know that not both of $c$ and $d$ can be zero. We first investigate the case where $c = 0$. Then $T$ reduces to $T \ z = \frac{a}{d} \ z + \frac{b}{d}$. Since the image of 0 is real, $\frac{b}{d}$ is real, and so is $T1 - T0 = \frac{a}{d}$. Together, these say that the coefficients can be taken to be real. If $d = 0$, then $T_\infty = \frac{a}{c}$ and $T1 - T_\infty = \frac{b}{c}$ are real, so the coefficients may be taken real. Finally, we consider the main case when neither $c$ nor $d$ is zero. Then $T0 = \frac{b}{d}$, $T_\infty = \frac{a}{c}$, and $T^{-1}_\infty = -\frac{d}{c}$ are all real. Taking the ratio of the first and third of these, $\frac{b}{c}$ is also real, and thus $a, b, d$ are real multiples of $c$, so we choose $a, b, c, d$ to be real.

A better way to approach this problem was used by several people in their solutions. Since a Möbius transformation is determined by its action on three distinct points in the extended plane. Let $x_1$, $x_2$, and $x_3$ be distinct real numbers whose images under $T$, say $r_1$, $r_2$, and $r_3$ are distinct finite real numbers also. Then by preservation of cross ratio, $(Tz, r_1, r_2, r_3) = (z, x_1, x_2, x_3)$. Writing this out explicitly with $w = Tz$,

$$\frac{w - r_2}{w - r_3} \cdot \frac{r_1 - r_3}{r_1 - r_2} = \frac{z - x_2}{z - x_3} \cdot \frac{x_1 - x_3}{x_1 - x_2},$$

which can be written as

$$\frac{w - r_2}{w - r_3} = \beta \frac{z - x_2}{z - x_3},$$

where $\beta$ is real. (It is the ratio of non-zero real ratios.) But now each side is a Möbius transformation with real coefficients, and the inverse of the transformation on the left also has real coefficients and the product of the matrix of the inverse and the matrix on the right has real coefficients. But this is a matrix for $T$.


If $Tz = \frac{az + b}{cz + d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$, where $\Gamma$ is the unit circle $\{z : |z| = 1\}$.

Using the symmetry principle, if $T$ maps $\Gamma$ to itself, then $T$ will map points symmetric in the unit circle to points symmetric in the unit circle. The points $w$ and $w^*$ are
symmetric in the unit circle when \( w^* \bar{w} = 1 \) or if \( w = 0 \) and \( w^* = \infty \). We write the condition that \((Tz)^* = Tz^*\) as

\[
Tz = \frac{1}{Tz^*} = \frac{1}{Tz^*} = \frac{1}{\frac{a\bar{z} + b}{c\bar{z} + d}} = \frac{\bar{c} + \bar{d}z}{\bar{a} + bz}
\]

Equality of these two expressions for \( T \) says there exists a non-zero complex number \( \lambda \) so that \( a = \lambda \bar{d} \), \( b = \lambda \bar{c} \), \( c = \lambda \bar{b} \), and \( d = \lambda \bar{a} \). Since either \( c \) or \( d \) is non-zero, then \(|\lambda| = 1\) for if \( c \neq 0 \) then \( b \neq 0 \) and \( bc = \lambda^2 \bar{c}b \) and thus \(|\lambda^2| = 1\), and similarly if \( d \neq 0 \).

Using this information we may rewrite

\[
\frac{az + b}{cz + d} = \frac{\lambda \bar{d}z + \lambda \bar{c}}{cz + d} = \lambda \frac{\bar{d}z + \bar{c}}{cz + d}.
\]

If \( d \neq 0 \) this may be written as

\[
Tz = \lambda \frac{\bar{d}z + \bar{c}}{d + \frac{\bar{c}}{d} z}
\]

while if \( d = 0 \) the transformation reduces to \( Tz = \lambda \frac{\bar{c}}{c} z \). In either case, the coefficient in front, \( \lambda \frac{\bar{d}}{d} \) or \( \lambda \frac{\bar{c}}{c} \) has modulus equal to 1. In the first case, we may write \( T \) as

\[
Tz = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}
\]

where \( \theta \) is real and \(|\alpha| \neq 1\). This second condition arises since \( T(-\frac{d}{c}) = \infty \) and so by the hypothesis that \( T \) maps the unit circle to the unit circle, we can not have \(|\frac{d}{c}| = 1\).

3. Conway, chapter 4, section 1, problem 13.

Find \( \int_\gamma z^{-\frac{1}{2}} dz \) where: (a) \( \gamma \) is the upper half of the unit circle from \(+1\) to \(-1\); (b) \( \gamma \) is the lower half of the unit circle from \(+1\) to \(-1\).

In the first case, \( \gamma(t) = e^{it} \) for \( t \in [0, \pi] \). The square root is the principal value, so

\[
\gamma(t)^{-\frac{1}{2}} = \frac{1}{e^{it/2}} = e^{-it/2}
\]

and the integral is

\[
\int_0^\pi e^{-it/2} e^{it/2} dt = i \int_0^\pi e^{it/2} dt = 2e^{i\pi/2} \bigg|_{t=0}^{t=\pi} = 2e^{i\pi/2} - 2 = 2i - 2.
\]

In the second case, \( \gamma(t) = e^{-it} \) for \( t \in [0, \pi] \). Again taking the principal branch of the square root, \( \gamma(t)^{-1/2} = e^{it/2} \) and the integral becomes \(-i \int_0^\pi e^{it/2} e^{-it} dt = \int_0^\pi -ie^{-it/2} dt.\)

The integrand is the conjugate of that appearing in the first case, so the result is \(-2i - 2\), the conjugate of the integral in part (a).

Let \( \gamma(t) = 1 + e^{it} \) for \( t \in [0, 2\pi] \) and find \( \int_{\gamma} (z^2 - 1)^{-1} \, dz \).

Substituting the given form of \( \gamma(t) \), the integral becomes

\[
\int_0^{2\pi} \frac{ie^{it}}{2e^{it} + e^{2it}} \, dt = \int_0^{2\pi} \frac{ie^{it}}{e^{it}(2 + e^{it})} \, dt.
\]

Resisting the impulse to divide out the factor of \( e^{it} \) in the numerator and denominator, we instead use partial fractions to write the integrand as

\[
\frac{i}{2} - i \frac{e^{it}}{2 + e^{it}}.
\]

The first part integrates to \( \pi i \), while the second has a primitive, \( -\frac{1}{2} \log(2 + e^{it}) \) along the curve (principal branch of the logarithm). Since the curve is closed, the second part integrates to 0. Thus the integral evaluates to \( \pi i \).


Let \( G \) be an open set and \( \gamma \) be a rectifiable curve in \( C \). Suppose that \( \phi : \{\gamma\} \times G \to C \) is continuous and define

\[
g(z) = \int_{\gamma} \phi(w, z) \, dw.
\]

Prove that \( g \) is continuous. If \( \frac{\partial g}{\partial z} \) exists for each \( (w, z) \in \{\gamma\} \times G \) and is continuous, then prove that \( g \) is analytic and that

\[
g'(z) = \int_{\gamma} \frac{\partial \phi}{\partial z}(w, z) \, dw.
\]

For the continuity, we estimate

\[
|g(z) - g(z')| = |\int_{\gamma} \phi(w, z) - \phi(w, z') \, dw|
\]

\[
\leq \int_{\gamma} |\phi(w, z) - \phi(w, z')| \, dw \leq \sup_{w \in \{\gamma\}} |\phi(w, z) - \phi(w, z')| V(\gamma).
\]

where \( V(\gamma) \) is the total variation of \( \gamma \). Now by uniform continuity of \( \phi \) restricted to \( \{\gamma\} \times K \) for any compact \( K \subset G \), given an \( \epsilon > 0 \) we may choose a \( \delta > 0 \) so that

\[
|\phi(w, z) - \phi(w', z')| < \frac{\epsilon}{V(\gamma)}
\]

when \( |(w, z) - (w', z')| < \delta \) and \( z \) and \( z' \) are in \( K \). Thus \( g \) is continuous restricted to every compact subset of \( G \) and so is continuous on \( G \). To prove the analyticity, we have to show that differentiation under the integral is valid, for then the first part of the problem and the formula \( g'(z) = \int_{\gamma} \frac{\partial \phi}{\partial z}(w, z) \, dw \) will prove the continuity, as we have assumed the continuity of the partial derivative. Now we estimate

\[
\left| \frac{g(z + h) - g(z)}{h} - \int_{\gamma} \frac{\partial \phi}{\partial z}(w, z) \, dw \right| = \left| \int_{\gamma} \frac{\phi(w, z + h) - \phi(w, z)}{h} - \frac{\partial \phi}{\partial z}(w, z) \, dw \right|
\]

\[
\leq \sup_{w \in \{\gamma\}} \left| \frac{\phi(w, z + h) - \phi(w, z)}{h} - \frac{\partial \phi}{\partial z}(w, z) \right| V(\gamma).
\]
To estimate the supremum, we use the fundamental theorem of calculus and the continuity of the partial derivative of $\phi$ with respect to $z$ to write

$$\frac{\phi(w, z + h) - \phi(w, z)}{h} = \frac{1}{h} \int_0^1 \frac{d}{dt} \phi(w, z + th) dt$$

$$= \frac{1}{h} \int_0^1 \phi_2(w, z + th) h dt = \int_0^1 \phi_2(w, z + th) dt$$

and so

$$|\frac{\phi(w, z + h) - \phi(w, z)}{h} - \frac{\partial \phi}{\partial z}(w, z)| = |\int_0^1 \phi_2(w, z + th) dt - \phi_2(w, z)|$$

$$= |\int_0^1 \phi_2(w, z + th) - \phi_2(w, z) dt|$$

$$\leq \int_0^1 |\phi_2(w, z + th) - \phi_2(w, z)| dt$$

$$\leq \sup_{0 \leq t \leq 1} |\phi_2(w, z + th) - \phi_2(w, z)|.$$

Now since $\phi_z$ is continuous, it is uniformly continuous on $\{\gamma\} \times K$ for compact subsets $K$ of $G$, and thus we may this last quantity uniformly small by making $h$ sufficiently small. Incorporating this estimate in one above, we see that we can take the derivative under the integral, and the other conclusions follow as indicated before.

6. Conway, chapter 4, section 2, problem 7 acd. Use the results of this section to evaluate the following line integrals.

(a) $\int_\gamma e^{iz} dz, \gamma = e^{it}$ for $t \in [0, 2\pi]$. By corollary 2.13, this is $2\pi i$ times the derivative of $f(z) = e^{iz}$ evaluated at $z = 0$. That is $2\pi i e^{i0} = -2\pi$.

(b) $\int_\gamma \sin z \frac{dz}{z^3}, \gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. By corollary 2.13, this is $2\pi i$ times the second derivative of $\sin z$ evaluated at $z = 0$ divided by $2!$. Since the second derivative of $\sin z$ is $-\sin z$ which evaluates to 0 at $z = 0$, the value of the integral is 0.

(c) $\int_\gamma \log z \frac{dz}{z^n}, \gamma(t) = 1 + \frac{1}{2} e^{it}$, for $t \in [0, 2\pi]$ and $n \geq 0$. The contour is contained in the open disk $|z - \frac{1}{2}| < \frac{1}{2}$ which is entirely contained in the right half plane $\Re z > 0$ and log $z$, the principal branch of the logarithm, is analytic there, so the integrand is analytic on an open disk containing the contour. By Proposition 2.15, the integral is equal to 0.