Complex Analysis    Spring 2001

Homework VI

Due Friday June 1

1. Conway, chapter 5, section 1, problem 1 b,h,i,j. Determine the nature of the isolated singularity at \( z = 0 \) of the following functions. If the function has a pole, find the singular part, for an essential singularity find the image of a small annulus.

   (b) \( f(z) = \frac{\cos z}{z} \) Using the series expansion for \( \cos z \), \( \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \) we have

   \[
   f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k-1}
   \]

   for \( z \neq 0 \). Since only one coefficient of a negative power of \( z \) is non-zero, \( f \) has a pole at \( z = 0 \) by corollary 1.18. The singular part is \( \frac{1}{z} \) arising from the \( k = 0 \) term in the sum.

   (h) \( f(z) = \frac{1}{1 - e^z} \) Since \( \lim_{z \to 0} (1 - e^z) = 0 \), the limit as \( z \to 0 \) of \( |f(z)| = \infty \) and so \( f(z) \) has a pole. From the series expansion for \( 1 - e^z = -\sum_{k=1}^{\infty} \frac{z^k}{k!} = -z \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} \) we see that \( 1 - e^z = -zb(z) \) where \( b(z) \) is analytic with \( b(0) = 1 \). Thus \( zf(z) = -\frac{1}{b(z)} \) has a removable singularity, and so \( f \) has a pole of order 1. Moreover, the singular part is \( -\frac{1}{z} \) since the Taylor expansion of \( \frac{1}{b(z)} \) starts with \( c_0 = \frac{1}{b(0)} \).

   (i) The Taylor expansion of \( \sin z \) converges for all \( z \in \mathbb{C} \), so \( \sin \frac{1}{z} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k+1} \).

   Then

   \[
   f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{z^{2k+1}}
   \]

   which shows that infinitely many Laurent coefficients of negative index are non-zero, and thus by corollary 1.18 the function has an essential singularity at \( z = 0 \). Application of the Casorati-Weierstrass theorem then says that the image of the annulus \( \{ z : 0 < |z| < \delta \} \) is dense.

   (j) This is similar to the preceding part. Again \( z^n \sin \frac{1}{z} \) will have infinitely many non-zero Laurent coefficients of negative index, and so has an essential singularity.

2. Conway, chapter 5, section 1, problem 4. Find the Laurent expansion of \( f(z) = \frac{1}{z(z-1)(z-2)} \) in the three annuli \( \text{ann}(0;0,1), \text{ann}(0;1,2), \text{ann}(0;2,\infty) \).

   This problem is most easily handled by partial fractions and manipulations with geometric series. A small simplification in this approach used by Abe Shepard is to note that the factor \( \frac{1}{z} \) can be left, and partial fractions applied only to \( \frac{1}{(z-1)(z-2)} \). Here

   \[
   \frac{1}{(z-1)(z-2)} = \frac{1}{1-z} + \frac{1}{z-2}
   \]
and these two terms can be treated as follows. For \(|z| < 1\), \(\frac{1}{1-z}\) has the geometric series expansion \(\sum_{k=0}^{\infty} z^k\) while
\[
\frac{1}{z - 2} = -\frac{1}{2} \frac{1}{1 - \frac{z}{2}} = -\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k.
\]
These have to be multiplied by \(\frac{1}{z}\) and summed. In the annulus of inner radius 1 and outer radius two, the expansion of \(\frac{1}{z-2}\) remains the same, since its radius of convergence is equal to 2, but the expansion for \(\frac{1}{1-z}\) must be replaced by
\[
\frac{1}{1-z} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}} = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k}
\]
which is valid for \(|z| > 1\). Finally in the annulus \(|z| > 2\) the expansion for \(\frac{1}{z-2}\) must be done in a similar manner, but the calculations are omitted.

3. Conway, chapter 5, section 1, problem 13.

(a) Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

If \(f(z) = c\) then \(f(\frac{1}{z}) = c\) so \(\lim_{z \to 0} zf(\frac{1}{z}) = \lim_{z \to 0} cz = 0\) and so \(f(\frac{1}{z})\) has a removable singularity at \(z = 0\) or \(f\) has a removable singularity at infinity. If \(f\) has a removable singularity at infinity, then \(\lim_{z \to \infty} f(z)\) exists as a finite complex number, and so \(f\) is bounded. Then by Liouville’s theorem, \(f\) is constant. Another proof, more in the spirit of this section, is to make the substitution of \(\frac{1}{z}\) for \(z\) in the Taylor expansion of \(f\), which is valid for all \(z \in \mathbb{C}\) since \(f\) is entire. Then \(f(\frac{1}{z}) = \sum_{k=0}^{\infty} a_k z^{-k}\) and the assumption that \(f(\frac{1}{z})\) has a removable singularity implies that \(a_k = 0\) for \(k \geq 1\) by corollary 1.18. Thus \(f(\frac{1}{z})\) is constant, and hence \(f\) is constant.

(b) Prove that an entire function has a pole of order \(m\) at infinity if and only if it is a polynomial of degree \(m\).

If \(f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0\) is a polynomial of degree \(m\), then \(f(\frac{1}{z}) = a_m z^{-m} + a_{m-1} z^{-(m-1)} + \cdots + a_1 z^{-1} + a_0\). By part b of corollary 1.18, \(f(\frac{1}{z})\) then has a pole of order \(m\) at \(z = 0\), and so \(f\) has a pole of order \(m\) at infinity. Conversely, suppose that \(f\) has a pole of order \(m\) at infinity, so \(f(\frac{1}{z})\) has a pole of order \(m\) at \(z = 0\). By corollary 1.18, \(f(\frac{1}{z})\) then has a Laurent expansion
\[
f(\frac{1}{z}) = p_m(\frac{1}{z}) + g(z),
\]
where \(p_m\) is a polynomial of degree \(m\) without constant term, and \(g\) is analytic at \(z = 0\). But since \(f\) is entire, \(g\) is analytic everywhere, and has a removable singularity at infinity, since \(\lim_{z \to \infty} g(z) = \lim_{z \to \infty} f(\frac{1}{z}) = f(0)\). (Here it was used that \(p_m\) has no constant term, so its limit at infinity is 0.) But then by part (a), \(g\) is constant, and so \(f(\frac{1}{z})\) is a polynomial in \(\frac{1}{z}\) of degree \(m\), and thus \(f\) is a polynomial of degree \(m\).
(c) Characterize those rational functions which have a removable singularity at infinity.

Any rational function \( r(z) = \frac{p(z)}{q(z)} \) may be written (by long division) in the form
\[
 r(z) = a(z) + \frac{p_1(z)}{q(z)}
\]
where \( a(z) \) is a polynomial (perhaps identically zero) and the degree of \( p_1 \) is less than the degree of \( q \). Since \( r \) has a removable singularity at infinity, \( r \) is bounded at infinity, and so is \( \frac{p_1(z)}{q(z)} \) by the degree condition, and thus \( a(z) \) must also be bounded. But by part (a), \( a(z) \) is then a constant, and so \( p(z) = aq(z) + p_1(z) \) is a polynomial of degree less than or equal to the degree of \( q(z) \). The converse is simple.

(d) Characterize those rational functions which have a pole of order \( m \) at infinity.

Proceeding as in the preceding part, we may write \( r(z) = a(z) + \frac{p_1(z)}{q(z)} \) where the degree of \( p_1 \) is less than the degree of \( q \). Since \( p_1q \) has a removable singularity at infinity, \( a(z) \) has a pole of order \( m \), and by part (b) must then have degree \( m \). Thus the degree of \( p \) must be \( m \) greater than the degree of \( q \). The converse is direct, again using part (b) and the form of \( r(z) \).

4. Prove that if \( f \) is entire with \( |f(z)| \leq K + m \log(1 + |z|) \) for some \( K, m > 0 \) then \( f \) has a removable singularity at infinity, and use this and the preceding problem to give another proof that such a function is constant.

To see that \( f \) has a removable singularity at infinity, we must evaluate the limit at \( z \to 0 \) of \( zf(\frac{1}{z}) \). Now using the given estimate for the modulus of \( f(z) \), \( |zf(\frac{1}{z})| \leq |z|(K + m \log(1 + \frac{1}{|z|})) \). The limit \( \lim z \to 0 K|z| = 0 \) is elementary, while the limit \( |z| \log(1 + \frac{1}{|z|}) \to 0 \) as \( z \to 0 \) follows from the real variable result, which is established by l'Hospital's rule.

\[
 \lim_{t \to 0^+} \frac{\log(1 + 1/t)}{1/t} = \lim_{s \to \infty} \frac{\log(1 + s)}{s} = \lim_{s \to \infty} \frac{1/(1 + s)}{1} = 0.
\]

5. Use our basic version of Rouché’s theorem to count the number of roots of \( p(z) = z^7 - 2z^5 + 6z^3 - z + 1 = 0 \) inside the unit disk.

On the boundary of the disk, \( |z| = 1 \),
\[
 |p(z)| \geq |6z^3| - |z^7 - 2z^5 - z + 1| \geq 6 - 5 = 1
\]
so \( p(z) \) has no zeros on the boundary. Also for \( z \) with \( |z| = 1 \),
\[
 |p(z) - 6z^3| = |z^7 - 2z^5 - z + 1| \leq 5 < 6 = |6z^3|,
\]
so by Rouché’s theorem, \( p(z) \) and \( 6z^3 \) have the same number of roots, counted with multiplicity. \( 6z^3 \) has only a triple root at \( z = 0 \), so \( p(z) \) has three roots (counted with multiplicity) inside the unit disk.
6. Conway, Chapter 5, section 3, problem 10 Let \( f \) be analytic in a neighborhood of the closed unit disk. If \( |f(z)| < 1 \) for \( |z| = 1 \), show that there is a unique \( z \) inside the unit disk with \( f(z) = z \). If \( |f(z)| \leq 1 \) for \( |z| = 1 \), what can be said?

From the given, \( |f(z) - z - (-z)| = |f(z)| < 1 = |z| \) for \( |z| = 1 \). The function \( g(z) = f(z) - z \) has no zeros on \( |z| = 1 \) since

\[
|g(z)| = |z - f(z)| \geq |z| - |f(z)| = 1 - |f(z)| > 0,
\]

and clearly \(-z\) has no zero on the boundary of the disk, so Rouché’s theorem applies, and \( f(z) - z \) and \(-z\) both have one zero in the unit disk. But a zero of \( f(z) - z \) is a fixed point for \( f(z) \) and our statement is proved.

If we now assume only that \( |f(z)| \leq 1 \) on \( |z| = 1 \), then the problem is harder. The example \( f(z) = z \) shows that there may be infinitely many fixed points in the interior, and the examples \( f(z) = z^n \) for \( n \geq 2 \) show that \( f \) may have a single interior fixed point and any finite number of fixed points on the boundary. In the other direction, if \( f(z) = \frac{1+z}{2} \), then \( |f(z)| \leq 1 \) on \( |z| = 1 \), but \( f \) has no fixed points in the disk. The first positive result which one can prove, and done by several of you, is to show that if \( f \) has no fixed points on the boundary, then \( f \) has a unique fixed point inside the disk.

The proof you presented goes as follows. From the hypothesis,

\[
|f(z) - z - (-2z)| = |f(z) + z| \leq |f(z)| + |z| \leq |2z|
\]

on \( |z| = 1 \). However, equality in the first application of the triangle inequality can only hold when \( f(z) \) is a positive multiple of \( z \), and the sum can only be \( 2 \) when both have modulus 1. This means that \( f(z) = z \) for some \( z \) on \( |z| = 1 \), but we assumed there were no fixed points on the boundary. Likewise, \( |f(z) - z| > 0 \) on the boundary. Thus Rouché’s theorem implies that \( f(z) - z \) and \(-2z\) have the same number of zeros inside the disk, and so \( f \) has a unique fixed point in the interior.

Using the Schwarz lemma, we can prove more. First, by the maximum principle, \( |f(z)| \leq 1 \) in the interior since this bound holds on the boundary. We suppose now that \( f \) has two distinct fixed points \( a \) and \( b \) in the interior. For \( |w| < 1 \) let \( \phi_w(z) = \frac{z-w}{1-\overline{w}z} \) be as on page 131. We know that \( \phi_w \) maps the interior of the unit disk to the interior, and the boundary to the boundary, and that \( \phi_w^{-1} = \phi_{-w} \). Let \( g(z) = \phi_a \circ f \circ \phi_{-a}(z) \). Then \( g(0) = \phi_a(f(a)) = \phi_a(a) = 0 \) and \( g(\phi_a(b)) = \phi_a(f(\phi_{-a}(\phi_a(b)))) = \phi_a(f(b)) = \phi_a(b) \). By the Schwarz lemma, \( g(z) = cz \) for some \( c \) with modulus 1, and since \( g \) has a non-zero fixed point, the constant \( c \) must be equal to 1. Thus \( g \) is the identity, and so \( f \) is the identity. We conclude that \( f \) can never have more than one interior fixed point, unless \( f \) is the identity map.