Q1. Prove the following facts:

(a) \( P[A \cup B] \geq P[A] \). (5%)
(b) \( P[A \cup B] \geq P[B] \). (5%)
(c) \( P[A \cap B] \leq P[A] \). (5%)
(d) \( P[A \cap B] \leq P[B] \). (5%)

1 Solution

(a) Since \( A \subset A \cup B \), \( P[A \cup B] \geq P[A] \).
(b) Since \( B \subset A \cup B \), \( P[A \cup B] \geq P[B] \).
(c) Since \( A \cap B \subset A \), \( P[A \cap B] \leq P[A] \).
(d) Since \( A \cap B \subset B \), \( P[A \cap B] \leq P[B] \).

Q2. In an experiment, \( A, B, C \) and \( D \) are events with probabilities \( P[A] = 1/4 \), \( P[B] = 1/8 \), \( P[C] = 5/8 \), and \( P[D] = 3/8 \). Furthermore, \( A \) and \( B \) are disjoint, while \( C \) and \( D \) are independent.

(a) Find \( P[A \cap B] \), \( P[A \cup B] \), \( P[A \cap B^c] \), and \( P[A \cup B^c] \). (5%)
(b) Are \( A \) and \( B \) independent? (5%)
(c) Find \( P[C \cap D] \), \( P[C \cap D^c] \), and \( P[C^c \cap D^c] \). (5%)
(d) Are \( C^c \) and \( D^c \) independent? (5%)
2 Solution

(a) Since \(A\) and \(B\) are disjoint, \(P[A \cap B] = 0\).
\[
\]
A Venn diagram should convince you that \(A \subset B^c\) so that \(A \subset B^c = A\). This implies \(P[A \cap B^c] = P[A] = 1/4\).

It also follows that \(P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8\).

(b) Events \(A\) and \(B\) are independent since \(P[A \cap B^c] \neq P[A]P[B]\).

(c) Since \(C\) and \(D\) are independent,
\[
P[C \cap D] = P[C]P[D] = 15/64.
\]
The next few items are a little trickier. From Venn diagram,, we see \(P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64\).

It follows that
\[
P[C \cup D^c] = P[C] + P[D] - P[C \cup D^c] = 5/8 + (1 - 3/8) - 25/64 = 55/64.
\]
Using DeMorgan’s law, we have
\[
P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64.
\]

(d) Since \(P[C^cD^c] = P[C^c]P[D^c], C\) and \(D\) are independent.

Q3. For independent events \(A\) and \(B\), prove that

(a) \(A\) and \(B^c\) are independent. (10%)
(b) \(A^c\) and \(B\) are independent (5%)
(c) \(A^c\) and \(B^c\) are independent (5%)

3 Solution

(a) For any events \(A\) and \(B\), we can write the law of total probability in the form of
\[
P[A] = P[A \cap B] + P[A \cap B^c].
\]
Since \(A\) and \(B\) are independent, \(P[A \cap B] = P[A]P[B]\). This implies
\[
\]
Thus, \(A\) and \(B^c\) are independent.

(b) Proving that \(A^c\) and \(B\) are independent is not really necessary. Since \(A\) and \(B\) are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of \(A\) and \(B\) proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels \(A\) and \(B\) reversed.
(c) To prove that $A^c$ and $B^c$ are independent, we apply the result of part (a) to the sets $A$ and $B^c$. Since we know from part (a) that $A$ and $B^c$ are independent, part (b) says that $A^c$ and $B^c$ are independent.

Q4. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or a negative (-) response. Suppose the test gives the correct answer 99% of the time. What is $P[-|H]$, the conditional probability that a person tests negative given that the person does have the HIV virus? What is $P[H|+]$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive? (20%)

4 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01 \quad (1)$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[H \cap +]}{P[+]} = \frac{P[H \cap +]}{P[H \cap +] + P[H^c \cap +]} \quad (2)$$

We can use Bayes’ formula to evaluate these joint probabilities.

$$P[H|+] = \frac{P[+|H]P[H]}{P[+|H]P[H] + P[+|H^c]P[H^c]} = \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} = 0.0194 \quad (3)$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Q5. Answer the following questions:

(a) Prove that $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ for any $A$ and $B$ (not necessarily disjoint). (10%)

(b) Now generalize: find a formula for $P[A \cup B \cup C]$ for any $A, B, C$ (not necessarily mutually exclusive). (10%)

5 Solution

(a) It can be easily checked that the sets $A$ and $B \cap \bar{A}$ are a partition of $A$. Then $(A \cup B) = A \cup (B \cap \bar{A})$ implies that $P(A \cup B) = P(B) + p(B \cap \bar{A})$. Similarly
set B can be partition into sets $A \cap B$ and $B \cap \bar{A}$: $B = (A \cap B) + (B \cap \bar{A})$ meaning that $P(B) = P(A \cap B) = P(B \cap \bar{A})$. Therefore,

$$P(A \cup B) = P(A) + P(B \cap \bar{A}) = P(A) + P(B) - P(A \cap B).$$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$. 