Observing Pairs

• Let us consider the following experiment: I sample a student in this class and observe the student’s weight and height.

• Let us denote the weight as $X$ and the height as $Y$—they are two random variables.

• We only observe if the student’s weight and height are below or above average. That means that both $X$ and $Y$ only take two values, e.g.,

$$P_X(x) = \begin{cases} 
1/2, & x = 0 \\
1/2, & x = 1 
\end{cases}, \quad P_Y(y) = \begin{cases} 
1/2, & y = 0 \\
1/2, & y = 1 \end{cases}$$

• Q: can the two PMFs summarize all the possible scenarios?
Observing Pairs

- What are the PMFs of $X$ and $Y$?

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<tr>
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<th>Height</th>
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<tr>
<td>Above average</td>
<td>0.1</td>
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<td>Below average</td>
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Below average  Above average
Observing Pairs

- What are the PMFs of $X$ and $Y$?

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<td>Weight</td>
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Joint PMF

- Clearly, if we are observing pairs, the **marginal** PMFs cannot summarize all the cases.

- You need something that is more powerful and is able to summarize all the different combinations of $X$ and $Y$.

- **Definition**: The joint PMF of $X$ and $Y$ is defined as

$$P_{X,Y}(x, y) = \begin{cases} P[X = x, Y = y], & x \in S_X, y \in S_Y, \\ 0, & \text{o.w.} \end{cases}$$

- Another important motivation: we usually want to predict one RV from another RV—and joint PMF can help us with this purpose.
  - e.g. medical history $\rightarrow$ heart attack
  - e.g., purchase history (iphone) $\rightarrow$ next thing to buy (airpod)
Properties

- **Property:** From the joint PMF $P_{X,Y}(x, y)$ one can easily derive the marginal PMFs $P_X(x)$ and $P_Y(y)$:

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y)$$
Properties

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Joint CDF

• **Q:** Now we know that from the joint PMF we can obtain the marginal PMFs easily. Can we go the other way around?

• **Definition:** The joint CDF is defined as

\[ F_{X,Y}(x, y) = P[X \leq x, Y \leq y]. \]

• Recall that we can compute \( F_X(x) \) from \( P_X(x) \). Can we compute joint CDF from joint PMF?
Joint CDF

- We are interested in \( P[X \leq x_1, Y \leq y_1] \)
Joint CDF

- We are interested in $P[X \leq x_1, Y \leq y_1]$

- from joint PMF to joint CDF: sum up all the probabilities in the shaded region.
from CDF to PMF

• Q: what if I wish to compute $P_{X,Y}(x_1, y_1)$ from $F_{X,Y}(x_1, y_1)$?

• Recall in the single-variable case: $P_X(x_1) = F_X(x_1) - F_X(x_1 - 1)$.

• For the joint case, we can do similar things.
from CDF to PMF

- Consider the example:
from CDF to PMF

- Consider the example:

\[ F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1 - 1, y_1) \]
from CDF to PMF

- Consider the example:

\[ F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1 - 1, y_1) - F_{X,Y}(x_1, y_1 - 1) \]
from CDF to PMF

• Consider the example:

\[ P_{X,Y}(x_1, y_1) = F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1 - 1, y_1) - F_{X,Y}(x_1, y_1 - 1) + F_{X,Y}(x_1 - 1, y_1 - 1) \]

\[ F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1 - 1, y_1) - F_{X,Y}(x_1, y_1 - 1) + F_{X,Y}(x_1 - 1, y_1 - 1) \]
Properties of Joint CDF

• Properties:

1. \(0 \leq F_{X,Y}(x, y) \leq 1\).
2. \(F_{X,Y}(\infty, \infty) = 1\).
3. \(F_{X,Y}(x, \infty) = F_X(x) = P[X \leq x, Y \leq \infty]\).
4. \(F_{X,Y}(\infty, y) = F_Y(y) = P[X \leq \infty, Y \leq y]\).
5. \(F_{X,Y}(\infty, y) = 0, F_{X,Y}(x, \infty) = 0\).
6. \(x_1 \leq x_2, y_1 \leq y_2 \implies F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)\).
Pairs of Continuous RVs

• First notice that the PDF of a single RV is a derivative of its CDF.

• Conversely, CDF is the running integral of PDF.

• Joint PDF is thus defined as what you integrate to get the joint CDF.

• **Definition:** If \( f_{X,Y}(x, y) \) satisfies

\[
P[X \leq x, Y \leq y] = F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v) \, du \, dv,
\]

then, it is the joint PDF of \( X \) and \( Y \).

• Here, we have

\[
f_{X,Y}(x, y) = \lim_{\Delta x \to 0, \Delta y \to 0} \left( \frac{P[x < X \leq x + \Delta x, y < Y \leq y + \Delta y]}{\Delta x \Delta y} \right) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}
\]
Example

- **Example**: Given the joint PDF of $X, Y$

  
  \[
  f_{X,Y}(x, y) = \begin{cases} 
  1, & x \in (0, 1), \ y \in (0, 1) \\
  0, & \text{o.w.}
  \end{cases}
  \]

  find the joint CDF.
• If $x < 0$ or $y < 0$, we have

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} 0 \, dx \, dy = 0$$
Example

- If $0 \leq x < 1$ and $0 \leq y < 1$, we have

$$F_{X,Y}(x, y) = \int_0^x \int_0^y 1 \, dx \, dy = xy$$
If $0 \leq x < 1$ and $y \geq 1$, we have

$$F_{X,Y}(x, y) = \int_{0}^{x} \int_{1}^{y} 1 \, dy \, dx = x(y - 1)$$
• If $0 \leq y < 1$ and $x \geq 1$, we have

$$F_{X,Y}(x, y) = \int_1^x \int_0^y 1 \, dy \, dx = y(x - 1)$$
Example

- If $x \geq 1$ or $y \geq 1$, we have

$$F_{X,Y}(x, y) = \int_0^1 \int_0^1 1 \, dx \, dy = 1$$
Example

• To summarize

\[
F_{X,Y}(x, y) = \begin{cases} 
0, & x < 0, \text{ or } y < 0, \\
xy, & 0 \leq x < 1, \text{ and } 0 \leq y < 1 \\
x(y - 1), & 0 \leq x < 1 \text{ and } y \geq 1 \\
y(x - 1), & 0 \leq y < 1 \text{ and } x \geq 1 
\end{cases}
\]