Q.1 Verify that \( \|x\|_1, \|x\|_\infty, \|x\|_2 \) and \((x^TPx)^{1/2}\) for \(P \in \mathbb{S}^n_{++}\) are norms. (25%)

Q.2 Assume that \(A \in \mathbb{S}^n\). Prove that
1. \(\det(A) = \prod_{i=1}^n \lambda_i\);
2. \(\text{tr}(A) = \sum_{i=1}^n \lambda_i\);
3. \(\|A\|_F^2 = \sum_{i=1}^n \lambda_i^2\),
where \(\lambda_1, \ldots, \lambda_n\) denote the eigenvalues of \(A\). (Hint: use the property \(\text{tr}(AB) = \text{tr}(BA)\).) (25%)

Q.3 Consider a case where \(x \in \mathbb{R}, y \in \mathbb{R}^n\) and \(h \in \mathbb{R}^n\). Let
\[f(x) = \|y - hx\|_2^2.\]
Prove that the optimal solution to
\[\min_x f(x)\]
is given by
\[x^* = \frac{h^T y}{\|h\|_2^2}.
\]
(Hint: plot out \(f(x)\) and make an observation.) (25%)

Q.4 Derive the gradients of the following functions:
1. \(f_1(x) = a^T x\) for \(a = [a_1, a_2, a_3]^T\);
2. \(f_2(x) = x^T Ax = \sum_{i=1}^n \sum_{j=1}^n A_{ij}x_i x_j\) for
\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix},
\]
where \(A = A^T\).
(Hint: your results should be consistent with the more general form:

\[ \nabla f_1(x) = a \quad \& \quad \nabla f_2(x) = 2Ax, \quad \forall A \in S^n, \]

but you should explicitly write out \( \partial f(x)/\partial x_i \) for \( i = 1, 2, 3 \) and verify the above.) (25%)