Solution for Q.1

(a) For any fixed $P$, $\|APx-b\|_2$ is convex function. As $f(x)$ is the maximum of a convex functions, it is a convex function.

(b) We can write $f(X)$ in the following way

$$f(X) = \lambda_{\min}(X) = \inf_{y \in \mathbb{R}^n, \neq 0} \frac{y^T X y}{y^T y}. \quad (1)$$

Assume

$$h(y) = \frac{y^T X y}{y^T y} = \frac{\text{tr}(Xyy^T)}{y^T y}. \quad (2)$$

$h(y)$ is a linear function. As $f(X)$ is a pointwise infimum of a concave functions $h(y)$, it is concave.

(c) Here

$$p(x, \omega) = \sum_{l=1}^{n} x_l \cos((n-1)\omega). \quad (3)$$

Then

$$f(x) = \int_0^{2\pi} \log p(x, \omega) d\omega \quad (4)$$

$$= \int_0^{2\pi} \log \sum_{l=1}^{n} x_l \cos((n-1)\omega) d\omega$$

$$= \int_0^{2\pi} h(x, \omega) d\omega$$

For each $\omega$, $h(x, \omega)$ is concave. As $f(x)$ is the integration of a concave function, it is concave.

(d) Here, $p(t)$ can be written as an affine function

$$p(t) = [1 \ t \ t^2 \ \cdots \ t^{n-1}]^T x. \quad (5)$$

The supremum of a convex function is convex and the infimum is a concave function. Their difference will be a convex function.
Solution for Q.2

(a) We show that the solution satisfies the first order optimality condition. The gradient of the objective function is given by

\[ \nabla f_o(x) = Px + q. \]  

(6)

Therefore, the optimality condition is that

\[ \nabla f_o(x)^T (y - x) = -1(y_1 - 1) + 2(y_2 + 1) \geq 0. \]  

(7)

This holds for all \(-1 \leq y_i \leq 1\).

Solution for Q.3

(a) We change the variable to \(y = Ax\). Then the equivalent problem will be

\[
\begin{align*}
\min & \quad c^T A^{-1} y \\
\text{subject to} & \quad y \geq b
\end{align*}
\]  

(8)

If \(A^{-T}c \leq 0\), the optimal solution is \(y = b\), with \(p^* = c^T A^{-1}b\). Otherwise, the LP is unbounded below.

(b) The equivalent LP problem is

\[
\begin{align*}
\min & \quad 1^T y \\
\text{subject to} & \quad -1 \leq x \leq y \\
& \quad -1 \leq Ax - b \leq 1
\end{align*}
\]  

(9)

We can also write \(x\) as \(x = x^+ - x^-\), and express the problem as

\[
\begin{align*}
\min & \quad 1^T x^+ + 1^T x^- \\
\text{subject to} & \quad -1 \leq Ax^+ - Ax^- - b \leq 1 \\
& \quad x^+ \geq 0, x^- \geq 0,
\end{align*}
\]  

(10)

where \(x^+, x^- \in \mathbb{R}^n\).

(c) We can rewrite the objective function as

\[
\|Ax - b\|_4 = \left( \sum_{i=1}^{m} (a_i^T x - b_i)^4 \right)^{1/4}.
\]  

(11)

Then, we can write the problem as

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} z_i^2 \\
\text{subject to} & \quad a_i^T x - b_i = y_i, \quad i = 1, \ldots, m \\
& \quad y_i^2 \leq z_i, \quad i = 1, \ldots, m.
\end{align*}
\]  

(12)
(d) For given $x$, the quadratic objective function is

$$
\frac{1}{2}(x^T P_0 x + \sup_{\|u\|_2 \leq 1} \sum_{i=1}^{K} u_i (x^T P_i x)) + q^T x + r
$$

$$
= \frac{1}{2}(x^T P_0 x) + \frac{1}{2}(\sum_{i=1}^{K} (x^T P_i x)^2)^{1/2} + q^T x + r.
$$

This is a convex function of $x$: each of the functions $x^T P_i x$ is convex since $P_i \geq 0$. The second term is a composition $h(g_i(x))$, where $g_i(x) = x^T P_i x$. The functions $g_i$ are convex and nonnegative. The function $h$ is convex, and for $y$, nondecreasing in each of its arguments. Therefore, the composition is convex. The resulting problem can be expressed as

$$
\begin{align*}
\min & \quad \frac{1}{2}(x^T P_0 x) + \|y\|_2 + q^T x + r \\
\text{subject to} & \quad \frac{1}{2}x^T P_i x \leq y_i, \quad i = 1, \cdots, K \\
& \quad Ax \leq b.
\end{align*}
$$

which can be further reduced to an SOCP

$$
\begin{align*}
\min & \quad u_t \\
\text{subject to} & \quad \left\| \begin{bmatrix} P_0^{1/2} x \\ 2u - 1/4 \end{bmatrix} \right\|_2 \leq 2u + 1/4 \\
& \quad \left\| \begin{bmatrix} P_0^{1/2} x \\ 2u - 1/4 \end{bmatrix} \right\|_2 \leq 2y_i + 1/4, \quad i = 1, \cdots, K \\
& \quad \|y\|_2 \leq t \\
& \quad Ax \leq b.
\end{align*}
$$

The variables are $x, u, t, \text{and } y$. Squaring both sides of the first inequality, we get

$$
\begin{align*}
x^T P_0 x + (2u - 1/4)^2 & \leq (2u + 1/4)^2, \\
i.e., \quad & x^T P_0 x \leq 2u.
\end{align*}
$$

Similarly, the other constraints are equivalent to $1/2x^T P_0 x \leq y_i$. 

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