1 Proof of Linear Independence of Vandemonde Matrices

Let $k$ be any positive integer, and consider the following matrix

$$
B = \begin{bmatrix}
1 & z_1 & z_1^2 & \cdots & z_1^{k-1} \\
1 & z_2 & z_2^2 & \cdots & z_2^{k-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & z_k & z_k^2 & \cdots & z_k^{k-1}
\end{bmatrix} \in \mathbb{C}^k,
$$

with $z_1, \ldots, z_k \in \mathbb{C}$. We will show that $B$ is nonsingular if $z_i$’s are distinct. For now, let us assume this to be true and focus on showing the linear independence of $A$. If $m \geq n$, we can represent $A$ by

$$A^T = [B \times ]$$

with $k = n$; here, “$\times$” means parts that do not matter. By the rank definition, we have $\text{rank}(A) = \text{rank}(A^T) \geq \text{rank}(B) = n$. Since we also have $\text{rank}(A) \leq n$, we obtain the result $\text{rank}(A) = n$. Moreover, if $m \leq n$ we can represent $A$ by

$$A = [B^T \times ]$$

with $k = m$. Following the same argument as above, we obtain $\text{rank}(A) = m$. Thus we have established the result that $A$ has full rank.

Now, we show that $B$ is nonsingular if $z_i$’s are distinct. Observe that

$$B\alpha = 0 \iff p(z_i) = 0, \ i = 1, \ldots, k \quad (1)$$

where

$$p(z) = \alpha_1 + \alpha_2 z + \alpha_3 z^2 + \cdots + \alpha_k z^{k-1}$$

denotes a polynomial of degree $k - 1$. On one hand, the condition on the R.H.S. of (1) implies that $z_1, \ldots, z_k$ are the roots of $p(z)$. On the other hand, it is known that a polynomial of degree $k - 1$ has $k - 1$ roots, and no more. Consequently, the above two statements contradict to each other if we have $z_i \neq z_j$ for all $i, j$ with $i \neq j$. Hence, we have shown that $B$ must be nonsingular if $z_i$’s are distinct.