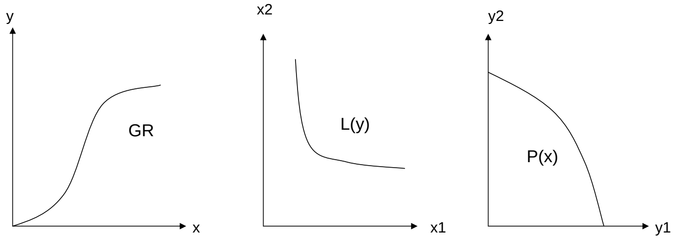
More Technical Efficiency and Data Envelopment Analysis (DEA)

based on OnFront, Reference Guide
Coelli, et al., *Introduction to Efficiency and Productivity*
Springer, 2005, Ch 6.

April 22, 2008

Technology: Set Representations

Equivalent Technology Sets: describe the relationship between all feasible inputs $x = (x_1, \ldots, x_N)$ and outputs $y = (y_1, \ldots, y_M)$



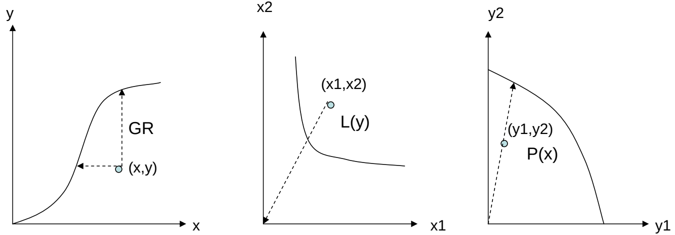
GR={(x,y):x can produce y}

L(y)={x: (x,y) is in GR}

P(x)={y: (x,y) is in GR}

If (x,y) is in GR, then x is in L(y) and y is in P(x)

Technical Efficiency



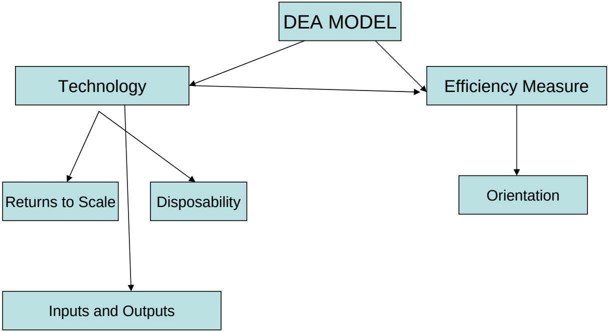
Fi(y,x)=min{lambda: lambda x is in L(y)}

Fo(x,y)=max{theta:theta y is in P(x)}

Estimating technical efficiency with DEA

- DEA uses linear programming to construct the technology and best practice frontier from the data in your sample
- Simultaneously, it estimates the distance to the best practice frontier for each observation
- There is a separate linear programming problem for each observation (DMU)

DEA Overview



Formalities

- $k = 1, \dots, K$, are the observations or decision making units (DMUs)
- $x^k = (x_{k1}, x_{k2}, \dots, x_{kN})$ are the inputs 1 to N for observation (DMU) k
- $y^k = (y_{k1}, y_{k2}, \dots, y_{kM})$ are the outputs 1 to M for observation (DMU) k
- $z = (z_1, z_2, \dots, z_K)$ are the intensity variables ('dot connectors'), which will be used to construct the best practice frontier

Input technical efficiency for each k=1,...,K, is defined as follows

$$F_i(y^k, x^k) = \min_{\lambda, z}\{\lambda : \lambda x^k \in L(y^k)\} \quad (1)$$

DEA (linear programming) problem for input technical efficiency, $F_i(y^k, x^k)$

For each observation k solve:

$$\begin{aligned} F_i(y^k, x^k) = \min_{\lambda, z} \{ \lambda \\ \text{subject to} \\ \sum_{k=1}^{K} z_k y_{km} &\geq y_{km}, m = 1, \ldots, M \\ \sum_{k=1}^{K} z_k x_{kn} &\leq \lambda x_{kn}, n = 1, \ldots, N \\ z_k &\geq 0, k = 1, \ldots, K. \end{aligned} \tag{2}$$

REMARK: The inequalities represent the technology, in this case $L(y^k)$.

How DEA constructs the technology sets, eg: L(y)

Recall: $L(y) = \{x = (x_1, \ldots, x_N)$: x can produce y=$(y_1, \ldots, y_M)\}$, in DEA, for observation k:

$$L(y^k) = \{(x_1, \ldots, x_N) : \tag{3}$$

$$\begin{aligned} \sum_{k=1}^{K} z_k y_{km} &\geq y_{km}, m = 1, \ldots, M \\ \sum_{k=1}^{K} z_k x_{kn} &\leq x_n, n = 1, \ldots, N \\ z_k &\geq 0, k = 1, \ldots, K\}. \end{aligned}$$

REMARKS: The inequalities allow for free (strong) disposability of both inputs and outputs. Note that we are *looking for* all possible x's that satisfy the constraints (x_n rather than x_{kn} on the RHS).

Now plot and use the following data to write out the constraints for the input set:

	capital	labor	widgets
	x_1	x_2	y
Firm A	30	40	50
Firm B	40	20	50
Firm C	40	30	50

$$
\begin{aligned}
L(50) = \{(x_1, x_2) : & & \qquad (4)\\
z_A 30 + z_B 40 + z_C 40 &\leq x_1 (capital)\\
z_A 40 + z_B 20 + z_C 30 &\leq x_2 (labor)\\
z_A 50 + z_B 50 + z_C 50 &\geq 50 (widgets)\\
z_A, z_B, z_C &\geq 0
\end{aligned}
$$

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

- $z_A = z_B = 1/2, z_C = 0$

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

- $z_A = z_B = 1/2, z_C = 0$
- capital$\geq$ 35, labor$\geq$ 30

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

- $z_A = z_B = 1/2, z_C = 0$
- capital≥ 35, labor≥ 30
- $z_A = z_B = 0, z_C = 1$

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

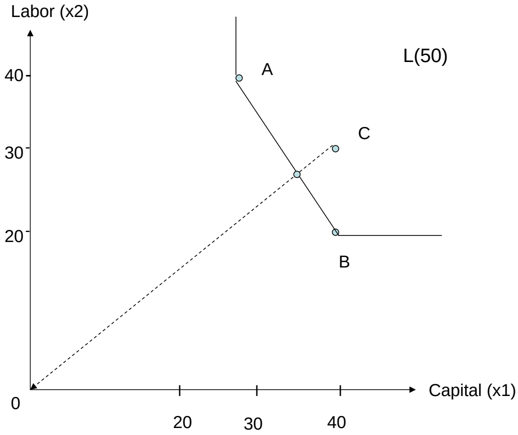
- $z_A = z_B = 1/2, z_C = 0$
- capital$\geq$ 35, labor$\geq$ 30
- $z_A = z_B = 0, z_C = 1$
- capital$\geq$ 40, labor$\geq$ 30

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

- $z_A = z_B = 1/2, z_C = 0$
- capital$\geq$ 35, labor$\geq$ 30
- $z_A = z_B = 0, z_C = 1$
- capital$\geq$ 40, labor$\geq$ 30
- $z_A = z_B = z_C = 1/3$

L(50) creates additional observations of capital and labor by allowing the z's to take any values greater than or equal to zero. What amount of capital and labor do you get if:

- $z_A = z_B = 1/2, z_C = 0$
- capital$\geq$ 35, labor$\geq$ 30
- $z_A = z_B = 0, z_C = 1$
- capital$\geq$ 40, labor$\geq$ 30
- $z_A = z_B = z_C = 1/3$
- capital$\geq$ 36.67, labor$\geq$ 30


Labor (x2)
L(50)
A
40
30
C
20
B
0
20
30
40
Capital (x1)

Now write out the the problem to solve for technical efficiency for Firm C.

$$
\begin{aligned}
F_i(y^C, x^C) = \min \lambda : \qquad\qquad\qquad\qquad & \qquad (5)\\
z_A 30 + z_B 40 + z_C 40 &\leq \lambda 40(\textit{capital})\\
z_A 40 + z_B 20 + z_C 30 &\leq \lambda 30(\textit{labor})\\
z_A 50 + z_B 50 + z_C 50 &\geq 50(\textit{widgets})\\
z_A, z_B, z_C &\geq 0
\end{aligned}
$$

solution: $F_i(y^C, x^C) = .91, z_A^* = .36, z_B^* = .64, z_C^* = 0$ How much capital and labor would firm C use if it were efficient?

How DEA constructs the technology sets, eg: GR

$$GR = \{(x_1, \ldots, x_N, y_1, \ldots, y_M) : (x_1, \ldots, x_N) \text{ can produce } (y_1, \ldots, y_M)\} \quad (6)$$

$$\begin{aligned} GR = \{(x_1, \ldots, x_N, y_1, \ldots, y_M) : & & \quad (7) \\ \sum_{k=1}^{K} z_k y_{km} &\geq y_m, m = 1, \ldots, M \\ \sum_{k=1}^{K} z_k x_{kn} &\leq x_n, n = 1, \ldots, N \\ z_k &\geq 0, k = 1, \ldots, K\}. \end{aligned}$$

REMARKS: The inequalities allow for free (strong) disposability of both inputs and outputs. Note that we are *looking for* all possible x's AND y's that satisfy the constraints (x_n and y_n) rather than x_{kn} and y_{km} on the RHS).

Now plot and use the following data to write out the constraints for the input-output set GR:

	output	input
	y	x
DMU A	4	3
DMU B	1	1
DMU C	3	2

$$
\begin{aligned}
GR = \{(x, y) :& \\
z_A 4 + z_B 1 + z_C 3 &\geq y(output) \\
z_A 3 + z_B 1 + z_C 2 &\leq x(input) \\
z_A, z_B, z_C &\geq 0\}
\end{aligned} \tag{8}
$$

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$

What does this look like?

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$

What does this look like?

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$
- $z_C = 5, z_B = z_A = 0$

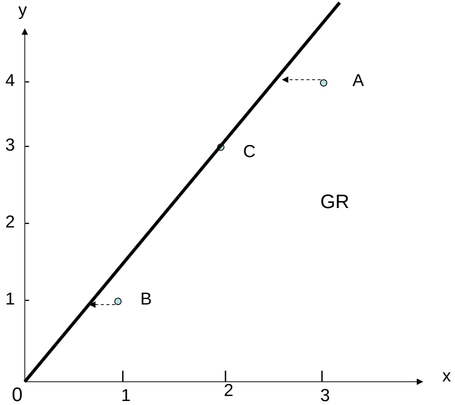
What does this look like?

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$
- $z_C = 5, z_B = z_A = 0$
- $y \leq 15, x \geq 10$

What does this look like?

What type of returns to scale does this technology exhibit?



Returns to Scale and Scale Efficiency

The DEA problems we have specified so far to estimate Farrell input and output based technical efficiency have restricted technology to satisfy:

- constant returns to scale
- strong disposability of inputs and outputs

The returns to scale of technology are determined by the restrictions on the intensity variables, i.e., the z's.

The disposability property arises from the inequalities on the input and output constraints.

Imposing Returns to Scale in DEA

Constant Returns to Scale:	$z_k \geq 0, k = 1, \ldots, K$
Nonincreasing Returns to Scale:	$z_k \geq 0, k = 1, \ldots, K$ $\sum_{k=1}^{K} z_k \leq 1$
Variable Returns to Scale:	$z_k \geq 0, k = 1, \ldots, K$ $\sum_{k=1}^{K} z_k = 1$

Imposing Nonincreasing Returns to Scale: EG

$$
\begin{aligned}
GR|N,S = \{(x,y) : \\
z_A 4 + z_B 1 + z_C 3 &\geq y(output) \\
z_A 3 + z_B 1 + z_C 2 &\leq x(input) \\
z_A, z_B, z_C &\geq 0 \\
z_A + z_B + z_C &\leq 1\}
\end{aligned} \tag{9}
$$

- N stands for Nonincreasing returns to scale
- S stands for Strong disposability

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$

What does this look like?

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$

What does this look like?

What amounts of input and output do you get if:

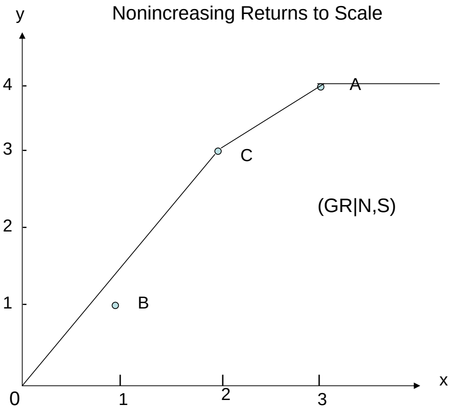
- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$
- $z_C = 1/3, z_A = z_B = 0$

What does this look like?

What amounts of input and output do you get if:

- $z_A = z_B = z_C = 0$
- $y \leq 0, x \geq 0$
- $z_C = 1/3, z_A = z_B = 0$
- $y \leq 1, x \geq 2/3$

What does this look like?


Nonincreasing Returns to Scale
y
4
3
2
1
0
1
2
3
x
A
C
B
(GR|N,S)

Imposing Variable Returns to Scale: EG

$$
\begin{aligned}
GR|V,S = \{(x,y) : & & \\
z_A 4 + z_B 1 + z_C 3 &\geq& y(output) \\
z_A 3 + z_B 1 + z_C 2 &\leq& x(input) \\
z_A, z_B, z_C &\geq& 0 \\
z_A + z_B + z_C &=& 1\}
\end{aligned} \qquad (10)
$$

- V stands for Variable returns to scale (increasing, constant and/or decreasing returns to scale)
- S stands for Strong disposability

What amounts of input and output do you get if:

- $z_A = z_C = 0, z_B = 1$

What does this look like?

What amounts of input and output do you get if:

- $z_A = z_C = 0, z_B = 1$
- $y \leq 1, x \geq 1$

What does this look like?

What amounts of input and output do you get if:

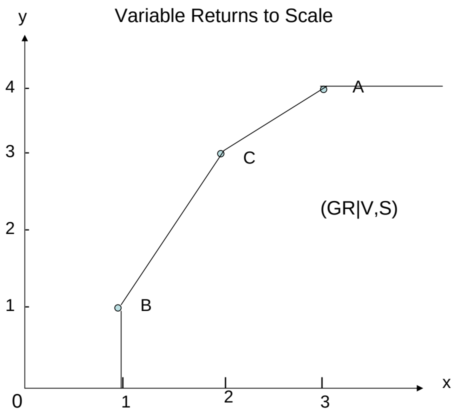
- $z_A = z_C = 0, z_B = 1$
- $y \leq 1, x \geq 1$
- $z_C = z_B = 1/2, z_A = 0$

What does this look like?

What amounts of input and output do you get if:

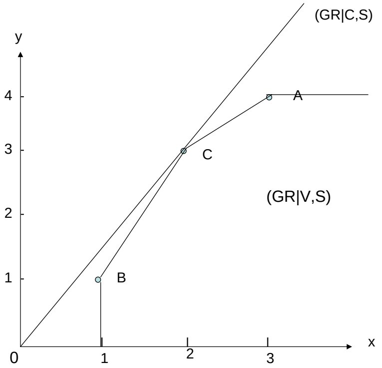
- $z_A = z_C = 0, z_B = 1$
- $y \leq 1, x \geq 1$
- $z_C = z_B = 1/2, z_A = 0$
- $y \leq 2, x \geq 1.5$

What does this look like?


Variable Returns to Scale
y
4
3
2
1
0
A
C
B
(GR|V,S)
1
2
3
x

Scale efficiency (SE)

Measures the 'gap' between the boundary of the input-output set (GR) under CRS and VRS:
SE = Farrell CRS technical efficiency/Farrell VRS technical efficiency.

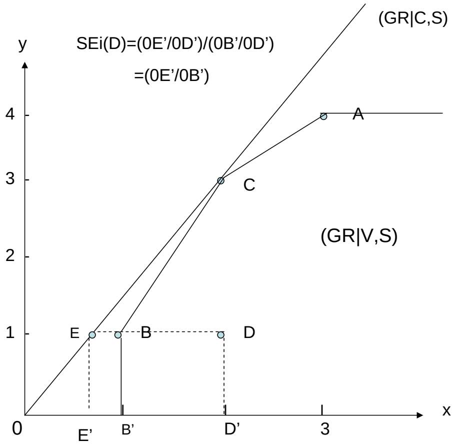


Remarks:

- The VRS technology $(GR|V, S)$ can never be 'bigger than' the CRS technology $(GR|C, S)$ for the same data set.
- We can measure the 'gap' between the technologies in either the input or output direction.

Input based scale efficiency, SE_i

$$SE_i = F_i(y,x|C,S)/F_i(y,x|V,S)$$



Output based scale efficiency, SE_o

$$SE_o = F_o(y, x|C, S)/F_o(y, x|V, S)$$