Solutions

MTH 342 Midterm 1

No notes nor calculators are allowed. Each question is worth 4 points for a total of 20 points.

1. Complete the following definitions:
Suppose $V$ is a vector space over $\mathbb{F}$.

(a) A subset $U$ of $V$ is called a **subspace** of $V$ if …

$U$ is also a vector space over $\mathbb{F}$ w/ the same $+$ and $0$.

(b) Suppose $U_1, U_2, \ldots, U_m$ are subspaces of $V$. The sum $U_1 + U_2 + \cdots + U_m$ is called a **direct sum** if …

Each $u \in U_1 + U_2 + \cdots + U_m$ can be written in exactly one way as $u = u_1 + \cdots + u_m$ where $u_i \in U_i$.

(c) A list $v_1, v_2, \ldots, v_n$ of vectors in $V$ is **linearly independent** if …

The only solution to $0 = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ with $a_i \in \mathbb{F}$ is $a_1 = a_2 = \cdots = a_n = 0$.

(d) A list $v_1, v_2, \ldots, v_n$ of vectors in $V$ **spans** $V$ if …

$$V = \text{span}(v_1, \ldots, v_n) = \{ \sum \alpha_i v_i \mid a_i \in \mathbb{F} \}.$$
2. Complete the statement of the theorems:

(a) A subset $U$ of $V$ is a subspace of $V$ if and only if $U$ satisfies the following three conditions:

- $\emptyset \subseteq U$
- If $u, v \in U$ then $u + v \in U$.
- If $u \in U$ and $k \in \mathbb{F}$ then $ku \in U$.

(b) In a finite-dimensional vector space, the length of every \underline{linearly independent} list of vectors is \\
\underline{less than or equal} to the length of every \underline{spanning} list of vectors.

(c) In a finite-dimensional vector space,

\underline{every spanning} list of vectors can be reduced to a \underline{basis}.

(d) In a finite-dimensional vector space,

\underline{every linearly independent} list of vectors can be extended to a \underline{basis}.
3. Suppose $U_1$ and $U_2$ are subspaces of $V$. Prove that the intersection $U_1 \cap U_2$ is a subspace of $V$.

We check the three conditions:

(1) $0 \in U_1 \cap U_2$

Since $U_1$ and $U_2$ are subspaces, $0 \in U_1$ and $0 \in U_2$; Thus $0 \in U_1 \cap U_2$.

(2) If $u, w \in U_1 \cap U_2$ then $u + w \in U_1 \cap U_2$

Since $U_1$ and $U_2$ are subspaces we have $u + w \in U_1$ and $u + w \in U_2$.

Thus $u + w \in U_1 \cap U_2$.

(3) If $u \in U_1 \cap U_2$ and $\alpha \in \mathbb{F}$ then $\alpha u \in U_1 \cap U_2$

Since $U_1$ and $U_2$ are subspaces we have $\alpha u \in U_1$ and $\alpha u \in U_2$.

Thus $\alpha u \in U_1 \cap U_2$. 
4. Suppose $V$ is a finite-dimensional vector space and $U$ is a subspace of $V$ such that $\dim U = \dim V$. Prove that $U = V$.

Say $\dim U = \dim V = n$.

We take a basis $u_1, \ldots, u_n$ for $U$. Since $u_1, \ldots, u_n$ is linearly independent, we can extend it to a basis for $V$. But since $\dim V = n$, this means we are adding in no new vectors. Thus $u_1, \ldots, u_n$ is also a basis for $V$.

Lastly, we then have

$U = \text{span}(u_1, \ldots, u_n) = V$. 
5. Suppose \( V \) is finite-dimensional and \( U \) is a subspace of \( V \). Prove that there is a subspace \( W \) of \( V \) such that \( V = U \oplus W \).

We begin by taking a basis \( u_1, \ldots, u_n \) for \( U \). We then extend it to \( u_1, \ldots, u_n, w_1, \ldots, w_r \) a basis for \( V \). We set

\[
W = \text{span}(w_1, \ldots, w_r).
\]

We first verify the sum \( U + W \) is direct. Suppose \( v \in U + W \) has two representations

\[
v = u + w, \quad \text{and} \quad v = \overline{u} + \overline{w},
\]

where \( u, \overline{u} \in U \) and \( w, \overline{w} \in W \).

Then write

\[
U = a_1u_1 + \cdots + a_nu_n, \quad \overline{U} = b_1u_1 + \cdots + b_nu_n,
\]

\[
w = c_1w_1 + \cdots + c_rw_r, \quad \overline{w} = d_1w_1 + \cdots + d_rw_r.
\]

Then

\[
0 = (a_1-b_1)u_1 + \cdots + (a_n-b_n)u_n + (c_1-d_1)w_1 + \cdots + (c_r-d_r)w_r,
\]

and since this is a basis for \( U \) we know each

\[
a_i = b_i \quad \text{and} \quad c_i = d_i.
\]

Thus \( u = \overline{u} \) and \( w = \overline{w} \).

So the sum \( U + W \) is direct.

Second we need to show \( U + W = V \).

But \( u_1, \ldots, u_n, w_1, \ldots, w_r \in U + W \), so

\[
V = \text{span}(u_1, \ldots, u_n, w_1, \ldots, w_r) \subseteq U + W. \text{ Thus, } V = U + W.
\]