Sets and Logic

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language...

—Galileo Galilei (1564–1642)

For the things of this world cannot be made known without a knowledge of mathematics.

—Roger Bacon (1214–1294)

1.1 Sets

A set is a collection of objects. The objects of the set are called the elements of the set. One way to specify a set is to list all the elements inside set brackets "[" and "]". For example, [Alabama, Alaska, Arizona, Arkansas] is a set with four elements. We may also specify a set in words. The set given above could be specified by stating "the set of all U.S. states that start with the letter A." It is convenient to give sets names, and conventionally, sets are named by capital letters. Thus, we may write $A = \{\text{Alabama, Alaska, Arizona, Arkansas}\}$. Alabama is an element of $A$. Birmingham, Atlanta, and
Wyoming are not elements of $A$. The symbol for "is an element of" is $\in$. Putting a slash through this symbol gives the symbol for "is not an element of." Thus, we may write

\[ \text{Alabama} \in A \]
\[ \text{Alaska} \in A \]
\[ \text{Birmingham} \notin A \]
\[ \text{Atlanta} \notin A \]
\[ \text{Wyoming} \notin A. \]

Let us consider the set consisting of the natural numbers less than 6, and let us call this set $B$. The previous cumbersome sentence may be shortened to this: Let $B = \{1, 2, 3, 4, 5\}$. Here we are listing the elements of $B$ in roster form rather than giving a verbal description of the elements. Counting down from 6, you may determine that the set of natural numbers less than 6 should be $\{5, 4, 3, 2, 1\}$. This is also correct. The elements of a set may be listed in any order. Thus, $B = \{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\} = \{3, 5, 2, 1, 4\}$, and there are many more correct representations of the set $B$.

Any set $U$ must be well-defined; that is, for every object $x$, there must be an unequivocal answer to the question "Is $x \in U"?" We may not always know the answer to this question, but we must know that an unequivocal answer exists. Consider the set $F$ of all living people who have an ancestor with the name Fletcher. Are you a member of this set? Though you may not know the answer, you should recognize that there is an indisputable answer—either yes or no. The set of good books, however, is not a well-defined set. The answer to the question "Is War and Peace a good book?" may be subject to dispute. The usage of the word good is subjective, and this makes the word an improper choice to use in specifying well-defined sets.

Two sets are equal if they contain exactly the same elements. The set $B = \{4, 3, 1, 5, 2\}$ and the set $\{\frac{3}{2}, \sqrt{4}, \sqrt{9}, 2^2, 5\}$ are equal since they contain exactly the same elements, namely $1 = \frac{3}{2}, 2 = \sqrt{4}, 3 = \sqrt{9}, 4 = 2^2, \text{ and } 5$. The set of kangaroos on the moon is a well-defined set that contains no elements. The set $\{\}\text{ containing no elements is called the empty set or null set and is denoted } \emptyset \text{ or } \{\}$. A set is finite if there is a whole number that tells the number of elements in the set. The set $B = \{1, 2, 3, 4, 5\}$ is finite, and the number of elements in $B$ is five.

1.1.1 DEFINITION The cardinality of a finite set $S$ is the number of elements in the set $S$ and is denoted $|S|$.

Counting the number of elements in a set may not be as easy as it sounds, especially if the set is described instead of listed. How many elements does the set of letters in the word throughout have? Stated another way, find the cardinality $|C|$ of the set $C = \{t, h, r, o, u, g, h, o, u, t\}$. To the question "Is $r \in C"?" we should answer "Yes." To the question "Is $t \in C"?" we should answer "Yes, yes." Though it is more emphatic, the affirmative outcome "Yes, yes" is not different from the affirmative outcome "Yes," so the element $t \in C$ only counts as one element, despite the fact that we listed it twice. If we let $D$ be the set of letters in the word trough, then $D = \{t, r, o, u, g, h\}$. The sets $C$ and $D$ have exactly the same elements, so $C = D$, and thus $|C| = |D| = 6$. Repeated
elements in a set should only be counted once. Recognizing the duplication is frequently more difficult than in this example.

Sometimes we may not be able to count the elements of a finite set. The set \( F \) of living people with an ancestor named Fletcher is a finite set, and though we do not know the exact number of elements in \( F \), we know \(|F|\) cannot exceed the current world population, and thus must be finite.

If a set is not finite, it is *infinite*. The set of natural numbers, for example, is infinite. We will not be able to list all the elements of an infinite set, but we may indicate an infinite set by a verbal description or by listing several of the elements in a clear pattern followed by an ellipsis ("..."). Some standard notation for some standard sets will illustrate this.

The set of natural numbers \( \mathbb{N} = \{1, 2, 3, 4, \ldots \} \)

The set of whole numbers \( \mathbb{W} = \{0, 1, 2, 3, \ldots \} \)

The set of integers \( \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots \} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

Another convenient way to specify a set symbolically is by *set-builder notation*, which we illustrate here. The notation \( \{x \mid x \in \mathbb{N} \text{ and } x < 6\} \) is read "the set of all \( x \) such that \( x \in \mathbb{N} \) and \( x < 6 \)." In general, \( \{x \mid \text{properties} \} \) is read "the set of all \( x \) such that \( x \) satisfies the properties \( \text{properties} \) stated." Unknown elements of a set are conventionally denoted by lowercase letters, such as the \( x \) above. If the elements \( x \) are to come from some specified set, we may include this information before the "pipe" symbol "|". The set \( \{x \mid x \in \mathbb{N} \text{ and } x < 6\} \) could be written as \( \{x \in \mathbb{N} \mid x < 6\} \) and read "the set of natural numbers \( x \) such that \( x < 6 \)." This is the set \( B = \{1, 2, 3, 4, 5\} \) we have seen earlier.

We may now introduce the notation for two other frequently used infinite sets.

The set of real numbers \( \mathbb{R} = \{x \mid x \text{ is a real number}\} \)

The set of rational numbers \( \mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\} \)

The set of rational numbers \( \mathbb{Q} \) consists of *Quotients* of integers, with the usual restriction that division by 0 is not allowed.

If we take an arbitrary set \( S \) and remove some, none, or all of its elements, the set \( T \) of remaining elements is called a *subset* of \( S \), and we write \( T \subseteq S \). Formally, a set \( T \) is a subset of \( S \) if and only if every element of \( T \) is also an element of \( S \). If \( T \) is a subset of \( S \), then \( S \) is a *superset* of \( T \) and we may write \( S \supseteq T \). The notation \( T \subseteq S \) may also be read "\( T \) is contained in \( S \)." We will illustrate this notation with some examples.

\[
\{\text{red, white, blue}\} \subseteq \{\text{red, white, blue, green}\}
\]

\[
\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5\}
\]

\[
\{2\} \subseteq \{1, 2, 3, 4, 5\}
\]

\[
\emptyset = \{\} \subseteq \{1, 2, 3, 4, 5\}
\]

\[
\{1, 2, 3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}
\]

\[
\{5, 6, 7\} \not\subseteq \{1, 2, 3, 4, 5\}
\]
The symbol $\not\subseteq$ used in the last example above means "is not a subset of." It is critical to use the correct terminology and symbols for subsets and elements of a set. Observe that $3 \in \{1, 3, 5\}$ but $3 \not\subseteq \{1, 3, 5\}$. Since $3$ is not a set, it cannot be a subset of anything. Similarly, $\{3\} \not\subseteq \{1, 3, 5\}$ but $\{3\} \not\subseteq \{1, 3, 5\}$.

If $T \subseteq S$ but $T \neq S$, we say $T$ is a proper subset of $S$ and write $T \subset S$. (Compare this notation to $<$ and $\subset$.) While $\{1, 3\}$ is a subset of $\{1, 2, 3, 4, 5\}$, denoted $\{1, 3\} \subseteq \{1, 2, 3, 4, 5\}$, we could be more explicit and say that it is a proper subset, denoted $\{1, 3\} \subset \{1, 2, 3, 4, 5\}$. Since $\{1, 2, 3, 4, 5\}$ is a subset of itself but not a proper subset of itself, we could write $\{1, 2, 3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$ but $\{1, 2, 3, 4, 5\} \not\subset \{1, 2, 3, 4, 5\}$.

Suppose $A \subseteq B$ and $B \subseteq C$. Then every element of $A$ is an element of $B$ and every element of $B$ is an element of $C$. It follows that every element of $A$ is an element of $C$; that is, $A \subseteq C$. Thus, $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$. We may depict this situation using a Venn diagram as shown in Figure 1.1. Venn diagrams provide informal graphical illustrations of shared elements of several sets.

![Figure 1.1](image)

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If $A \subseteq B$ and $B \subseteq A$, then every element of $A$ is an element of $B$ and every element of $B$ is an element of $A$. Thus, $A$ and $B$ have exactly the same elements, so $A = B$. This is a standard way to show that two sets are equal: show $A$ is contained in $B$ and show $B$ is contained in $A$. (For example, see Example 1.2.2 in the next section.)

Every set is a subset of itself, and the empty set $\emptyset$ is a subset of any set. Thus, for any set $S$, we have $\emptyset \subseteq S \subseteq S$. This seems to show that every set $S$ has at least two subsets, namely the empty set and itself. This is true unless $S = \emptyset$, in which case these "two" subsets are really one and the same. We can properly state that every nonempty set $S$ has at least two subsets, namely $\emptyset$ and $S$. Counting the number of subsets of a given set is an important problem we will consider from several approaches in the chapters that follow. Let us count all the subsets of a few small sets. To count them, we need a systematic way to find all the subsets.

**1.1.2 EXAMPLE** How many subsets does the set $\{1, 2\}$ have?

**Solution** There is one subset of $\{1, 2\}$ with zero elements: $\emptyset$.
There are two subsets of $\{1, 2\}$ with one element: $\{1\}$ and $\{2\}$.
There is one subset of $\{1, 2\}$ with two elements: $\{1, 2\}$.
This gives a total of $1 + 2 + 1 = 4$ subsets of the two-element set $\{1, 2\}$.
1.1.3 **Example**  How many subsets does the set \{1, 2, 3\} have?

**Solution**  There is one subset of \{1, 2, 3\} with zero elements: \emptyset.
There are three subsets of \{1, 2, 3\} with one element: \{1\}, \{2\}, and \{3\}.
There are three subsets of \{1, 2, 3\} with two elements: \{1, 2\}, \{1, 3\}, and \{2, 3\}.
There is one subset of \{1, 2, 3\} with three elements: \{1, 2, 3\}.
This gives a total of \(1 + 3 + 3 + 1 = 8\) subsets of the three-element set \{1, 2, 3\}.  

The elements of a set may take any form. That is, we may take sets of any kind of objects. We may form sets of words, such as \{black, white\}, or sets of letters, such as \{b, l, a, c, k, w, h, i, t, e\}. We may form sets of numbers, such as \{2, 4, 6, 8, 10\}, or even sets of sets, such as \{\{2, 4, 6, 8, 10\}, \{3, 6, 9\}, \{4, 8\}, \{5, 10\}\}. To avoid confusion about the context of the word set, a set whose elements are sets will be called a collection of sets or a family of sets. Collections of sets are typically denoted with script capital letters. Thus, \(\mathcal{C} = \{\{2, 4, 6, 8, 10\}, \{3, 6, 9\}, \{4, 8\}, \{5, 10\}, \{6\}\}\) is a collection of five sets.
We have \{3, 6, 9\} \in \mathcal{C}, \{4, 8\} \in \mathcal{C}, \text{ and } \{6\} \in \mathcal{C}, \text{ but } 6 \notin \mathcal{C}, 3 \notin \mathcal{C}, \{3\} \notin \mathcal{C}, \text{ and } \{2, 4\} \notin \mathcal{C}.

A subcollection of a collection \(\mathcal{C}\) is a collection \(\mathcal{D}\) such that every set in the collection \(\mathcal{D}\) is also a set in the collection \(\mathcal{C}\). Thus, \(\{\{4, 8\}, \{5, 10\}\} \subseteq \mathcal{C}\) says that \(\{\{4, 8\}, \{5, 10\}\}\) is a subcollection of \(\mathcal{C}\). The definition of subcollection is precisely the definition of subset but with a shift of terminology to compensate for the fact that our "set" of "elements" is, in this case, called a "collection" of "sets." Some other examples may clarify these definitions.

Consider the collection of all subsets of \{1, 2\}. In Example 1.1.2 we found all the elements of this collection. The collection of all subsets of \{1, 2\} is \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}. This is an important construction and has a special name.

1.1.4 **Definition**  The collection of all subsets of a given set \(S\), denoted \(\mathcal{P}(S)\). Thus, \(\mathcal{P}(S) = \{A | A \subseteq S\}\).

Example 1.1.3 shows that the power set of \{1, 2, 3\} is
\[\mathcal{P}({1, 2, 3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}\.

We may consider subcollections of \(\mathcal{P}({1, 2, 3})\) such as the collection \(\mathcal{D}\) of all subsets of \{1, 2, 3\} that contain the element 2:
\[\mathcal{D} = \{A \in \mathcal{P}({1, 2, 3}) | 2 \in A\} = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}\.

The collection \(\mathcal{F}\) of subsets of \{1, 2, 3\} that have cardinality 2 is
\[\mathcal{F} = \{A \in \mathcal{P}({1, 2, 3}) | |A| = 2\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\.

We have \(\mathcal{D} \subseteq \mathcal{P}({1, 2, 3})\) and \(\mathcal{F} \subseteq \mathcal{P}({1, 2, 3})\). Observe the distinction between "containing two elements" and "containing the element 2." As with any set, we may consider the cardinality of a set of sets—that is, of a collection. Here we have |\(\mathcal{D}\)| = 4, |\(\mathcal{F}\)| = 3 and |\(\mathcal{P}({1, 2, 3})\)| = 8.

The following example will reinforce the importance of distinguishing between an element of a set and a subset of a set.
1.1.5 Example

Let \( A = \{\text{Alabama, Alaska, Arizona, Arkansas}\} \)
\[
E = \emptyset
\]
\[
I = \{\text{Illinois, Indiana, Iowa}\}
\]
\[
O = \{\text{Ohio, Oklahoma, Oregon}\}, \text{ and}
\]
\[
U = \{\text{Utah}\}.
\]
Now, if we let \( \mathcal{V} = \{A, E, I, O, U\} \), then \( \mathcal{V} \) is a collection of five sets and we have
\[
\text{Alaska} \in A
\]
\[
\{\text{Alaska, Arizona}\} \subseteq A
\]
\[
\text{Alaska} \notin \mathcal{V}
\]
\[
\{\text{Alaska, Arizona}\} \notin \mathcal{V}
\]
\[
\{\text{Alaska, Arizona}\} \notin \mathcal{V}
\]
\[
I = \{\text{Illinois, Indiana, Iowa}\} \in \mathcal{V}
\]
\[
I \notin \mathcal{V}
\]
\[
\{O, E, U\} \subseteq \mathcal{V}
\]
\[
\{E, U\} = \{\emptyset\}, \{\text{Utah}\} \subseteq \mathcal{V}
\]
\[
\{U\} \subseteq \mathcal{V}.
\]
Note that Utah \( \in U \) and \( \{\text{Utah}\} \subseteq U \) (in fact, \( \{\text{Utah}\} = U \)), but Utah \( \notin U \). Furthermore, \( U \in \mathcal{V} = \{A, E, I, O, U\} \) and \( \{\{\text{Utah}\}\} = \{U\} \subseteq \mathcal{V} \), but \( \{\text{Utah}\} = U \notin \mathcal{V} \), Utah \( \notin \mathcal{V} \), and Utah \( \notin \mathcal{V} \). There are also some subtleties involving the empty set in this example. From the definition of \( \mathcal{V} \), we see that \( E = \emptyset \in \mathcal{V} \). The empty set, however, is a subset of any set, and in particular, the empty collection is a subcollection of any collection. For our collection \( \mathcal{V} \), we have \( \emptyset \subseteq \mathcal{V} \). Now we have shown that \( E = \emptyset \in \mathcal{V} \) and \( E = \emptyset \subseteq \mathcal{V} \).

This is a rare occurrence. Only in extraordinary circumstances will an element of a set also be a subset of that set. Note that \( E = \emptyset \subseteq \mathcal{V} \) and also \( \{E\} = \{\emptyset\} \subseteq \mathcal{V} \), but \( \emptyset \neq \{\emptyset\} \).

Generally, \( x \neq \{x\} \), and there is no exception for \( x = \emptyset \). While \( \emptyset \) has no elements, \( \{\emptyset\} \) has one element, namely \( \emptyset \). Before leaving this example, we should note that the collection \( \mathcal{V} \) of the five sets \( A, E, I, O, \) and \( U \) is not the same as the set \{\text{Alabama, Alaska, Arizona, Arkansas, Illinois, Indiana, Iowa, Ohio, Oklahoma, Oregon, Utah}\} of the 11 states that start with a vowel. In the next section we will see that this latter set is the union of the collection \( \mathcal{V} \).

Large collections of sets are often expressed using an "index" for each set. For example, suppose a certain class meets for 36 days. Let \( S_1 \) be the set of students present on the first day, \( S_2 \) be the set of students present on the second day, and in general, let \( S_k \) be the set of students present on the \( k \)-th day (\( k \in \{1, 2, 3, \ldots, 36\} \)). The subscript \( k \) is called the index (plural: indices). The collection \( \mathcal{S} \) of all the sets \( \{S_1, S_2, \ldots, S_{36}\} \)
may be represented as an indexed collection using various notations:
\[ S = \{ S_k \mid k = 1, 2, 3, \ldots, 36 \} \]
\[ = \{ S_k \mid k \in \{ 1, 2, 3, \ldots, 36 \} \} \]
\[ = \{ S_k \mid k \in I \} \text{ where } I = \{ 1, 2, 3, \ldots, 36 \} \]
\[ = \{ S_k \}_{k=1}^{36} \]
\[ = \{ S_k \}_{k \in I} \text{ where } I = \{ 1, 2, 3, \ldots, 36 \}. \]

The index \( k \) is a "dummy variable"—we could just as well use \( i, j, \lambda \), or any other symbol. The set of values that the index may assume is called the index set. In the example at hand, the index set is \( I = \{ 1, 2, 3, \ldots, 36 \} \).

**EXERCISES**

1. (a) True or false? \( \{ \text{Red, White, Blue} \} = \{ \text{White, Blue, Red} \} \).
   (b) What is wrong with this statement: Red is the first element of the set \( \{ \text{Red, White, Blue} \} \)?

2. Which has the larger cardinality? The set of letters in the word **MISSISSIPPI** or the set of letters in the word **FLORIDA**?

3. Fill in the blank with the appropriate symbol, \( \in \) or \( \subseteq \).
   (a) \( \{ 1, 2, 3 \} \quad \_ \quad \{ 1, 2, 3, 4 \} \)
   (b) \( 3 \quad \_ \quad \{ 1, 2, 3, 4 \} \)
   (c) \( \{ 3 \} \quad \_ \quad \{ 1, 2, 3, 4 \} \)
   (d) \( \{ a \} \quad \_ \quad \{ \{ a \}, \{ b \}, \{ a, b \} \} \)
   (e) \( \emptyset \quad \_ \quad \{ \{ a \}, \{ b \}, \{ a, b \} \} \)
   (f) \( \{ \{ a \}, \{ b \} \} \quad \_ \quad \{ \{ a \}, \{ b \}, \{ a, b \} \} \)

4. Draw a Venn diagram showing the proper relationship between these sets: \( \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{W}, \text{ and } \mathbb{Z} \).

5. (a) How many subsets does the empty set have?
   (b) How many subsets does the set \( \{ 1 \} \) have?
   (c) Noting the number of subsets of a two-element set and of a three-element set from Examples 1.1.2 and 1.1.3, how many subsets do you think a four-element set \( \{ 1, 2, 3, 4 \} \) would have?
   (d) List all the subsets of the four-element set \( \{ 1, 2, 3, 4 \} \).
   (e) How many subsets do you think a five-element set would have? A six-element set? An \( n \)-element set?

6. Determine whether the sets below are well-defined or not. For each well-defined set, state whether it is finite or infinite.
   (a) The set of women pregnant with twins at some time during this year.
   (b) The set of kangaroos in Australia.
   (c) The set of tall buildings.
   (d) The set of grains of sand on the earth.
   (e) The set of even integers.
   (f) The set of hairs on your head.
   (g) The collection of all subsets of the set of hairs on your head.
   (h) The set of people who shook hands with George Washington.
7. (a) Are there well-defined sets in Exercise 6 for which we may not know the answer to the question “Is \( x \) an element of this set?” for every object \( x \)?
(b) Are there finite sets in Exercise 6 for which we do not know the cardinality?

8. Let \( S_1 = \{o, n, e\}, S_2 = \{t, w, o\}, S_3 = \{t, h, r, e, e\}, \) and so on.
(a) Find all \( k \in \{1, 2, \ldots, 10\} \) with \(|S_k| = 4\).
(b) Find distinct indices \( j, k \in \mathbb{N} \) with \( S_j = S_k \).
(c) Find the smallest value of \( k \in \mathbb{N} \) with \( a \in S_k \).
(d) Let \( \mathcal{S} = \{S_k\}_{k=1}^{16} \). Determine whether the following statements are true or false.

\[
\begin{align*}
\text{i.} & \quad S_{13} = \{n, e, i, t, h, e, r\} \\
\text{ii.} & \quad \{n, e, i\} \subseteq S_{20} \\
\text{iii.} & \quad S_1 \subseteq \mathcal{S} \\
\text{iv.} & \quad S_3 \subseteq \mathcal{S} \\
\text{v.} & \quad \emptyset \notin \mathcal{S} \\
\text{vi.} & \quad \emptyset \subseteq \mathcal{S} \\
\text{vii.} & \quad S_1 \subseteq S_{11} \\
\text{viii.} & \quad S_1 \subseteq S_{21} \\
\text{ix.} & \quad S_1 \subseteq S_{21} \\
\text{x.} & \quad \{n, i, e\} \in \mathcal{S} \\
\text{xi.} & \quad \{f, o, u, r\} \subseteq \mathcal{S} \\
\text{xii.} & \quad u \in S_{40} \\
\text{xiii.} & \quad \mathcal{P}(S_0) \subseteq \mathcal{P}(S_{19}) \\
\text{xiv.} & \quad \mathcal{P}(S_0) \subseteq \mathcal{P}(S_{19}) \\
\text{xv.} & \quad \{s, t\} \in \mathcal{P}(S_6) \\
\text{xvi.} & \quad w \in \mathcal{P}(S_2)
\end{align*}
\]

9. For \( k \in \{1, 2, \ldots, 20\} \), let \( D_k = \{x \mid x \text{ is a prime number that divides } k\} \) and let \( \mathcal{D} = \{D_k \mid k \in \{1, 2, \ldots, 20\}\} \).
(a) Find \( D_1, D_2, D_{10}, \) and \( D_{20} \).
(b) True or false:

\[
\begin{align*}
\text{i.} & \quad D_2 \subseteq D_{10} \\
\text{ii.} & \quad D_7 \subseteq D_{10} \\
\text{iii.} & \quad D_{10} \subseteq D_{20} \\
\text{iv.} & \quad \emptyset \in \mathcal{D} \\
\text{v.} & \quad \emptyset \subseteq \mathcal{D} \\
\text{vi.} & \quad 5 \in \mathcal{D} \\
\text{vii.} & \quad \{5\} \in \mathcal{D} \\
\text{viii.} & \quad \{4, 5\} \in \mathcal{D} \\
\text{ix.} & \quad \{3\} \subseteq \mathcal{D} \\
\text{x.} & \quad \mathcal{P}(D_0) \subseteq \mathcal{P}(D_6) \\
\text{xi.} & \quad \mathcal{P}(\{3, 4\}) \subseteq \mathcal{D} \\
\text{xii.} & \quad \{2, 3\} \in \mathcal{P}(D_{12})
\end{align*}
\]

(c) Find \(|D_{10}|\) and \(|D_{19}|\).
(d) Find \(|\mathcal{D}|\).

10. Give an example of an indexed collection \( \mathcal{S} = \{S_k\}_{k=1}^{16} \) with \(|\mathcal{S}| = 3 \).

1.2

Set Operations

There are some standard set operations used to derive new sets from given sets.

Intersection and Union

Given sets \( S \) and \( T \), the intersection of \( S \) and \( T \), denoted \( S \cap T \), is the set of elements which are in both \( S \) and \( T \). The union of sets \( S \) and \( T \), denoted \( S \cup T \), is the set of all elements which are in either \( S \) or \( T \) or both.

\[
S \cap T = \{x \mid x \in S \text{ and } x \in T\}
\]
\[
S \cup T = \{x \mid x \in S \text{ or } x \in T\}
\]
It is clear from the definitions that for any sets $S$ and $T$, $S \cap T = T \cap S$ and $S \cup T = T \cup S$. The Venn diagrams in Figure 1.2 depict the sets $S$ and $T$, the intersection $S \cap T$, and the union $S \cup T$.

![Venn Diagrams](image)

**Figure 1.2**
Intersections and unions.

For example, suppose $A = \{1, 3, 5, 7\}$, $B = \{3, 4, 5, 6\}$, and $C = \{2, 4\}$. Then we have

- $A \cap B = \{3, 5\}$
- $A \cup B = \{1, 3, 4, 5, 6, 7\}$
- $A \cap C = \emptyset$
- $A \cup C = \{1, 2, 3, 4, 5, 7\}$
- $B \cap C = \{4\}$
- $B \cup C = \{2, 3, 4, 5, 6\}$

Two sets with no "overlap," such as $A$ and $C$ above, are said to be disjoint. Formally, sets $S$ and $T$ are disjoint if $S \cap T = \emptyset$.

Let us consider another example. Let $U$ be the set of students enrolled at Ottawa University. In this example, we will only consider subsets of this set $U$. Such a set $U$ containing all the objects to be considered is called the universal set for the problem in question.

Let $H = \{x \in U \mid x \text{ has black hair}\}$.

Let $E = \{x \in U \mid x \text{ has green eyes}\}$. 
Now $H$ is the set of Ottawa University students with black hair and $E$ is the set of Ottawa University students with green eyes.

$$H \cap E = \{x \in U | x \in H \text{ and } x \in E\}$$

= the set of Ottawa University students with black hair and green eyes.

$$H \cup E = \{x \in U | x \in H \text{ or } x \in E\}$$

= the set of Ottawa University students with black hair or green eyes.

Observe that those Ottawa University students with black hair and green eyes qualify to be an element of $H \cup E$ in two ways. If a waitress asks you if you would like french fries or a baked potato, she is using an “exclusive or”: She means you may select one or the other but not both. In mathematics, the word or is interpreted as an “inclusive or,” so saying “P or Q” means “P or Q or both.” If a student is admitted to the Ottawa University black-haired green-eyed student union $H \cup E$ because she has black hair, she will not be thrown out if she also happens to have green eyes. Recognizing that some students may qualify as a member of $H \cup E$ by two criteria is important in counting the number of elements in the union $H \cup E$. Suppose we know that Ottawa University has 127 black-haired students and 73 green-eyed students; that is, suppose $|H| = 127$ and $|E| = 73$. Without further information, we cannot determine the number $|H \cup E|$ of students with black hair and green eyes. If we simply add $|H|$ and $|E|$, the dually qualified students—those with black hair and green eyes; that is, the elements of $H \cap E$—are counted twice (Fig. 1.3). To find the correct answer, we must add $|H|$ and $|E|$, then subtract the number $|H \cap E|$ that were counted twice. Thus we have

$$|H \cup E| = |H| + |E| - |H \cap E|.$$ 

![Figure 1.3](image)

$|H| + |E|$ counts the elements of $H \cap E$ twice. To compensate, subtract $|H \cap E|$ once.

Without further information on the size of the overlap $H \cap E$, we may only find a possible range of values for $|H \cup E|$. At one extreme, if $H$ and $E$ do not overlap at all—that is, if $H$ and $E$ are disjoint (Fig. 1.4)—then there are no students with both black hair and green eyes, so no one is counted twice and

$$|H \cup E| = |H| + |E| - |H \cap E|$$

$$= |H| + |E| - |\emptyset|$$

$$= 127 + 73 - 0 = 200.$$
At the other extreme, the largest overlap $H \cap E$ occurs if all 73 green-eyed students also have black hair. In this case, $E \subseteq H$, so $H \cap E = E$ and
\[
|H \cup E| = |H| + |E| - |H \cap E|
= |H| + |E| - |E|
= |H| = 127.
\]

Equivalently, if $E \subseteq H$ (Fig. 1.5), then $H \cup E = H$ and therefore $|H \cup E| = |H| = 127$.

Thus, if we only know $|H| = 127$ and $|E| = 73$, we may conclude that the intersection $H \cap E$ has $n$ elements for some $n$ satisfying $0 \leq n \leq 73$. The formula $|H \cup E| = |H| + |E| - |H \cap E|$ now tells us that the union $H \cup E$ has from 127 to 200 elements. The exact answer can be determined only if we know $|H \cap E|$ exactly. If we determine that there are exactly 32 black-haired, green-eyed students at Ottawa University, then we would know that $|H \cup E| = |H| + |E| - |H \cap E| = 127 + 73 - 32 = 168$.

In the discussion above on $|H \cup E|$, we implicitly encountered the fact that the number of elements of a subset $T$ of set $S$ cannot exceed the number of elements of $S$. That is, if $T \subseteq S$, then $|T| \leq |S|$. In particular, since $H \cap E \subseteq H$ and $H \cap E \subseteq E$, we have $|H \cap E| \leq |H|$ and $|H \cap E| \leq |E|$. Similarly, since $S \subseteq H \cup E$ and $E \subseteq H \cup E$, we have $|H| \leq |H \cup E|$ and $|E| \leq |H \cup E|$.

We summarize our results here.

1.2.1 Theorem Suppose $S$ and $T$ are finite sets. Then

(a) $|S \cup T| = |S| + |T| - |S \cap T|$.

(b) If $S$ and $T$ are disjoint, then $|S \cup T| = |S| + |T|$.

(c) If $S \subseteq T$, then $|S| \leq |T|$.

We may define the intersection or union of more than two sets. If $\mathcal{S} = \{S_i | i \in I\}$ is a collection of sets $S_i$ indexed by the set $I$, then we define the intersection of the collection $\mathcal{S}$ to be the intersection of all the sets in the collection:

$$\bigcap_{i \in I} \mathcal{S} = \bigcap_{i \in I} S_i = \{x | x \in S_i \text{ for every } i \in I\}.$$

The intersection of a collection is thus the set of elements common to all of the sets in the collection.
The union of the collection \( \mathcal{C} = \{S_i \mid i \in I\} \) is the union of all the sets in the collection:

\[
\bigcup_{i \in I} \mathcal{C} = \bigcup_{i \in I} S_i = \{x \mid x \in S_i \text{ for at least one } i \in I\}.
\]

The union of a collection consists of all the elements that belong to at least one of the sets in the collection.

If the index set \( I \) is finite, say \( I = \{1, 2, 3, \ldots, n\} \), then we may write

\[
\bigcap_{i \in I} S_i = \bigcap_{i=1}^n S_i = S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_n
\]

and a similar notation holds for unions.

We say that the sets of a collection \( \{S_i \mid i \in I\} \) are mutually disjoint (or pairwise disjoint) if \( S_i \neq S_j \) implies \( S_i \cap S_j = \emptyset \), or equivalently, if \( S_i \cap S_j \neq \emptyset \) implies \( S_i = S_j \). Thus, \( \{S_i \mid i \in I\} \) is a collection of mutually disjoint sets if every pair of distinct sets in the collection is disjoint.

A collection of sets \( \{S_i \mid i \in I\} \) is called a nested collection if for any \( i, j \in I \), either \( S_i \subseteq S_j \) or \( S_j \subseteq S_i \).

### 1.2.2 Example

Given below are three collections of intervals on the real line. For each collection, determine whether the sets of the collection are mutually disjoint. Determine whether the collection is nested. Find the union of the collection and the intersection of the collection.

\[\mathcal{A} = \{(n, n + 1) \mid n \in \mathbb{Z}\}\]

\[\mathcal{B} = \{(n, n + 2) \mid n \in \mathbb{Z}\}\]

\[\mathcal{C} = \{(-\frac{1}{n}, 1 + \frac{1}{n}) \mid n \in \mathbb{N}\}\]

**Solution**

The sets of \( \mathcal{A} \) are mutually disjoint, for if \( m \) and \( n \) are distinct integers, then the intervals \((m, m + 1)\) and \([n, n + 1)\) are disjoint. The sets of \( \mathcal{B} \) are not mutually disjoint, for given any integer \( n \), the number \( n + 1.5 \) is an element of two distinct sets \((n, n + 2)\) and \((n + 1, n + 3)\) of \( \mathcal{B} \). The sets of \( \mathcal{C} \) are not mutually disjoint, for \( 0 \) is contained in the intersection of any pair of sets of \( \mathcal{C} \).

Neither \( \mathcal{A} \) nor \( \mathcal{B} \) is a nested collection, for each contains pairs of nonempty disjoint sets. \( \mathcal{C} \) is a nested collection, for given any \( m, n \in \mathbb{N} \), we have either \( m \leq n \) or \( n \leq m \). Without loss of generality, assume \( m \leq n \). Then \((-\frac{1}{n}, 1 + \frac{1}{n}) \subseteq (-\frac{1}{m}, 1 + \frac{1}{m})\).

As every real number \( x \) is contained in the interval \([n, n + 1]\) where \( n \) is the greatest integer less than or equal to \( x \), we see that \( \mathbb{R} \subseteq \bigcup \mathcal{A} \). On the other hand, since \([n, n + 1] \subseteq \mathbb{R}\) for every integer \( n \), we have \( \bigcup \mathcal{A} = \bigcup_{n \in \mathbb{Z}} [n, n + 1] \subseteq \mathbb{R} \), and thus \( \bigcup \mathcal{A} = \mathbb{R} \). Similarly, \( \bigcup \mathcal{B} = \mathbb{R} \). As the intervals of the collection \( \mathcal{C} \) were nested with the intervals getting smaller as the index increases, we find that the largest interval in \( \mathcal{C} \) corresponds to the smallest index. That is, every interval of \( \mathcal{C} \) is nested inside the first interval \((-\frac{1}{1}, 1 + \frac{1}{1}) = (-1, 2)\). Thus, \( \bigcup \mathcal{C} = (-1, 2) \).

Since both collections \( \mathcal{A} \) and \( \mathcal{B} \) contain pairs of disjoint intervals, we have \( \bigcap \mathcal{A} = \emptyset = \bigcap \mathcal{B} \). Since \( \mathcal{C} \) is a nested collection, \( \bigcap \mathcal{C} \) would equal the smallest interval of
if \( \mathcal{C} \) had a smallest interval. However, \( \mathcal{C} \) has no smallest interval. It is easy to see that every element of \([0, 1]\) is contained in every interval \((-\frac{1}{n}, 1 + \frac{1}{n})\) of \( \mathcal{C} \) and that every number outside of the interval \([0, 1]\) is excluded from some interval of \( \mathcal{C} \). Thus, \( \bigcap \mathcal{C} = [0, 1] \).

### Complements

Suppose \( A \) is the set of month names that contain an \( a \), \( R \) is the set of month names that contain an \( r \), and \( Y \) is the set of month names that contain a \( y \). The universal set in this setting would be the set \( U \) of all twelve month names, and we have

\[
A = \{\text{January, February, March, April, May, August}\}
\]

\[
R = \{\text{January, February, March, April, September, October, November, December}\}
\]

\[
Y = \{\text{January, February, May, July}\}
\]

The intersection \( A \cap R \cap Y \) is the set of month names that contain an \( a \), an \( r \), and a \( y \); and thus, \( A \cap R \cap Y = \{\text{January, February}\} \). The union of \( A \), \( R \), and \( Y \) is \( A \cup R \cup Y = \{\text{January, February, March, April, May, July, August, September, October, November, December}\} \). Observe the placement of the 12 months in the eight regions of the Venn diagram in Figure 1.6.

![Figure 1.6](image)

\( A \cap R \cap Y = \{\text{January, February}\}; \ A \cup R \cup Y = \{\text{January, February, March, April, May, July, August, September, October, November, December}\} \).

In the previous example, the 11-element subset \( A \cup R \cup Y \) of the 12-element universal set \( U \) could be described by specifying which element of \( U \) was left out. The set of elements left out of a set \( S \) is called the complement of the set \( S \).

\section*{1.2.3 Definition}

If \( S \) is any subset of the universal set \( U \), then the complement of \( S \) in \( U \) (Fig. 1.7), denoted \( S^c \) or \( U \setminus S \), is the set \( S^c = U \setminus S = \{x \in U \mid x \notin S\} \).
From the definition or the Venn diagram, it should be clear that \((S^c)^c = S\).

The notation \(U \setminus S\) has the advantage that it explicitly names the universal set, which we may want to alter or restrict at times. Suppose Al moves to the gulf coast so he can catch and eat fresh oysters every month, except December and January, which are too cold for Al. The set of months that Al eats fresh oysters may be described as \(U \setminus \{\text{December, January}\}\), where \(U\) is the universal set of the 12 months. However, once Al learns from the locals that oysters should only be eaten in months with an \(r\), the set of months that Al eats fresh oysters may be described as \(R \setminus \{\text{December, January}\}\), where \(R\) is the set of months that contain an \(r\).

1.2.4 **DEFINITION**  The set \(R \setminus S = \{x \in R \mid x \notin S\}\) is called the *complement* of \(S\) in \(R\). To emphasize that the set \(R\) may not be the entire universal set, sometimes \(R \setminus S\) is called the *relative complement* of \(S\) in \(R\).

To find \(R \setminus S\), we start with all the elements of \(R\) and throw out any that are also in \(S\) (Fig. 1.8). For this reason, \(R \setminus S\) is typically called a *set difference*. For example, if \(R\) is the set of right-handed people and \(S\) is the set of people under 5 feet tall, then \(R \setminus S\) is the set of right-handed people who are 5 feet tall or taller.

It is easy to show that if \(R\) and \(S\) are subsets of the universal set \(U\), then

\[
R \setminus S = R \cap S^c.
\]

The proof is left as an exercise.

The set \(\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}\) is the set of real numbers that are not rational. This set is called the set of *irrational numbers*.
Two important set-theoretic properties describing interactions of intersections, unions, and complements are given below. A more complete discussion and proofs of these statements will appear in Sections 1.4 and 1.5.

**De Morgan’s Laws:** If \( \{S_i\}_{i \in I} \) is a collection of sets, then
\[
\left( \bigcup_{i \in I} S_i \right)^c = \bigcap_{i \in I} (S_i^c)
\]
and
\[
\left( \bigcap_{i \in I} S_i \right)^c = \bigcup_{i \in I} (S_i^c).
\]
These may be stated as the complement of a union of sets is the intersection of the complements and the complement of an intersection of sets is the union of the complements.

**Distributive Laws:** For sets \( A, B, \) and \( C, \)
\[
(A \cup B) \cap C = (A \cap C) \cup (B \cap C)
\]
and
\[
(A \cap B) \cup C = (A \cup C) \cap (B \cup C).
\]

**Cartesian Products**

Suppose you have three vehicles: a pickup, a Buick, and a Jaguar. You have two options for Saturday evening entertainment: the opera or bowling. To decide what to drive and where to go, you must choose one element from the set \( V = \{\text{pickup, Buick, Jaguar}\} \) of vehicles and one element from the set \( E = \{\text{opera, bowling}\} \) of excursion options. If you decide to drive the pickup, there remain two options for the excursion. We will list these two possible outcomes as ordered pairs (pickup, opera) and (pickup, bowling). Similarly, the other possible outcomes are (Buick, opera), (Buick, bowling), (Jaguar, opera), and (Jaguar, bowling). We can visualize the decision process and the possible outcomes by a *tree diagram* as shown in Figure 1.9.

![Figure 1.9](image)

**A tree diagram.**

The set of outcomes is the six-element set \( \{(\text{pickup, opera}), (\text{pickup, bowling}), (\text{Buick, opera}), (\text{Buick, bowling}), (\text{Jaguar, opera}), (\text{Jaguar, bowling})\} \) consisting of every possible ordered pair whose first coordinate is an element of \( V \) and whose second
coordinate is an element of $E$. This set is called the Cartesian product of the sets $V$ and $E$, denoted $V \times E$.

1.2.5 **DEFINITION** The Cartesian product of two nonempty sets $A$ and $B$, denoted $A \times B$, is the set

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

of all ordered pairs whose first coordinate is an element of $A$ and whose second coordinate is an element of $B$.

The most familiar Cartesian product is the product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$. By considering "solution sets" of equations, we have a link between algebraic equations like $y = 3x + 1$ and the geometric subset $\{(x, y) | x \in \mathbb{R} \land y = 3x + 1\}$ of the plane. This link is the foundation of analytic geometry and was utilized by René Descartes (1596–1650), Pierre de Fermat (1601–1665), and others. Cartesian products are named in honor of René Descartes.

In the example above, $|V| = 3$, $|E| = 2$ and $|V \times E| = 3 \cdot 2 = 6$. If $A$ and $B$ are any finite sets, we may count the elements of $A \times B = \{(a, b) | a \in A, b \in B\}$ by considering the tree diagram for $A \times B$. At the first branching, there is one limb for each element of $A$. At the second branching, we append $|B|$ limbs onto the end of each of the $|A|$ existing limbs. Thus, the total number of limb tips on the right of the tree diagram is $|B| \cdot |A| = |A| \cdot |B|$. This gives us the following result.

1.2.6 **THEOREM** If $A$ and $B$ are nonempty finite sets, then $|A \times B| = |A| \cdot |B|$.

A more formal proof of the theorem above follows on page 22. Using "$	imes$" to represent multiplication of real numbers as well as the Cartesian product of sets, the result can be restated as $|A \times B| = |A| \times |B|$.

1.2.7 **EXAMPLE** Jimmy and Stacy want to go to the movies on Wednesday, Thursday, Friday, or Saturday of next week. There are three movies playing at the cinema: *Amadeus*, *Beethoven*, and *Critter from the Brown Lake*. How many possible outcomes do they have for the selection of a day and a movie? Draw a tree diagram to illustrate the possible outcomes.

**Solution** Jimmy and Stacy must choose a day from the set $D = \{W, T, F, S\}$ and a movie from the set $M = \{A, B, C\}$. The set of all their possible outcomes is $\{(d, m) | d \in D, m \in M\} = D \times M$. Thus, they have $|D \times M| = |D| \cdot |M| = 4 \cdot 3 = 12$ possible outcomes, as illustrated in the tree diagram of Figure 1.10.

Extending the problem above, suppose that each of the movies has two showtimes each night—7 PM and 9 PM. If Jimmy and Stacy wish to select a day, movie, and time, they must select one element from $D$ as above, one element from $M$ as above, and one element from the set $T = \{7, 9\}$ of possible showtimes. Their selection will be an ordered triple $(d, m, t)$, where $d \in D$, $m \in M$, and $t \in T$. The set of all such ordered triples is the Cartesian product of $D$, $M$, and $T$: $D \times M \times T = \{(d, m, t) | d \in D, m \in M, t \in T\}$. 
The elements of $D \times M \times T$ can be illustrated by a tree diagram with three branching stages (Fig. 1.11). The first two branchings correspond to the selection of day and movie as shown in Example 1.2.7 above. Having selected a particular $d \in D$ and a particular $m \in M$, there are two time options, 7 or 9; thus, each existing branch forks into two branches at the third branching stage.

The total number of (day, movie, time) options is $4 \cdot 3 \cdot 2 = 24$. That is,

$$|D \times M \times T| = |D| \cdot |M| \cdot |T|.$$ 

Our motivation for defining Cartesian products of three sets and the method for counting the elements in a Cartesian product of three sets extend to Cartesian products of any finite number of sets.

**1.2.8 Definition** If $S_1$, $S_2$, $S_3$, ..., $S_n$ are nonempty sets, then the Cartesian product $S_1 \times S_2 \times S_3 \times \cdots \times S_n$ is the set \{$(s_1, s_2, \ldots, s_n) | s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$\} of all ordered $n$-tuples whose $k$-th coordinate $s_k$ is an element of the $k$-th set $S_k$ for each $k = 1, 2, \ldots, n$.

We may denote the Cartesian product $S_1 \times S_2 \times S_3 \times \cdots \times S_n$ by the notation $\prod_{i=1}^n S_i$. The symbol $\prod$ is the Greek letter pi, corresponding to the first letter of the word product. This notation is comparable to the "sigma notation" $\sum_{i=1}^n a_i$ for the sum $a_1 + a_2 + \cdots + a_n$. Observe that the Greek sigma $\sum$ corresponds to the initial letter of the word sum.
1.2.9 THEOREM If $S_1, S_2, S_3, \ldots, S_n$ are nonempty finite sets, then

$$|S_1 \times S_2 \times S_3 \times \cdots \times S_n| = |S_1| \cdot |S_2| \cdot |S_3| \cdots \cdot |S_n|.$$

A formal proof of this theorem will be presented in Section 2.2. For now, it is enough to keep in mind our arguments based on what the tree diagram for the Cartesian product would look like. Observe that if $n = 2$, the theorem reduces to Theorem 1.2.6.

If $S_1 = S_2 = S_3 = \cdots = S_n = S$, then the Cartesian product

$$S_1 \times S_2 \times S_3 \times \cdots \times S_n = S \times S \times \cdots \times S \times S$$

is denoted $S^n$. The familiar examples of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$ and $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ follow this convention. If $S$ is a finite set, the theorem above implies $|S^n| = |S|^n$.

EXERCISES

1. Let $S = \{1, 3, 5, 7, 9\}$, $T = \{1, 2, 3, 4, 5\}$, and $V = \{3, 6, 9\}$. List the elements of the specified sets.

(a) $S \cap T$  
(b) $S \cup T$  
(c) $S \cap V$  
(d) $S \cup V$  
(e) $(T \cap V) \cup S$  
(f) $T \cap (V \cup S)$  
(g) $V \times T$  
(h) $V \times (T \cap S)$
2. Let \( U \) be the set of 52 cards in a standard deck. (For a description of a standard deck, see the paragraph following Example 4.5.2 on page 158.) Let \( S \) be the set of spades, \( D \) the set of diamonds, \( A \) the set of aces, and \( K \) the set of kings. Tell which cards belong to each set below, and find the cardinality of each set.

(a) \( A \cap D \)  
(b) \( S \cap D \)  
(c) \( A \cap (S \cup D) \)  
(d) \( (A \cup K) \cap (S \cup D) \)  
(e) \( (A \cap S) \cup (K \cap D) \)  
(f) \( D^c \)  
(g) \( K \cap S^c \)  
(h) \( K \cap (S \cup D)^c \)  
(i) \( (A \cup K)^c \cap S \)  
(j) \( A \times K \)  
(k) \( A \times K^c \)  
(l) \( S \times S^c \)  
(m) \( S \setminus K \)  
(n) \( K \setminus S \)  
(o) \( (K \setminus S) \times S \)

3. Let \( M_3 = \{3, 6, 9, 12, 15, \ldots\} \) and \( M_5 = \{5, 10, 15, 20, 25, \ldots\} \). Describe:

(a) \( M_3 \cap M_5 \)  
(b) \( M_3 \setminus M_5 \)  
(c) \( \mathbb{N} \setminus M_3 \)  
(d) \( M_3 \cup M_5 \)

4. For each collection given below, determine whether the sets of the collection are mutually disjoint. Also, determine whether the collection is nested, and find the union and intersection of the collection. (We use interval notation below:

\[ (a, b) = \{x \in \mathbb{R} | a < x < b\} \]

(a) \( \mathcal{A} = \{\left[\frac{1}{n}, n + 1\right] | n \in \mathbb{N}\} \)  
(b) \( \mathcal{B} = \{\{\frac{1}{n}, n\} | n \in \mathbb{N}\} \)  
(c) \( \mathcal{C} = \{(n, \infty) | n \in \mathbb{N}\} \)

5. Let the collections \( \mathcal{A} \) through \( \mathcal{E} \) be defined as in Exercise 4. Determine whether the following statements are true or false.

(a) \( \emptyset \in \mathcal{A} \)  
(b) \( \emptyset \subseteq \mathcal{A} \)  
(c) If \( B \) is any element in \( \mathcal{B} \), then \( 0 \in B \).

(d) \( \mathcal{B} \subseteq \mathcal{A} \)  
(e) \( \mathcal{C} \subseteq \mathcal{D} \)  
(f) \( \mathcal{B} \cap \mathcal{C} = \emptyset \)  
(g) \( \mathcal{B} \cap \mathcal{E} = \emptyset \)  
(h) \( (-2, 2) \subseteq \mathcal{E} \)  
(i) \( (-\sqrt{3}, \sqrt{3}) \in \mathcal{E} \)  
(j) \( \bigcap \mathcal{E} \in \mathcal{B} \)

6. If \( R \) and \( S \) are subsets of the universal set \( U \), show that \( R \setminus S = R \cap S^c \).

7. Show that \( A \times (B \cap C) = (A \times B) \cap (A \times C) \).

8. For any two finite sets \( S \) and \( T \), show that the average of \(|S|\) and \(|T|\) does not exceed \(|S \cup T|\).

9. Given a collection of sets \( \{S_i | i \in I\} \), is the condition that

\[ S_i \cap S_j \neq \emptyset \quad \text{implies} \quad i = j \]

equivalent to the condition that

\[ S_i \cap S_j \neq \emptyset \quad \text{implies} \quad S_i = S_j \]?

Explain.
10. At Scottsville Junior High, 43 eighth graders are taking Algebra, 32 are taking Spanish, and 7 are taking Algebra and Spanish. How many are taking Algebra or Spanish?

11. A craftsman produces 100 wooden trucks. He classifies the best ones as “premium quality” and charges a higher price for them. He first inspects the trucks for painting imperfections and excludes 31 trucks on this criterion. Then he inspects the remaining trucks for woodworking errors (wobbly wheels) and excludes 19 more. He remarks that more trucks had woodworking errors than painting irregularities. How can this be?

12. At a diner, the special consists of your choice of meatloaf or chicken, served with mashed potatoes and your choice of one vegetable from seven on the menu and one dessert from five on the menu. How many ways can you place an order for the special?

13. Four runners, A, B, C, and D, are in a race. Draw a tree diagram showing all the possible outcomes (with no ties).

14. For each room rented, a hotel records a three-symbol code consisting of a 1, 2, 3, or 4+ to specify the number of occupants, an S or N to specify smoking or nonsmoking, and an S or O to specify street view or ocean view. How many different codes are there? Illustrate them with a tree diagram.

15. Next semester, Luis wants to take only physics and chemistry on Mondays. Sections of physics are offered at 11:00, 12:00, 1:00, and 2:00 on Mondays, and Monday sections of chemistry are offered at 2:00, 3:00, and 4:00. Draw a tree diagram showing all possible outcomes for his Monday class schedule.

16. A combination lock has a dial with the numbers 1, 2, . . . , 60. To open the lock, you must turn the dial clockwise to a specified number, then turn the dial counterclockwise to a second specified number, then turn the dial clockwise to a third specified number. The specified numbers are called the combination key.
   (a) How many combination keys are possible?
   (b) How many combination keys are possible if the first and third numbers must be even?

1.3 Partitions

Large sets are often divided into categories. The set of students at a large 4-year university is divided into the four categories of freshman, sophomore, junior, and senior. The set of cards in a standard deck is divided into the four suits: hearts, clubs, diamonds, and spades. The set of people is divided into the categories of male and female.

The “categories” in these cases are mutually disjoint since no element can belong to more than one category. Together, the categories make up the entire original set since every element of the original set falls into one of the categories. Each category is
nonempty. A division of a set into categories satisfying these three conditions provides a partition of the set (Fig. 1.12).

1.3.1 **DEFINITION** A partition of a nonempty set $S$ is a collection $\mathscr{C} = \{B_i\}_{i \in I}$ of nonempty mutually disjoint subsets of $S$ with $\bigcup_{i \in I} B_i = S$. The sets $B_i \in \mathscr{C}$ are called the blocks of the partition.

1.3.2 **EXAMPLE** The collection $\{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$ is a partition of the set $S = \{1, 2, 3, 4, 5, 6\}$ since the blocks $B_1 = \{1, 2\}$, $B_2 = \{3, 4, 5\}$ and $B_3 = \{6\}$ are nonempty, mutually disjoint, and their union $\{1, 2\} \cup \{3, 4, 5\} \cup \{6\}$ is $S$.

1.3.3 **EXAMPLE** The set of capitals of the 50 U.S. states is partitioned into the set $H$ of state capitals in the Hawaii time zone, the set $A$ of state capitals in the Alaska time zone, the set $P$ of state capitals in the Pacific time zone, the set $M$ of state capitals in the Mountain time zone, the set $C$ of state capitals in the Central time zone, and the set $E$ of state capitals in the Eastern time zone.

No state capital falls into more than one time zone, so the collection $\{H, A, P, M, C, E\}$ is mutually disjoint. The sets $H, A, P, M, C,$ and $E$ are all nonempty. (Honolulu $\in H$, so $H \neq \emptyset$. The proofs that $A, P, M, C,$ and $E$ are nonempty are left to the reader.)
Finally, $H \cup A \cup P \cup M \cup C \cup E$ does give the set of all state capitals, since each of the 50 state capitals is in one of the named time zones.

The example above would not work if we replaced “state capitals” by “states,” for some states are not contained in a single time zone.

1.3.4 **Example** At a county fair watermelon-eating contest, participants under 10 years of age are to report to Tent $A$, participants 10–14 years of age are to report to Tent $B$, participants over 14 years of age are to report to Tent $C$, and female participants are to report to Tent $D$. The sets of participants instructed to report to Tents $A$, $B$, $C$, and $D$ do not form a partition of the participants: Female participants are instructed to report to two different tents, so the set of female participants is either empty or not disjoint from the other sets. Even if Tent $D$ above were omitted, the remaining three sets still might not form a partition, for one of the sets could be empty.

Partitions may provide an easy way to count the elements of a set $S$. Suppose $B_1, B_2, \ldots, B_n$ are the distinct blocks of a partition of a finite set $S$. Since each block $B_i$ is nonempty, we have $|B_i| > 0$ for $i = 1, 2, \ldots, n$. Because the blocks are mutually disjoint, we have $|B_1 \cup B_2 \cup \cdots \cup B_n| = |B_1| + |B_2| + \cdots + |B_n|$. Furthermore, since $B_1 \cup B_2 \cup \cdots \cup B_n = S$, we have $|B_1| + |B_2| + \cdots + |B_n| = |S|$. This should not come as a surprise, but it is worth noting the exact steps in the verification.

In Theorem 1.2.6, we saw that for finite nonempty sets $A$ and $B$, $|A \times B| = |A| \cdot |B|$. We justified this result by counting the number of paths through the tree diagram for $A \times B$. We now present a more formal proof, which essentially partitions the paths through the tree diagram into those starting from the first branch, those starting from the second branch, and so on.

**Proof of Theorem 1.2.6** If $A$ and $B$ are nonempty finite sets, then $|A \times B| = |A| \cdot |B|$. Suppose $A$ has $n$ distinct elements $a_1, a_2, \ldots, a_n$. Partition $A \times B$ into blocks $C_1 = \{(a_1, b) | b \in B\}$, $C_2 = \{(a_2, b) | b \in B\}$, $\ldots, C_n = \{(a_n, b) | b \in B\}$. We will check that this is indeed a partition. Since $B \neq \emptyset$, there is an element $b_0 \in B$ and thus, for any $k = 1, 2, \ldots, n$, the block $C_k$ is nonempty, since $(a_k, b_0) \in C_k$. The blocks are mutually disjoint, for if some element $(a, b)$ were in two different blocks $C_j$ and $C_k$ ($j \neq k$), then this would say that the first coordinate $a$ of $(a, b)$ should equal both $a_j$ and $a_k$, which cannot happen since $a_j \neq a_k$. Finally, $(a, b) \in A \times B$ implies $a \in A = \{a_1, a_2, \ldots, a_n\}$ and thus $(a, b) = (a_k, b) \in C_k$ for some $k \in \{1, 2, \ldots, n\}$. This shows that every element of $A \times B$ is in some block; that is, $\bigcup_{k=1}^{n} C_k = A \times B$. Thus, $\{C_k\}_{k=1}^{n}$ is a partition of $A \times B$. It now follows that

$$|A \times B| = |C_1| + |C_2| + \cdots + |C_n|.$$ 

But, each block $C_k = \{(a_k, b) | b \in B\}$ is a set of distinct elements indexed by the set $B$ and therefore has the same cardinality as $B$. Substituting $|B|$ for each $|C_k|$ above gives

$$|A \times B| = |B| + |B| + \cdots + |B| = n|B|.$$ 

$n$ times

Recalling that $n = |A|$, we have $|A \times B| = |A| \cdot |B|$. $\blacksquare$
EXERCISES

1. (a) Under what circumstances does the collection $\{ R, S \setminus R \}$ form a partition of $R \cup S$?
   (b) Under what circumstances does the collection $\{ B_i \mid i \in I \}$ form a partition of $\bigcup_{i \in I} B_i$?

2. If $\{ B_1, B_2, B_3 \}$ is a partition of $S$ and $B_4 = B_1$, is $\{ B_1, B_2, B_3, B_4 \}$ a partition of $S$?
   (See Exercise 1.2.9.)

3. Are the following collections necessarily partitions of the indicated sets? Justify your answers.
   (a) Let $S$ be the set of students at a university, and let $\mathcal{C} = \{ B_0, B_1, B_2, B_3, B_4, B_5 \}$ where
       $B_0$ = the set of students with no credit cards,
       $B_1$ = the set of students with exactly one credit card,
       $B_2$ = the set of students with exactly two credit cards,
       $B_3$ = the set of students with exactly three credit cards,
       $B_4$ = the set of students with exactly four credit cards,
       $B_5$ = the set of students with five or more credit cards.
   (b) Let $B$ be the set of books in the Library of Congress, and let $\mathcal{C} = \{ S, L \}$, where
       $S$ is the set of books in the Library of Congress with fewer than 200 pages
       and $L$ is the set of books in the Library of Congress with 200 or more pages.
   (c) Let $A$ be the set of dresses, and let $\mathcal{C} = \{ S, D, P \}$, where
       $S$ = the set of dresses with stripes,
       $D$ = the set of dresses with dots,
       $P$ = the set of dresses with no stripes.
   (d) Let $ST$ be the set of 50 U.S. states, and let $\mathcal{C} = \{ A, B, C, \ldots, Z \}$, where
       $A$ = the set of states starting with the letter $A$,
       $B$ = the set of states starting with the letter $B$,
       and so on.
   (e) Let $Q$ be the set of quadrilaterals, and let $\mathcal{P} = \{ R, S, T, O \}$, where
       $R$ = the set of rectangles,
       $S$ = the set of squares,
       $T$ = the set of trapezoids,
       $O = Q \setminus (R \cup S \cup T)$.

4. For $m \in \mathbb{N}$, let $C_m = \{ x \in \mathbb{R} \mid m - 1 \leq x^2 < m \}$. Is $\mathcal{C} = \{ C_m \mid m \in \mathbb{N} \}$ a partition of $\mathbb{R}$?

5. Determine which of the following collections form a partition of $\mathbb{R}^2$.
   (a) For $b \in \mathbb{R}$, let $l_b = \{ (x, y) \in \mathbb{R}^2 \mid y = b \}$, and let $\mathcal{C} = \{ l_b \mid b \in \mathbb{R} \}$.
   (b) For $m \in \mathbb{R}$, let $L_m = \{ (x, y) \in \mathbb{R}^2 \mid y = mx \}$, and let $\mathcal{D} = \{ L_m \mid m \in \mathbb{R} \}$.
   (c) For $t \geq 0$, let $S_t = \{ (x, y) \in \mathbb{R}^2 \mid \max \{ |x|, |y| \} = t \}$, and let
       $\mathcal{E} = \{ S_t \mid t \in [0, \infty) \}$.
   (d) For $r \geq 0$, let $C_r = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \}$, and let
       $\mathcal{F} = \{ C_r \mid r \in [0, \infty) \}$.
(e) For \( a \in \mathbb{R} \), let \( P_a = \{(x, y) \in \mathbb{R}^2 \mid y = ax^2\} \), and let \( \mathcal{G} = \{P_a \mid a \in \mathbb{R}\} \).

(f) Let \( \mathbb{I} = \mathbb{R} \setminus \mathbb{Q} \), and let \( \mathcal{H} = \{\mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times \mathbb{I}, \mathbb{I} \times \mathbb{Q}\} \).

6. Give partitions of \( \mathbb{R} \) having (a) one block, (b) two blocks, (c) three blocks, (d) infinitely many blocks.

7. Let \( U \) be the set of 52 cards in a standard deck. (See p. 158 for a description of a standard deck.) Give partitions of \( U \) having (a) 2 blocks, (b) 4 blocks, (c) 11 blocks, (d) 52 blocks.

8. Assume \( \mathcal{B} = \{B_i \mid i \in I\} \) is a partition of \([0, \infty)\). For \( i \in I \), define \( C_i = \{x \in \mathbb{R} \mid x^2 \in B_i\} \). Verify that \( \mathcal{C} = \{C_i \mid i \in I\} \) is a partition of \( \mathbb{R} \).

9. If \( \mathcal{P} = \{B_i \mid i \in J\} \) is a partition of set \( S \) and each block \( B_j \) of this partition is further partitioned into \( \mathcal{D}_j = \{C_i \mid i \in I_j\} \), then the collection \( \mathcal{D} = \bigcup_{j \in J} \mathcal{D}_j \) of the blocks of the blocks of the partition \( \mathcal{P} \) forms a partition of the set \( S \), and we say partition \( \mathcal{D} \) of \( S \) is finer than partition \( \mathcal{P} \), or \( \mathcal{P} \) is a refinement of \( \mathcal{D} \), or partition \( \mathcal{P} \) is coarser than partition \( \mathcal{D} \).

Two partitions \( \mathcal{C} \) and \( \mathcal{D} \) of a set \( S \) are given. Determine whether \( \mathcal{C} \) is finer than \( \mathcal{D} \), \( \mathcal{D} \) is finer than \( \mathcal{C} \), or neither.

(a) \( S = [0, 1]; \mathcal{C} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}, \mathcal{D} = \{[0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1]\} \)

(b) \( S = \mathbb{N}; \mathcal{C} = \{\text{even natural numbers}, \text{odd natural numbers}\}, \mathcal{D} = \{\text{even natural numbers < 99}, \text{even natural numbers > 99}, \text{odd natural numbers}\} \)

(c) \( S = \mathbb{R}; \mathcal{C} = \{\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}, \mathcal{D} = \{(-\infty, 0), [0, \infty)\} \)

(d) \( S = \mathbb{R}; \mathcal{C} = \{\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}, \mathcal{D} = \{x \mid x \in \mathbb{R}\} \)

10. Refer to the definition in Exercise 9. For each partition below, give a finer partition and a partition that is neither finer nor coarser.

(a) The partition \( \mathcal{P} = \{\text{males}, \text{females}\} \) of the set of living people.

(b) The partition \( \mathcal{C} \) of all triangles in the plane into the set of acute triangles, the set of right triangles, and the set of obtuse triangles.

(c) The partition \( \mathcal{D} = \{(-\infty, 0), [0, \infty)\} \) of \( \mathbb{R} \).

11. Refer to the definition in Exercise 9.

(a) Given any nonempty set \( S \), show that the partition \( \mathcal{D} = \{\{x\} \mid x \in S\} \) of \( S \) into singleton sets (sets with one element) is finer than any partition of \( S \).

(b) What is the coarsest partition of a set \( S \)?

(c) If \( \mathcal{P} \) and \( \mathcal{D} \) are partitions of set \( S \), each having a finite number of blocks, and if \( \mathcal{P} \) is finer than \( \mathcal{D} \), how do \( |\mathcal{P}| \) and \( |\mathcal{D}| \) compare?

(d) If \( \mathcal{P} \) and \( \mathcal{D} \) are partitions of set \( S \), each having a finite number of blocks, and if \( |\mathcal{P}| \leq |\mathcal{D}| \), does it follow that one of \( \mathcal{P} \) or \( \mathcal{D} \) is a refinement of the other? Explain.

12. In calculus, the definite integral \( \int_a^b f(x) \, dx \) of a nonnegative function \( f \) represents the area under the curve \( y = f(x) \) over the interval \( x \in [a, b] \). It is approximated by partitioning the interval \([a, b]\) into small subintervals, assuming \( f(x) \) is constant over each subinterval, and summing the areas of the resulting rectangles.
Calculus texts typically define a partition of an interval \([a, b]\) to be a set \(\{x_0, x_1, \ldots, x_n\}\) of real numbers with \(a = x_0 < x_1 < \cdots < x_n = b\). Explain how such a set gives a partition of \([a, b]\) in the sense defined in this section.

### 1.4 Logic and Truth Tables

**Logic**

*Statements* are sentences that are either true or false. We may denote statements by capital letters. The statement

\[ P = \text{Snow is frozen water} \]

is true. The statement

\[ Q = 1.72 \text{ is an integer} \]

is false. The *negation* of a statement \(P\), denoted \(\sim P\) and read “not \(P\),” is the statement “\(P\) fails,” “\(P\) does not occur,” “\(P\) is false,” or “it is not the case that \(P\) happens.” With \(P\) as above, the negation of \(P\) is

\[ \sim P = \text{It is not the case that snow is frozen water} \]

or, in plain English,

\[ \sim P = \text{Snow is not frozen water} \]

Because \(P\) was true, \(\sim P\) is false. The negation of the statement \(Q\) above is

\[ \sim Q = \text{It is not the case that 1.72 is an integer} \]

or

\[ \sim Q = 1.72 \text{ is not an integer} \]

Because \(Q\) was false, \(\sim Q\) is true. The *truth value* of any statement \(P\) is either “true” or “false,” depending on whether the statement \(P\) is true or false. Two statements are *equivalent* if they always have the same truth values. (For example, “\(x + 5 = 8\)” is equivalent to “\(2x = 6\),” for both are true if \(x = 3\) and false if \(x \neq 3\).) A statement and its negation always have opposite truth values. That is, if \(P\) is true, then \(\sim P\) is false, and if \(P\) is false, then \(\sim P\) is true.

Just as a double negative is positive, the negation of the negation of a statement \(P\) is again \(P\). That is, for any statement \(P\), we have \(\sim(\sim P) = P\). For example, if \(P\) is the statement “The pool is dry,” then \(\sim P = \text{The pool is not dry},\) and \(\sim(\sim P) = \text{It is false that the pool is not dry} = \text{The pool is dry} = P\).

We may form *compound statements* by using the words *and* or *or*. If \(P\) and \(Q\) are two input statements, the statement “\(P\) and \(Q\)” is denoted \(P \land Q\) and called the *conjunction* of statements \(P\) and \(Q\). The statement “\(P\) or \(Q\)” is denoted \(P \lor Q\) and is called the *disjunction* of statements \(P\) and \(Q\). Statements that do not involve \(\land, \lor,\) or \(\sim\) are called *simple statements*. 