MTH 355 First Day

De­view Sylla­bus

Sec­tion 4.1

• A set is a col­lec­tion of ob­jects. The ob­jects of a set are the el­e­ments of a set.

• Set do not ac­count for or­der & rep­et­i­tion of el­e­ments.

• We use words to de­scribe a set or list the el­e­ments inside brack­ets {}.

Ex: We can spe­cify a set w/ 4 el­e­ments as

\{Alabama, Alaska, Arizona, Arkansas\}.

We can also de­scribe this as the set con­tain­ing the cur­rent U.S. states whose name be­gins w/ an “A”.

As in most math­e­mat­ics, we can as­sign vari­ables as names. Say

\textbf{B = \{Alabama, Alaska, Arizona, Arkansas\}.}
We use epsilon to specify an element in a set.

Since Alabama, Alaska, Arizona, and Arkansas are elements of B, we write

*Alabama ∈ B*,
*Alaska ∈ B*,
*Arizona ∈ B*, and
*Arkansas ∈ B*.

Oregon is not an element of B. To specify this, we use a slash through the epsilon. In our example,

*Oregon /∈ B*.

Listing the elements explicitly and in brackets is sometimes called roster form.

Recall sets do not account for order.

Example: 

*C = \{a, b, c, d\}*

and 

*D = \{b, a, d, c\}.*

Then yes, 

*C = D*.

Also, sets do not account for repetitions.

Example: 

*A = \{1, 2, 3\}*, 

and 

*B = \{1, 1, 1, 2, 1, 3, 1, 1\}.*

Then yes, 

*A = B*.
In general, two sets are equal if they contain exactly the same elements.

In later classes you may investigate this more thoroughly, but for now you should think of the only restriction on sets being that they must be well-defined in the sense that there is a definitive yes/no answer to whether a given element is in a set. You do not need to know the answer for a given element, just that it is certainly yes or no.

Example Let A be the set of names of the people in this class. Even though you may not know everyone's names, so you do not know if a given name is an element of A, you do know the answer for a given name is yes or no.

We must avoid subjective statements to define a set. We can't say something like, let B be the set of good books, as people can disagree whether or not book is good.
A set can have no elements, this set is called the empty set or null set and is denoted by \( \emptyset \).

A finite set is a set for which a whole number tells the number of elements of a set. The cardinality of a finite set is the number of elements in the set. For a finite set \( S \), the cardinality of \( S \) is denoted by \( |S| \).

Remember sets do not count repetitions!

**Ex.** Let \( C \) be the set of letters in the word “throughout”. Then \( C = \{ t, h, r, o, u, g, h, o, u, t \} \).

But also \( C = \{ t, h, r, o, u, g \} \).

So \( |C| = 6 \), not 10!
We may also write this as
\[ A = \{ x \in \mathbb{Z} \mid 1 \leq x \leq 10 \}. \]

In general, set builder notation works as
\[ \{ x \mid \text{well defined description of conditions on } x \} \]
for \( x \) to be an element of the set.

If the elements are to come from a specific set, say \( S \), then one can also write
\[ \{ x \in S \mid \text{conditions on } x \text{ for } x \text{ to be an element of the set} \}. \]

Two infinite sets that are either difficult to write as a pattern or use ellipses, or impossible to do so, are

The set of rational numbers = \( \mathbb{Q} = \{ a/b \mid a, b \in \mathbb{Z}, b \neq 0 \} \)
The set of real numbers = \( \mathbb{R} = \{ x \mid x \text{ is a real number} \} \).

If you have taken MTH 311, then you know properly defining what is a real number takes a little bit of time.
Given two sets \( S \) and \( T \), we can ask if one contains only elements from the other.

If every element of \( S \) is an element of \( T \), then \( S \) is a **subset** of \( T \), and we write \( S \subseteq T \). We can also instead say \( T \) is a superset of \( S \) and write \( T \supseteq S \).

That

When \( S \) is a subset of \( T \), we may also say \( S \) is contained in \( S \), but this is not the same as \( S \) is an element of \( T \),
when \( S \) is not a subset of \( T \), we write \( S \not\subseteq T \).

Therefore

When \( S \subseteq T \), but \( S \not\subseteq T \), we write \( S \subset T \)
and say \( S \) is a **proper subset** of \( T \).

**Example**

True:
\[
\{a, b, c, 3\} \subseteq \{a, b, c, 3, 4\}
\]
\[
\{a, b, c\} \subset \{a, b, d, 3\}
\]
\[
\{a, b\} \subseteq \{a, b, 3\}
\]

False:
\[
\{a, b, c\} \subseteq \{a, b, 3\}
\]
\[
\{a, b, 3\} \subset \{a, b, c, d\}
\]
\[
\{a, b, 3\} \not\subseteq \{a, b, c\}
\]
\[
\{a, b, c\} \not\subseteq \{a, b, c\}
\]
If a set is not finite, then it is infinite. We will revisit the cardinality of infinite sets later. We can describe an infinite set in words, but we certainly cannot write out all elements explicitly as in finite sets. When there is a clear pattern to the elements, we can use ellipses.

\[ \mathbb{N} = \{1, 2, 3, 4, \ldots\} \]
\[ \mathbb{W} = \{0, 1, 2, 3, 4, \ldots\} \]
\[ \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\} \]
\[ = \{0, -3, -2, -1, 0, 1, 2, 3, \ldots\} \]

With finite sets & even more so with infinite sets, this roster form quickly becomes cumbersome. For this reason we have set-builder notation:

**Ex.** Let \( A \) be the set of integers from 4 to 10.

So \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \).

In set builder notation, \( A = \{ x \in \mathbb{Z} \mid 1 \leq x \leq 10, x \neq 0\} \).
Set containment is sometimes drawn as Venn diagrams.

Ex. Suppose A \subset B and B \subset C. Then A \subset C.

The following are false:
- A \not\subset \emptyset
- \emptyset \subseteq \emptyset
- a \in \emptyset
- \emptyset \in \emptyset
- \emptyset \subseteq \emptyset
- \emptyset \in \emptyset

\emptyset, a, b, c, \emptyset a, b, c, \emptyset a, b, c , \emptyset a, b, c , \emptyset a, b, c.

Estimated here: 7/10, 25%
Further examples w/ subsets

Ex. How many subsets does \(\{1,2,3\}\) have?

Let's list them.
There is one w/ 0 elements: \(\emptyset\)
There are 3 w/ 1 element: \(\{1\}, \{2\}, \text{and} \{3\}\)
There are 3 w/ 2 elements: \(\{1,2\}, \{1,3\}, \text{and} \{2,3\}\)
There is 1 w/ 3 elements: \(\{1,2,3\}\).

In total we have 8 subsets of \(\{1,2,3\}\).

For a given set \(S\), the power set of \(S\),
denoted by \(\mathcal{P}(S)\), is the set of all subsets of \(S\).

Notice this means a set can contain sets!
This means the difference between \(\subseteq\) and \(\in\)
can be tricky.

Ex. From the previous example
\[
\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}
\]
and $|\mathcal{P}(\{1, 2, 3\})| = 8$.

Here $\{1\} \not\in \mathcal{P}(\{1, 2, 3\})$, $1 \not\in \mathcal{P}(\{1, 2, 3\})$

$\{2\} \not\in \mathcal{P}(\{1, 2, 3\})$, $2 \not\in \mathcal{P}(\{1, 2, 3\})$.

$\emptyset \in \mathcal{P}(\{1, 2, 3\})$, $\emptyset \subseteq \mathcal{P}(\{1, 2, 3\})$.

Sometimes when dealing with a large number of sets, we index them.

For example, this class meets 20 times. For $k$ from one to 20, let $S_k$ denote the set of students present on day $k$. So $S_1$ is the set of students present on the first day and so on.

Let $A = \{S_1, S_2, \ldots, S_{20}\}$. There are various ways to write $A$,

$\begin{align*}
A &= \{S_k \mid k=1, 2, 3, \ldots, 20\}, \\
A &= \{S_k \mid k \in \{1, 2, 3, \ldots, 20\}\}, \\
A &= \{S_k \mid k \in \mathbb{Z}, 1 \leq k \leq 20\}, \\
A &= \{S_k \mid k \in \mathbb{Z}^2 \text{ and } x = \{1, 2, 3, \ldots, 20\}\}, \\
A &= \{S_k \mid k=1, \ldots\}
\end{align*}$
\[ A = \sum_{k \in I} x_k \quad \text{where} \quad I = \{1, 2, 3, \ldots, n\}. \]

\( A \) is an indexed set, or indexed collection, and \( I \) is the index set.