Calculus texts typically define a partition of an interval \([a, b]\) to be a set \(\{x_0, x_1, \ldots, x_n\}\) of real numbers with \(a = x_0 < x_1 < \cdots < x_n = b\). Explain how such a set gives a partition of \([a, b]\) in the sense defined in this section.

### 1.4 Logic and Truth Tables

#### Logic

*Statements* are sentences that are either true or false. We may denote statements by capital letters. The statement

\[ P = \text{Snow is frozen water} \]

is true. The statement

\[ Q = 1.72 \text{ is an integer} \]

is false. The *negation* of a statement \(P\), denoted \(\sim P\) and read “not \(P\),” is the statement “\(P\) fails,” “\(P\) does not occur,” “\(P\) is false,” or “it is not the case that \(P\) happens.” With \(P\) as above, the negation of \(P\) is

\[ \sim P = \text{It is not the case that snow is frozen water} \]

or, in plain English,

\[ \sim P = \text{Snow is not frozen water}. \]

Because \(P\) was true, \(\sim P\) is false. The negation of the statement \(Q\) above is

\[ \sim Q = \text{It is not the case that 1.72 is an integer} \]

or

\[ \sim Q = 1.72 \text{ is not an integer}. \]

Because \(Q\) was false, \(\sim Q\) is true. The *truth value* of any statement \(P\) is either “true” or “false,” depending on whether the statement \(P\) is true or false. Two statements are *equivalent* if they always have the same truth values. (For example, “\(x + 5 = 8\)” is equivalent to “\(2x = 6\),” for both are true if \(x = 3\) and false if \(x \neq 3\).) A statement and its negation always have opposite truth values. That is, if \(P\) is true, then \(\sim P\) is false, and if \(P\) is false, then \(\sim P\) is true.

Just as a double negative is positive, the negation of the negation of a statement \(P\) is again \(P\). That is, for any statement \(P\), we have \(\sim(\sim P) = P\). For example, if \(P\) is the statement “The pool is dry,” then \(\sim P = \text{“The pool is not dry,” and } \sim(\sim P) = \text{“It is false that the pool is not dry” = “The pool is dry” = } P\).

We may form *compound statements* by using the words *and* or *or*. If \(P\) and \(Q\) are two input statements, the statement “\(P\) and \(Q\)” is denoted \(P \land Q\) and called the *conjunction* of statements \(P\) and \(Q\). The statement “\(P\) or \(Q\)” is denoted \(P \lor Q\) and is called the *disjunction* of statements \(P\) and \(Q\). Statements that do not involve \(\land, \lor, \text{ or } \sim\) are called *simple statements.*
The examples below will illustrate this. For the statements \( P, Q, \) and \( R \) below, we give several other statements built from these using negation, conjunction, and disjunction.

\[
P = \text{Milk contains fat.} \\
Q = \text{Milk contains calcium.} \\
R = \text{Milk is green.}
\]

\[
\lnot P = \text{Milk contains no fat.} \\
\lnot Q = \text{Milk contains no calcium.} \\
\lnot R = \text{Milk is not green.}
\]

\[
P \land Q = P \land Q = \text{Milk contains fat and calcium.} \\
P \land R = P \land R = \text{Milk contains fat and is green.} \\
P \land \lnot R = P \land \lnot R = \text{Milk contains fat and is not green.} \\
P \lor Q = P \lor Q = \text{Milk contains fat or calcium.} \\
P \lor R = P \lor R = \text{Milk contains fat or milk is green.} \\
\lnot Q \lor R = \lnot Q \lor R = \text{Milk contains no calcium or milk is green.} \\
P \lor \lnot P = P \lor \lnot P = \text{Milk contains fat or contains no fat.} \\
P \land \lnot P = P \land \lnot P = \text{Milk contains fat and contains no fat.}
\]

The last two statements above are of special interest. For any statement \( P \), either \( P \) or \( \lnot P \) must be true, so the statement “\( P \lor \lnot P \)” must always be true. A statement such as \( P \lor \lnot P \), which is always true, is called a tautology. Also, it is impossible for both \( P \) and \( \lnot P \) to be true, so \( P \land \lnot P \) will always be false for any statement \( P \). A statement such as \( P \land \lnot P \), which is always false, is called a contradiction.

Continuing our example, let us consider statements combining \( \lnot, \lor, \) and \( \land \). The negation of \( P \land R \) is

\[
\lnot(P \land R) = \text{It is not the case that (milk contains fat and is green).}
\]

For \( P \land R \) to be true, both \( P \) and \( R \) must be true. The negation \( \lnot(P \land R) \) says it is not the case that both \( P \) and \( R \) are true. Observe that “not both \( P \) and \( R \)” means one of \( P \) or \( R \) must fail. Thus, the statement for \( \lnot(P \land R) \) above may be rephrased as

Either milk contains no fat or milk is not green.

The words both and either are used above for emphasis. In mathematics, “Both \( P \) and \( R \)” is equivalent to “\( P \) and \( R \),” and “Either \( S \) or \( T \)” is equivalent to “\( S \) or \( T \).” Thus, the rephrased statement of \( \lnot(P \land R) \) above is

\[
\lnot(P \land R) = \text{Milk contains no fat or milk is not green.} \\
= (\lnot P) \lor (\lnot R) \\
= \lnot P \lor \lnot R
\]

The last equality above is based on precedence of operation conventions. Just as mul-
Multiplication precedes addition in calculating $3 \cdot 4 + 2 \cdot 3$, negations precede conjunctions and disjunctions in logical statements. After negations are performed, conjunctions and disjunctions are performed from left to right.

For the statements $P$ and $R$ as above, we have shown that $\sim(P \land R)$ is equivalent to the statement $\sim P \lor \sim R$. This holds for any statements $P$ and $R$. The formula $\sim(P \land R) = \sim P \lor \sim R$ simply says that “Not both $P$ and $R$” is equivalent to “Either $P$ fails or $Q$ fails.”

Now, let us consider the negation of an “or” statement, such as $\sim(P \lor Q)$. Since $P \lor Q$ means “Either $P$ or $Q$ occurs,” $\sim(P \lor Q)$ means “Not (either $P$ or $Q$)” or, in better English, “Neither $P$ nor $Q$ occurs.” But, “Neither $P$ nor $Q$” means “Not $P$ and not $Q$,” so $\sim(P \lor Q) = \sim P \land \sim Q$. This holds for any statements $P$ and $Q$. For the statements $P$ and $Q$ as given in the last example, we have

$$\sim(P \lor Q) = \text{It is not the case that (milk contains fat or calcium).}$$

For “$P$ or $Q$” to fail, both $P$ and $Q$ must fail:

$$\sim(P \lor Q) = \sim P \land \sim Q = \text{Milk contains no fat and no calcium.}$$

The previous two paragraphs tell us how to negate an “and” statement and how to negate an “or” statement. These results are called De Morgan’s Laws, named for the British mathematician Augustus De Morgan (1806–1871).

### 1.4.1 Theorem (De Morgan’s Laws)

For any statements $P$ and $Q$, we have

$$\sim(P \land Q) = \sim P \lor \sim Q$$

and

$$\sim(P \lor Q) = \sim P \land \sim Q.$$ 

Keep in mind what De Morgan’s Laws say in ordinary language: Not both $P$ and $Q$ means one or the other (or both) is lacking, so NOT ($P$ and $Q$) means NOT $P$ or NOT $Q$. Similarly, NOT ($P$ or $Q$) means neither $P$ nor $Q$, which is equivalent to (NOT $P$ and NOT $Q$).

De Morgan’s Laws are most frequently stated in set theoretic terms, as we will now illustrate. Consider the complement $(A \cap B)^c$ of the set $A \cap B$. The following statements are equivalent:

- $x \in (A \cap B)^c$
- $x \notin A \cap B$
- $\sim(x \in A \cap B)$
- $\sim(x \in A \land x \in B)$
- $\sim(x \in A \text{ or } x \in B)$
- $\sim(x \in A \lor x \in B)$
- $x \notin A \text{ or } x \notin B$
- $x \notin A^c \text{ or } x \notin B^c$
- $x \in A^c \lor B^c$
- $x \in A^c \cup B^c$
- Definition of complement
- Definition of $\notin$
- Definition of $\cap$
- Definition of $\lor$
- Definition of complement
- Definition of $\cup$
Thus, \( x \in (A \cap B)^c \) if and only if \( x \in A^c \cup B^c \), so the sets \((A \cap B)^c\) and \(A^c \cup B^c\) have exactly the same elements—that is, \((A \cap B)^c = A^c \cup B^c\). A similar argument shows that \((A \cup B)^c = A^c \cap B^c\).

We will formally state these results.

1.4.2 THEOREM (De Morgan’s Laws for Sets) For any sets \(A\) and \(B\), we have

\[
(A \cap B)^c = A^c \cup B^c
\]

and

\[
(A \cup B)^c = A^c \cap B^c.
\]

The second set theoretic result above is suggested by the Venn diagrams shown in Figure 1.13.

![Figure 1.13](image)

De Morgan’s Law: \(A^c \cap B^c = (A \cup B)^c\).

Suppose a friend tells you she is going to McGregor’s Restaurant to order either a hamburger and fries or a cheeseburger and fries. This could be restated by saying she will definitely order fries and will order either a hamburger or a cheeseburger. This bit of common sense may be stated formally as

\[(\text{Hamburger and Fries}) \text{ OR (Cheeseburger and Fries)}\]

\[= (\text{Hamburger or Cheeseburger}) \text{ and Fries,}\]

or

\[(H \land F) \lor (C \land F) = (H \lor C) \land F.\]

Observing that \(A \land B = B \land A\) and \(A \lor B = B \lor A\) for any statements \(A\) and \(B\), we may rearrange the equation above to get

\[F \land (H \lor C) = (F \land H) \lor (F \land C).\]

This result holds not only for these statements \(F, H,\) and \(C\) but for any statements \(A, B,\) and \(C\).

\[A \land (B \lor C) = (A \land B) \lor (A \land C).\]

Recall that the fact that \(a \cdot (b + c) = a \cdot b + a \cdot c\) for any real numbers \(a, b,\) and \(c\) is referred to by saying that multiplication distributes over addition. The last equation is stated by saying that the operation \(\land\) distributes over the operation \(\lor\). The paragraphs
that follow show that we will not need to remember which of the operations \( \land \) or \( \lor \) distributes over the other—each distributes over the other.

Let us consider another culinary example. Suppose a restaurant has issued two types of coupons. The first coupons are good for a free bowl of chili or soup. The second coupons are good for a free bowl of chili or a baked potato. John has one coupon of each type and wishes to use them both. Thus, John may select (chili or soup) and (chili or potato). His possible choices may be illustrated in a tree diagram, as shown in Figure 1.14.

Figure 1.14
Possible uses of two coupons.

Observe that three of the four outcomes include chili. Thus, John will have chili or the only remaining outcome, which is (soup and potato). Thus, the situation of the two coupons

(chili or soup) and (chili or potato)

implies

chili or (soup and potato).

It is easy to see that the latter statement implies the former as well. This suggests that for any statements \( C, S, \) and \( P, \)

\[
(C \lor S) \land (C \lor P) = C \lor (S \land P).
\]

We summarize the previous results. These distributive laws are most easily proved using truth tables, which is our next topic.

### 1.4.3 Theorem (Distributive Laws)

For statements \( A, B, \) and \( C, \)

\[
(A \lor B) \land C = (A \land C) \lor (B \land C)
\]

and

\[
(A \land B) \lor C = (A \lor C) \land (B \lor C).
\]

If \( A, B, \) and \( C \) are sets, let \( A_1, B_1, \) and \( C_1 \) be the statements \( x \in A, x \in B, \) and \( x \in C, \) respectively. Applying the distributive laws stated above to statements \( A_1, B_1, \) and \( C_1 \) gives the distributive laws in the set theoretic form below.
1.4.4 **THEOREM (Distributive Laws for Sets)** For sets $A$, $B$, and $C$,

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

and

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

**Truth Tables**

A convenient way to document common sense arguments as given above is to include the information in a *truth table*. Truth tables should not be used as a substitute for common sense, but they may clarify some complicated logical situations. Truth tables list the truth values of statements. For compound statements built from simple statements using $\sim$, $\land$, or $\lor$, we list one row for each possible configuration of truth values of all the simple statements. Let us first consider the truth table for $\sim P$. Given an arbitrary statement $P$, it may be either true or false. We will list these two possible outcomes in the left column. The negation $\sim P$ will have the opposite truth value from $P$. This is indicated in the truth table below.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\sim P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
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</tbody>
</table>

Truth table for $\sim P$

Given two statements $P$ and $Q$, there are four possibilities for the sequence of truth values for $P$ and $Q$. These four possibilities are listed in the left two columns of the truth tables below. Of the four possible truth value combinations, $P \land Q$ is true only when both $P$ and $Q$ are true. This information appears in the third column of the truth table for $P \land Q$. Similarly, $P \lor Q$ is true if either $P$ or $Q$ (or both) is true. This information appears in the third column of the truth table for $P \lor Q$ below.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \land Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
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</tbody>
</table>

Truth table for $P \land Q$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

Truth table for $P \lor Q$

To construct a truth table for $\sim P \lor \sim Q$, observe that $\sim P \lor \sim Q$ is derived from the simple statements $P$ and $Q$ by using $\sim$ and $\lor$. We must start the truth table with all possible truth value outcomes for the pair of simple statements $P$ and $Q$ in the left two columns. Before applying the disjunction $\lor$ to the statements $\sim P$ and $\sim Q$, we need columns for $\sim P$ and $\sim Q$. These are the third and fourth columns of the truth table below. The entries of the $\sim P$ column are obtained by toggling the truth values of $P$, ...
and the entries of the column for \( \sim Q \) are obtained by toggling the truth values of \( Q \). Finally, the \( \sim P \lor \sim Q \) column is obtained by applying \( \lor \) to the \( \sim P \) and \( \sim Q \) columns.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \sim P )</th>
<th>( \sim Q )</th>
<th>( \sim P \lor \sim Q )</th>
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<tbody>
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</table>

Truth table for \( \sim P \lor \sim Q \)

We may extend the truth table above to contain truth values for other statements. If we wish to include a column for \( \sim(P \land Q) \), we must first include a column for \( P \land Q \), then negate that column, as shown below.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \sim P )</th>
<th>( \sim Q )</th>
<th>( \sim P \lor \sim Q )</th>
<th>( P \land Q )</th>
<th>( \sim(P \land Q) )</th>
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<tbody>
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<td>T</td>
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</table>

Truth table for \( \sim P \lor \sim Q \) extended to include \( \sim(P \land Q) \)

Observe that the column for \( \sim P \lor \sim Q \) and that for \( \sim(P \land Q) \) are identical. Regardless of the truth values of the input propositions \( P \) and \( Q \), the truth values of \( \sim P \lor \sim Q \) and \( \sim(P \land Q) \) will always be the same. One may use this as a basis for the definition of logically equivalent statements. Two statements are logically equivalent if they produce identical truth values for all possible combinations of truth values for the input statements. Thus, the truth table above provides a proof of De Morgan’s Law \( \sim(P \land Q) = \sim P \lor \sim Q \).

1.4.5 **EXAMPLE** Construct the truth table for \( P \land \sim P \) and for \( P \lor \sim P \).

**Solution** This problem only involves one input statement \( P \), which may be either true or false. Thus, we will need only two rows to list all the possible truth value outcomes of the input statement. We toggle the \( P \) column to find the truth values of \( \sim P \) as recorded in the second column of the table below. The third and fourth columns of the table below are obtained by applying \( \land \) and \( \lor \) to the \( P \) and \( \sim P \) columns.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \sim P )</th>
<th>( P \land \sim P )</th>
<th>( P \lor \sim P )</th>
</tr>
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<tbody>
<tr>
<td>T</td>
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Truth table for \( P \land \sim P \) and \( P \lor \sim P \)
Recall that a statement that is false in any situation (i.e., whose truth table column contains only F's) is a contradiction, and a statement that is true in any situation (i.e., whose truth table column contains only T's) is a tautology. The truth table above illustrates the facts that \( P \land \sim P \) is a contradiction and that \( P \lor \sim P \) is a tautology.

### 1.4.6 **EXAMPLE**

Given three statements \( P, Q, \) and \( R, \) find the truth table for \((P \land \sim Q) \lor R\).

**Solution**
The set of ordered triples of truth values for the three simple statements \( P, Q, \) and \( R \) is the Cartesian product \( \{T_P, F_P\} \times \{T_Q, F_Q\} \times \{T_R, F_R\} \) and, thus, has \( 2 \times 2 \times 2 = 8 \) elements (see Theorem 1.2.9). This implies that we will need eight rows to include all the possible “input truth values.” The left three columns of the table below show all these possibilities. (These rows may be listed in any order, but if you wish to compare two truth tables, it is convenient to have the “input truth values” listed in the same order in both tables.) Before constructing the column for \((P \land \sim Q) \lor R\), we first need a column for \( P \land \sim Q \), and before this column can be constructed, we need a column for \( \sim Q \). The columns are shown below.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( R )</th>
<th>( \sim Q )</th>
<th>( P \land \sim Q )</th>
<th>( (P \land \sim Q) \lor R )</th>
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Truth table for \((P \land \sim Q) \lor R\)

### EXERCISES

1. Consider the statements \( S, G, \) and \( H \) below.

   \( S = \) Susan studies.

   \( G = \) Susan gets good grades.

   \( H = \) Susan gets help when needed.

Write each of the following sentences symbolically.

(a) Susan studies but does not get good grades.
(b) Susan gets help when needed or she does not study.
(c) It is not true that Susan studies and gets good grades.
(d) Either Susan studies and gets good grades or Susan does not get help when needed.
(e) Susan studies or does not study, and she gets good grades.
(f) Susan studies, gets help when needed, and gets good grades.
(g) Susan studies and gets help when needed or else she does not get good grades.
2. Consider the statements $P$, $Q$, and $R$ below.

$P =$ Presidential candidates must be 35 years of age or older.

$Q =$ Presidential candidates must be citizens of the United States.

$R =$ Presidential candidates must have $27$ million.

Write the statements below in words.

(a) $\sim P$

(b) $P \land Q$

(c) $Q \lor R$

(d) $\sim (Q \lor R)$

(e) $(P \land Q) \lor R$

(f) $\sim (P \land Q \land R)$

3. Consider the statements $R$, $W$, and $M$ below.

$R =$ It rains.

$W =$ Marco goes for a walk.

$M =$ Marco goes to a movie.

Write each of the following statements symbolically. Negate each statement symbolically using De Morgan’s Laws, and rewrite the negation in words.

(a) It rains or Marco goes for a walk.

(b) It does not rain and Marco goes for a walk.

(c) It does not rain or Marco goes to a movie.

(d) Marco does not go to a movie and it does not rain.

4. Use Theorem 1.4.1 to show that $(A \cup B)^c = A^c \cap B^c$ for any sets $A$ and $B$.

5. Illustrate De Morgan’s Law $(A \cap B)^c = A^c \cup B^c$ using Venn diagrams.

6. Use a truth table to verify the distributive property $A \land (B \lor C) = (A \land B) \lor (A \land C)$.

7. (a) Verify the equation $\sim (P \land Q) \land \sim Q = \sim Q$ using a truth table.

(b) Give a verbal justification for the equation in part (a).

8. (a) Verify the equation $(P \land Q) \lor \sim P = \sim P \lor Q$ using a truth table.

(b) Give a verbal justification for the equation in part (a).

9. Use a truth table to verify that $(P \land \sim Q) \lor (\sim P \land \sim Q) \lor (\sim P \lor Q)$ is a tautology.

10. Shown below is a truth table with the column headings missing. Fill in column headings having the indicated truth values.

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<thead>
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11. Verify that $(P \land Q) \lor (\sim P \land \sim Q) = (P \lor \sim Q) \land (Q \lor \sim P)$. 
12. Construct a truth table with columns for $(P \lor Q) \land R$ and $P \lor (Q \land R)$. Is the placement of the parentheses in $P \lor Q \land R$ important?

13. Verify that $\neg[(P \lor \neg Q) \land R] = (\neg P \land Q) \lor \neg R$.

14. Shown below is a truth table with the column headings missing. Fill in column headings having the indicated truth values, using only $P$, $Q$, $R$, $\land$, and $\neg$.

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1.5 **Quantifiers**

If a statement involves a variable, the truth value of the statement may depend on the value of the variable. For example, if $x$ is a real number, the statement “$x^2 = 4$” is true for the values $x = \pm 2$ and false for all other values of $x$. Any such statement whose truth value depends on the value of the variable in question is called a **conditional statement**. The statement “$(a + b)^2 = a^2 + 2ab + b^2$” is not a conditional statement—it is always true for any values of $a$ and $b$. Such a statement involving variables which is always true regardless of the choice of values for the variables is called an **identity**.

With statements involving variables, we may want to quantify when the statement is valid. For the conditional statement “$x^2 = 4$,” we might state “There exist values of $x$ for which $x^2 = 4$.” The fact that the statement “$(a + b)^2 = a^2 + 2ab + b^2$” is an identity might be stated as “For all values of $a$ and $b$, we have $(a + b)^2 = a^2 + 2ab + b^2$.” The phrases **there exist** and **for all** appear so frequently that we have special symbols for them:

$\exists$ means **there exists**

$\forall$ means **for any, for all, for every, or for each**

The symbol $\exists$ is called the **existential quantifier**, and the symbol $\forall$ is called the **universal quantifier**. Let us see how they are used.

Suppose $\{S_i\}_{i \in I}$ is a collection of sets. Recall that

$$\bigcap_{i \in I} S_i = \{x \mid x \text{ is an element of each of the sets } S_i\}$$

and

$$\bigcup_{i \in I} S_i = \{x \mid x \text{ is an element of at least one of the sets } S_i\}.$$
We may write these expressions using quantifiers. Observe that
\[ \bigcap_{i \in I} S_i = \{ x \mid x \in S_i \text{ for every } i \in I \} = \{ x \mid \forall i \in I, x \in S_i \} \]
and
\[ \bigcup_{i \in I} S_i = \{ x \mid \text{there exists an index } i \in I \text{ such that } x \in S_i \} = \{ x \mid \exists i \in I \text{ such that } x \in S_i \}. \]

We may restate the fact that the equation \( x^2 = -12 \) has no real solutions as “for any real number \( x \), \( x^2 \neq -12 \)” or “\( \forall x \in \mathbb{R}, x^2 \neq -12 \).”

The statement
\[ \forall n \in \mathbb{N} \exists m \in \mathbb{N} \text{ such that } m > n \]
says that for every natural number \( n \) there exists a natural number \( m \) that is greater than \( n \). This statement is true and is called the **Archimedean property**. Observe that a different choice of \( n \) might require a different choice of \( m \), but no matter what natural number \( n \) we chose, there does exist a larger natural number \( m \) (take \( m = n + 1 \), for example).

For expressions involving more than one quantifier, we must be very careful to place the quantifiers in the correct order. The statement above is not the same as the statement
\[ \exists m \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, m > n. \]

This latter statement says there exists a natural number \( m \) that is greater than every natural number. This is clearly false. Any candidate \( m \) for the largest integer fails to meet the criterion of the statement above, for \( m \neq m \). Thus, the order in which the quantifiers are placed is critical. If the first quantifier in a statement is \( \exists \), we will call the statement a “\( \exists \)-statement” (read “‘there exists’ statement”). Analogously, any statement whose first quantifier is \( \forall \) will be called a “\( \forall \)-statement” (read “‘for all’ statement”).

To prove a \( \exists \)-statement, we need only exhibit one element that was claimed to exist. To prove the statement
\[ \exists z \in \mathbb{R} \text{ such that } \forall a \in \mathbb{R}, z + a = a, \]
we must only find a real number \( z \) with the property that adding it to any other number \( a \) gives a sum identically equal to \( a \). Clearly \( z = 0 \) is the number we need, for \( 0 + a = a \) for every real number \( a \). Having found one element meeting the requirements called for, we have proved the statement above. (Because zero satisfies this property, zero is called the **additive identity element**.)

Proving a \( \forall \)-statement requires verification for every possibility, and this is usually more difficult than proving a \( \exists \)-statement. For example, consider the statements

1. Every passenger on the plane has a $10 bill.
2. There is a passenger on the plane who has a $10 bill.

If we let \( P \) be the set of passengers on the plane, we may rephrase these statements as
\[ \forall p \in P, \text{ } p \text{ has a } $10 \text{ bill.} \quad (1.1) \]
\[ \exists p \in P \text{ such that } p \text{ has a } $10 \text{ bill.} \quad (1.2) \]
Verifying the $\exists$-statement above can be accomplished by finding a single passenger with a $10$ bill. You could approach the proof of the $\exists$-statement (1.2) by asking loudly on the plane “Does anyone have a $10$ bill?”

The verification of the $\forall$-statement (1.1), however, would require something much more difficult and intrusive than merely having one passenger exhibit a $10$ bill. The verification of statement (1.1) would require having every passenger exhibit a $10$ bill.

Disproving a $\forall$-statement is much easier than proving a $\forall$-statement. All that it would take to show that statement (1.1) is false would be a single passenger who did not have a $10$ bill. That is, to disprove a $\forall$-statement, we only need to find a single counterexample—a single case where the claimed property fails.

Disproving a statement means proving that the statement is false, or equivalently, proving the negation of the statement. To disprove statement (1.1) above, we wish to prove

It is not the case that $(\forall p \in P, \ p \text{ has a }$ $10 \text{ bill})$, or

$$\sim(\forall p \in P, \ p \text{ has a } 10 \text{ bill}).$$

By our discussion above, these statements are equivalent to

There exists a passenger who does NOT have a $10$ bill, or

$$\exists p \in P \text{ such that } \sim(p \text{ has a } 10 \text{ bill}).$$

Summarizing, we have shown that

$$\sim(\forall p \in P, \ p \text{ has a } 10 \text{ bill}) = \exists p \in P \text{ such that } \sim(p \text{ has a } 10 \text{ bill}).$$

Observe that the negation of this $\forall$-statement was a $\exists$-statement involving negation.

Now let us consider what is required to disprove a $\exists$-statement, using the statement (1.2) as our example. To disprove statement (1.2) we must show that there does not exist a passenger on the plane with a $10$ bill. In plain English, we would say we must show that NO passenger has a $10$ bill. Equivalently, for every passenger $p$ on the plane, we must verify that passenger $p$ does not have a $10$ bill. This may be written as $\forall p \in P, \sim(p \text{ has a } 10 \text{ bill}).$ This statement is equivalent to disproving statement (1.2), that is, equivalent to the negation of statement (1.2). Thus we have

$$\sim(\exists p \in P \text{ such that } p \text{ has a } 10 \text{ bill}) = \forall p \in P, \sim(p \text{ has a } 10 \text{ bill}).$$

Observe that the negation of this $\exists$-statement was a $\forall$-statement involving negation.

In general, suppose $Q$ is any property (such as “has a $10$ bill”). The statements “There exists an $x$ such that $Q$ holds” and “For any $x$, $Q$ holds” may be written as “$\exists x$ such that $Q$” and “$\forall x, Q$,” respectively. The negations of these statements are given below.

**Negation of Quantified Statements:**

$$\sim(\exists x \text{ such that } Q) = \forall x, \sim Q$$

$$\sim(\forall x \ , \ Q ) = \exists x \text{ such that } \sim Q$$

Observe that the negation of any $\exists$-statement is a $\forall$-statement involving negation and that the negation of any $\forall$-statement is a $\exists$-statement involving negation. We have noted this for the particular statements given above. In words, the equations above simply say: 

*To disprove that there exists $x$ such that $Q$ holds, we must show that $Q$ fails for all $x$.*
To disprove that \( Q \) holds for all \( x \), we must only exhibit a single counterexample; that is, we need only find one \( x \) for which \( Q \) fails.

1.5.1 **EXAMPLE**  Determine whether each statement below is true or false. Give the negation of each statement.

(a) \( \forall x \in \mathbb{R}, x^2 > 0 \)
(b) \( \forall A \in \mathcal{P}(Q), \emptyset \subseteq A \)
(c) \( \exists m \in \mathbb{N} \) such that \( \forall n \in \mathbb{N}, m \cdot n = n \)

**Solution** (a) To disprove the statement that for any real number \( x \), its square is positive, we would need to exhibit a real number \( x \) whose square is not positive. There does exist a real number \( x \) whose square is \( \leq 0 \), namely \( x = 0 \). Thus, the statement is false. The negation of statement (a) is

\[
\neg (\forall x \in \mathbb{R}, x^2 > 0) = \exists x \in \mathbb{R} \text{ such that } \neg (x^2 > 0) \]

\[
= \exists x \in \mathbb{R} \text{ such that } x^2 \leq 0.
\]

(b) Recall that \( \mathcal{P}(Q) \) is the collection of all subsets of the set \( Q \), so saying \( A \in \mathcal{P}(Q) \) is equivalent to saying \( A \subseteq Q \). Now statement (b) simply says \( \forall A \subseteq Q, \emptyset \subseteq A \), which is clearly true. The negation of statement (b) is

\[
\neg (\forall A \in \mathcal{P}(Q), \emptyset \subseteq A) = \exists A \in \mathcal{P}(Q) \text{ such that } \emptyset \not\subseteq A.
\]

(c) Statement (c) says that there is some natural number \( m \) with the property that its product with any natural number \( n \) is identically \( n \). To prove this \( \exists \)-statement, we need only exhibit such an element \( m \). There does exist such an element \( m \), called the **multiplicative identity**, and \( m = 1 \). Thus, statement (c) is true. The negation of the \( \exists \)-statement (c) is the \( \forall \)-statement

\[
\forall m \in \mathbb{N}, \neg (\forall n \in \mathbb{N}, m \cdot n = n)
\]

This last statement, however, may be taken further, for it contains (as a subproblem) the negation of a \( \forall \)-statement. Performing this second negation, we have

\[
\neg (\exists m \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, m \cdot n = n) = \forall m \in \mathbb{N}, \neg (\forall n \in \mathbb{N}, m \cdot n = n)
\]

\[
= \forall m \in \mathbb{N}, \exists n \in \mathbb{N} \text{ such that } \neg (m \cdot n = n)
\]

\[
= \forall m \in \mathbb{N}, \exists n \in \mathbb{N} \text{ such that } m \cdot n \neq n.
\]

We have stated De Morgan’s Laws for two sets (Theorem 1.4.2). We are now ready to prove the results for arbitrary collections of sets.

1.5.2 **THEOREM (De Morgan’s Laws)**  If \( \{S_i\}_{i \in I} \) is a collection of sets, then

\[
\left( \bigcup_{i \in I} S_i \right)^c = \bigcap_{i \in I} (S_i^c)
\]

and

\[
\left( \bigcap_{i \in I} S_i \right)^c = \bigcup_{i \in I} (S_i^c).
\]
That is, the complement of a union of sets is the intersection of the complements, and the complement of an intersection of sets is the union of the complements.

**Proof** We will only prove the first statement. The following statements are equivalent.

\[ x \in (\bigcup_{i \in I} S_i)^c \]
\[ x \not\in \bigcup_{i \in I} S_i \quad \text{Definition of complement} \]
\[ \sim(x \in \bigcup_{i \in I} S_i) \quad \text{Definition of } \sim \]
\[ \sim(\exists i \in I \text{ such that } x \in S_i) \quad \text{Definition of } \exists \]
\[ \forall i \in I, \sim(x \in S_i) \quad \text{Negation} \]
\[ \forall i \in I, x \not\in S_i \quad \text{Definition of } \not\in \]
\[ \forall i \in I, x \in S_i^c \quad \text{Definition of complement} \]
\[ x \in \bigcap_{i \in I} (S_i^c) \quad \text{Definition of } \bigcap \]

Thus, the elements of \((\bigcup_{i \in I} S_i)^c\) are exactly the elements of \(\bigcap_{i \in I}(S_i^c)\), so

\[ (\bigcup_{i \in I} S_i)^c = \bigcap_{i \in I}(S_i^c). \]

**EXERCISES**

1. Write each sentence below as a statement using \(\forall\) or \(\exists\).
   
   (a) For every positive real number \(\epsilon\), there is a natural number \(n\) with \(\frac{1}{n} < \epsilon\).
   
   (b) Every even number greater than 2 is the product of an even number and a prime number.
   
   (c) For every positive real number \(\epsilon\), there is a positive real number \(\delta\) such that \(x^2 < \epsilon\) whenever \(|x| < \delta\).
   
   (d) There exists an integer \(m\) with the property that for every integer \(x\), there exists an integer \(y\) with \(xy = m\).
   
   (e) There is always some prime number strictly between any given natural number \(n > 1\) and its square.

2. Determine whether each statement below is true or false. Give the negation of each statement.
   
   (a) \(\forall x \in \mathbb{R} \exists a \in \mathbb{R} \text{ with } |x| < a\)
   
   (b) \(\exists a \in \mathbb{R} \text{ such that } \forall x \in \mathbb{N}, a < x\)
   
   (c) \(\forall x \in \mathbb{R} \exists y \in \mathbb{R} \text{ such that } xy = 1\)
   
   (d) \(\exists b \in \mathbb{R} \text{ such that } \forall a \in \mathbb{N}, |a - b| \leq 100\)
   
   (e) \(\forall a \in \mathbb{R}, \sqrt{a^2} = a\)
   
   (f) \(\forall a \in [0, \infty), \exists x \in \mathbb{R} \text{ such that } x^2 = a \text{ and } -x^2 = a\)

3. Determine whether each statement below is true or false. Give the negation of each statement.
   
   (a) \(\exists x \in \mathbb{Z} \text{ such that } \forall y \in \mathbb{Z}, \frac{y}{x} \in \mathbb{Z}\)
   
   (b) \(\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z} \text{ such that } \frac{b}{a} \in \mathbb{Z}\)
(c) $\forall u \in \mathbb{N}, \exists v \in \mathbb{N} \setminus \{u\}$ such that $\frac{u}{v} \in \mathbb{N}$

(d) $\forall u \in \mathbb{N}, \exists v \in \mathbb{N} \setminus \{u\}$ such that $\frac{u}{v} \in \mathbb{N}$

(e) $\forall a \in \mathbb{N}, \exists b, c \in \mathbb{N}$ such that $ab = c^3$

4. Determine whether each statement below is true or false. Give the negation of each statement.

(a) $\mathcal{P}([1, 2, 3, \ldots, n]) \subset \mathcal{P}(\mathbb{N}) \forall n \in \mathbb{N}$

(b) $\exists S \subseteq \mathbb{R}$ such that $\forall k \in \mathbb{N}, |(k-1, k) \cap S| = 1$

(c) $\forall k \in \mathbb{N}, \exists S \in \mathcal{P}([1, 2, \ldots, k])$ such that $S \neq \emptyset$ and $\forall x, y \in S, x - y$ is even

(d) $\forall k \in \mathbb{N}, \exists S \in \mathcal{P}([1, 2, \ldots, k])$ such that $S \neq \emptyset$ and $\forall x, y \in S, x - y$ is odd

(e) $\forall$ set $S, \emptyset \subset \mathcal{P}(S)$

5. Show that $(\bigcap_{i \in I} S_i)^c = \bigcup_{i \in I} (S_i^c)$, and justify each step.

6. A set $S$ of real numbers has the Archimedean property if and only if

$\forall a, b \in S, \exists n \in \mathbb{N}$ such that $na > b$.

(a) Write the negation of the defining statement for the Archimedean property of a set $S$.

(b) Which of the following sets of numbers have the Archimedean property?

i. $\mathbb{R}$  ii. $\mathbb{N}$  iii. $\mathbb{Z}$  iv. $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$

1.6 Implications

"Then you should say what you mean," the March Hare went on.

"I do," Alice hastily replied. "At least—at least I mean what I say—that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "Why you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"

—Lewis Carroll (1832–1898)

A statement of the form "If $P$ then $Q$" is called an implication. The implication "If $P$ then $Q$" may be rephrased as "$P$ implies $Q$," denoted $P \Rightarrow Q$, or as "$Q$ is implied by $P$," denoted $Q \Leftarrow P$. In an implication $P \Rightarrow Q$, the statement $P$ is called the antecedent and the statement $Q$ is called the consequence. Below are some implications.

If $x^2 = 9$, then $x = -3$.

antecedent  consequence

If $x = 5$, then $x^2 = 78.6$.

antecedent  consequence

If pigs can fly, then $4 + 2 = 42$.

antecedent  consequence

If $4 + 2 = 42$, then pigs can fly.

antecedent  consequence
Some of these implications are true and some of them are not. Sometimes statements that do not look like implications can be rephrased as implications:

My plants will die if they get no water.
If my plants get no water, then they will die.

All crows are black.
If it is a crow, then it is black.

You cannot go to the movie unless you clean your room.
If you do not clean your room, then you cannot go to the movie.

The path is muddy whenever it rains.
If it rains, then the path is muddy.

The implication $P \Rightarrow Q$ is not the same as the implication $Q \Rightarrow P$:

If you are a lawyer, then you can read. \[ P \Rightarrow Q \]

If you can read, then you are a lawyer. \[ Q \Rightarrow P \]

The converse of an implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$.
If the antecedent and consequence are conditional statements, the truth value of an implication is independent of the truth value of its converse: Such an implication and its converse may both be true, both false, or one true and one false, depending on the statements $P$ and $Q$ involved. The examples below will illustrate this.

Implication: If $x + 2 = 5$, then $x = 3$. True
Converse: If $x = 3$, then $x + 2 = 5$. True

Implication: If $2x = 6$, then $x = 5$. False
Converse: If $x = 5$, then $2x = 6$. False

Implication: If $x = -3$, then $x^2 = 9$. True
Converse: If $x^2 = 9$, then $x = -3$. False (x might equal 3 or -3)

To provide examples of all possible outcomes of the truth values of an implication and its converse, we still need an implication that is false but whose converse is true. The implication below provides such an example. Compare it with the last implication above.

Implication: If $x^2 = 9$, then $x = -3$. False
Converse: If $x = -3$, then $x^2 = 9$. True
This example illustrates that the converse of the converse of $P \Rightarrow Q$ is again $P \Rightarrow Q$. Furthermore, when using the word *converse*, we must be sure it is understood what implication we are taking the converse of. Because each of the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ is the converse of the other, we may simply say that these statements are converses without going into the detail of which is the converse of which.

It is important to recognize what an implication implies and what it does not. Let us return to the implication

$$
\text{If you are a lawyer, then you can read.}
$$

Let us assume for now that this is a true implication. The implication says nothing about people who are not lawyers. Maybe they can read, or maybe not. If you are *not* a lawyer—that is, if $\sim P$—then the implication $P \Rightarrow Q$ tells us nothing. Similarly, if you can read, the implication above tells us nothing about your profession: Maybe you are a lawyer, or maybe not. The implication “If $P$ then $Q$” should not be expected to provide any consequence if $Q$ occurs. However, given the implication above, we can conclude something if you cannot read. Given a true implication $P \Rightarrow Q$, if $Q$ does not occur, then $P$ could not have occurred (for had $P$ occurred, $Q$ would have). In the example at hand, it follows that if you cannot read, then you are not a lawyer.

Thus, if the implication $P \Rightarrow Q$ is true, then the implication $(\sim Q) \Rightarrow (\sim P)$ is also true. Our arguments above hold not only for the particular statements $P$ and $Q$ given there but for any statements $P$ and $Q$.

In the standard rules of precedence, negations ($\sim$) are performed first, followed by conjunctions ($\land$) and disjunctions ($\lor$) taken in order from left to right, and implications ($\Rightarrow$ or $\Leftarrow$) are taken last. Thus, $\sim P \lor Q \Rightarrow R \land \sim S$ is understood to mean $((\sim P) \lor Q) \Rightarrow (R \land (\sim S))$. In the case at hand, $(\sim Q) \Rightarrow (\sim P)$ will be written $\sim Q \Rightarrow \sim P$.

### 1.6.1 Definition
Given an implication $P \Rightarrow Q$, the implication $\sim Q \Rightarrow \sim P$ is called the *contrapositive* of $P \Rightarrow Q$.

We have shown that if the implication $P \Rightarrow Q$ is true, then its contrapositive $\sim Q \Rightarrow \sim P$ is true. We will now show that if the contrapositive $\sim Q \Rightarrow \sim P$ is true, then the implication $P \Rightarrow Q$ is true. We will use a sly trick. Let us consider the contrapositive of the contrapositive of $P \Rightarrow Q$. That is, let us consider the contrapositive of $\sim Q \Rightarrow \sim P$, which is $\sim(\sim P) \Rightarrow \sim(\sim Q)$. Recalling that $\sim(\sim P) = P$ and $\sim(\sim Q) = Q$, we see that the contrapositive of the contrapositive of $P \Rightarrow Q$ is $P \Rightarrow Q$. Now we are ready to apply this observation to the problem at hand. We have shown above that the contrapositive of any true implication is true. Thus, if $\sim Q \Rightarrow \sim P$ is true, its contrapositive is true, and its contrapositive is $P \Rightarrow Q$.

Summarizing, we have shown that if an implication is true, then its contrapositive is true, and if the contrapositive is true, then the implication is true. Thus, any implication and its contrapositive will always have the same truth value. We have proved the following result.

### 1.6.2 Theorem
An implication and its contrapositive are logically equivalent.
Let us review our definitions.

Implication: \( P \Rightarrow Q \)

Converse: \( Q \Rightarrow P \)

Contrapositive: \( \sim Q \Rightarrow \sim P \)

For those who love definitions, there is another related implication. Given an implication \( P \Rightarrow Q \), the converse of its contrapositive is \( \sim P \Rightarrow \sim Q \) and is called the inverse of \( P \Rightarrow Q \). In applications, the converse and contrapositive are far more important than the inverse.

Let us also note the following:

The negation of the negation of \( P \) is \( P \); that is, \( \sim (\sim P) = P \).

The converse of the converse of \( (P \Rightarrow Q) \) is again \( (P \Rightarrow Q) \).

The contrapositive of the contrapositive of \( (P \Rightarrow Q) \) is again \( (P \Rightarrow Q) \).

The inverse of the inverse of \( (P \Rightarrow Q) \) is again \( (P \Rightarrow Q) \).

### 1.6.3 EXAMPLE
Give the converse and contrapositive of the implications below.

(a) If \( 1 + 1 = 3 \), then horses have feathers.
(b) If it will snow today, then I will not swim today.
(c) If you will visit Austria, then you will speak German.

#### Solution

(a) Implication: If \( 1 + 1 = 3 \), then horses have feathers.
   
   Converse: If horses have feathers, then \( 1 + 1 = 3 \).

   Contrapositive: If horses do not have feathers, then \( 1 + 1 \neq 3 \).

(b) Implication: If it will snow today, then I will not swim today.
   
   Converse: If I will not swim today, then it will snow today.

   Contrapositive: If I will swim today, then it will not snow today.

(c) Implication: If you will visit Austria, then you will speak German.
   
   Converse: If you will speak German, then you will visit Austria.

   Contrapositive: If you will not speak German, then you will not visit Austria.

We know that each of these implications has the same truth value as its contrapositive, but the truth value of the converse is linked to the particular statements \( P \) and \( Q \) and may not be determined from the truth value of the implication \( P \Rightarrow Q \).

We have seen all this without discussing how to determine whether an implication \( P \Rightarrow Q \) is true or false. In our discussion above, we assumed that the implication "If you are a lawyer, then you can read" was true. Is this implication really true? What would it take to show that this implication is false? The only way this particular implication could fail would be if there were a lawyer somewhere who cannot read. In general terms,
an implication \( P \Rightarrow Q \) will be false if and only if \( P \) occurs yet \( Q \) fails—that is, if and only if \( P \) is true yet \( Q \) is false. In determining the truth of an implication, remember that there is only one way it could be false: The antecedent (the “if part”) is true, yet the consequence (the “then part”) fails.

Consider the following example:

\[
\begin{align*}
\text{If} & \quad \text{pigs can fly}, \quad \text{then} \quad 7 + 2 = 9. \\
\phantom{\text{If}} & \quad P \quad Q
\end{align*}
\]

There are three statements included above: the antecedent \( P \), the consequence \( Q \), and the implication \( P \Rightarrow Q \). In this implication, the antecedent \( P = \text{“pigs can fly”} \) is false, and the consequence \( Q = \text{“}7 + 2 = 9\text{”} \) is true. To determine whether the implication \( P \Rightarrow Q \) is true, remember that the only way the implication can be false is if \( P \) is true yet \( Q \) is false. In the example at hand, \( P \) is never true, so the implication \( P \Rightarrow Q \) never fails. This is a true implication.

Thus, an implication \( P \Rightarrow Q \) is automatically true if \( P \) is always false. If \( P \) is guaranteed to never happen, we can truthfully state that any consequence would follow if \( P \) did happen. Thus, the implication

\[
\begin{align*}
\text{If} & \quad \text{pigs can fly}, \quad \text{then} \quad 7 + 2 = 483.9 \\
\phantom{\text{If}} & \quad P \quad Q
\end{align*}
\]

is a true implication. The consequence \( 7 + 2 = 483.9 \) is clearly false, but the implication \( P \Rightarrow R \) does not say \( 7 + 2 = 483.9 \) is true. The implication only says \( 7 + 2 = 483.9 \) would be true if pigs could fly. Since pigs cannot fly, the implication is not very applicable but is true. The implication \( P \Rightarrow R \) would be false only if pigs could fly and \( 7 + 2 \neq 483.9 \).

Let us address this issue of the truth of an implication using logical notation. To construct the truth table for the implication \( P \Rightarrow Q \), we list all possible outcomes for the truth values of the input statements \( P \) and \( Q \) in the left two columns. The implication \( P \Rightarrow Q \) is false if and only if \( P \) is true yet \( Q \) is false. This allows us to complete the third column in the truth table below.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \Rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

\[ \{ \text{If the antecedent} \ P \text{ fails, the implication} \ P \Rightarrow Q \text{ is automatically true.} \]
that \( \sim(P \Rightarrow Q) = P \land \sim Q \) by constructing a truth table for these statements, as given below.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \Rightarrow Q )</th>
<th>( \sim(P \Rightarrow Q) )</th>
<th>( \sim Q )</th>
<th>( P \land \sim Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
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<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Because the columns for \( \sim(P \Rightarrow Q) \) and \( P \land \sim Q \) are identical, this proves that these statements are logically equivalent.

The most logically complex implications \( P \Rightarrow Q \) are those in which the antecedent \( P \) and consequence \( Q \) are conditional statements, and hence may take on different truth values. If \( P \) and \( Q \) are conditional statements, the implication \( P \Rightarrow Q \) is considered true only if the implication holds in all possible instances, and it is considered false if there is a single instance in which the implication fails. For example, the implication

\[
x = 5 \Rightarrow x^2 = 0
\]

does happen to be true in the instance that \( x = 2 \), in which case the implication becomes \( 2 = 5 \Rightarrow 2^2 = 0 \). However, the implication is clearly false in the one suggested instance \( x = 5 \); hence, the implication is a false implication. Indeed, as an implication \( P \Rightarrow Q \) will be true in all instances in which \( P \) is false, we need only consider the (naturally suggested) cases in which \( P \) is true.

Now let us compare an “if” statement with an “only if” statement. With

\[ P = \text{I will try the escargot} \]

and

\[ Q = \text{You will try the escargot}, \]

the statement “\( P \) if \( Q \)” is

I will try the escargot if you will, \hspace{1cm} (1.3)

and the statement “\( P \) only if \( Q \)” is

I will try the escargot only if you will. \hspace{1cm} (1.4)

Writing it in the standard if–then order, statement (1.3) becomes

If you will try the escargot, then I will try the escargot. \hspace{1cm} (1.3')

If \( Q \) then \( P \).

\[ Q \Rightarrow P. \]

Under implication (1.3'), I still have the option to try the escargot myself even if you do not. Under implication (1.4), however, if you do not try the escargot, I have no option—
am obligated not to try the escargot. Thus, statement (1.4) is equivalent to

If you will not try the escargot, then I will not try the escargot.

(1.4')

If \( \sim Q \) then \( \sim P \).

\[ \sim Q \Rightarrow \sim P. \]

Replacing this last implication by its contrapositive, which is logically equivalent, gives

\[ P \Rightarrow Q. \]

(1.4'')

Comparing statements (1.3) and (1.3'), we see that "\( P \) if \( Q \)" is equivalent to \( (Q \Rightarrow P) \), and comparing statements (1.4) and (1.4''), we see that "\( P \) only if \( Q \)" is equivalent to \( (P \Rightarrow Q) \). Similar reasoning shows that this holds for any statements \( P \) and \( Q \):

\[
(P \text{ if } Q) = (Q \Rightarrow P)
\]

\[
(P \text{ only if } Q) = (P \Rightarrow Q).
\]

In particular, note that "\( P \) if \( Q \)" and "\( P \) only if \( Q \)" are converses.

We may argue that "\( P \) only if \( Q \)" is \( (P \Rightarrow Q) \) directly without phrasing the implication as its contrapositive. Suppose \( P \) occurs only if \( Q \) does, and suppose we know that \( P \) occurs. Then \( Q \) must have occurred. Thus, if "\( P \) only if \( Q \)" is true, then "If \( P \) then \( Q \)" is true. Arguing the other direction, if the statement "If \( P \) then \( Q \)" is true, then clearly \( P \) occurs only if \( Q \) does—that is, clearly "\( P \) only if \( Q \)" is true.

If \( P \) implies \( Q \) and \( Q \) implies \( P \), then \( P \) and \( Q \) will have precisely the same truth values—either both true or both false. Thus, if \( P \Rightarrow Q \) and \( Q \Rightarrow P \), then \( P \) and \( Q \) are logically equivalent. This situation may be described as \( (P \Rightarrow Q) \land (P \Leftarrow Q) \), or simply \( P \iff Q \). Since "\( P \Rightarrow Q \)" may be stated as "\( P \) only if \( Q \)" and "\( Q \Rightarrow P \)" may be stated as "\( P \) if \( Q \)" we may read \( P \iff Q \) as "\( P \) if and only if \( Q \)". A common abbreviation for "if and only if" is iff. It will be important to remember that \( P \iff Q \) is a double implication: \( P \) implies \( Q \), and \( P \) is implied by \( Q \).

If \( P \Rightarrow Q \), then we say \( P \) is a sufficient condition for \( Q \), and \( Q \) is a necessary condition for \( P \). If \( R \) is a necessary and sufficient condition for \( Q \), then \( R \iff Q \).

1.6.4 **EXAMPLE** A set \( S \) of real numbers satisfies the Archimedean property if and only if

\[ \forall a, b \in S, \exists n \in \mathbb{N} \text{ such that } na > b. \]

Find necessary and sufficient conditions for a set \( S \subseteq \mathbb{R} \) to satisfy the Archimedean property.

**Solution** If \( S \) contains any number \( a \leq 0 \), then \( na \leq a \ \forall n \in \mathbb{N} \), so \( S \) cannot have the Archimedean property. Thus, a necessary condition for \( S \) to have the Archimedean property is that \( S \) contains only positive elements. Furthermore, the condition that \( S \) contains only positive elements is sufficient to insure that \( S \) has the Archimedean property, for then if \( a, b \in S \subseteq (0, \infty) \), then there exists \( m \in \mathbb{N} \) with \( a > \frac{1}{m} > b \), and multiplying by any natural number \( n > mb > 0 \) gives \( na > (mb)(\frac{1}{m}) = b \).

1.6.5 **EXAMPLE** Verify that \( (x, y, z) = (a, b, c) \) in \( \mathbb{R}^3 \) if and only if

\[ \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = 0. \]
Solution To verify a double implication, we must verify each implication separately.

(⇒): If \((x, y, z) = (a, b, c)\), then
\[
\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = \sqrt{0+0+0} = 0.
\]

(⇐): If \(\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = 0\), then \((x-a)^2 + (y-b)^2 + (z-c)^2 = 0\). A sum of three nonnegative numbers can be zero only if each of the three numbers is zero; thus, \((x-a)^2 = (y-b)^2 = (z-c)^2 = 0\). It follows that \((x-a) = (y-b) = (z-c) = 0\); that is, \(x = a, y = b, \) and \(z = c\).

EXERCISES

Exercises 1–4 refer to the statements \(I, H, S, \) and \(U\) below.

- \(I\) = Interest rates go down.
- \(H\) = More people buy houses.
- \(S\) = The stock market goes up.
- \(U\) = Unemployment goes up.

1. With \(I, H, S, \) and \(U\) as above, write each implication, its converse, and its contrapositive in words.

   (a) \(I \Rightarrow S\)  
   (b) \(\sim U \Rightarrow H\)  
   (c) \(S \Rightarrow (I \land H)\)

2. With \(I, H, S, \) and \(U\) as above, write each statement as a sentence.

   (a) \(H \Rightarrow \sim U\)  
   (b) \(I \iff H\)  
   (c) \(I \Rightarrow (H \land \sim U)\)
   (d) \((\sim H \land \sim S) \Rightarrow U\)  
   (e) \(\sim (S \Rightarrow U)\)

3. Write each sentence in symbolic form, using \(I, H, S, \) and \(U\) as above.

   (a) Interest rates go down only if unemployment goes up.
   (b) The stock market goes up if more people buy houses.
   (c) Unemployment does not go up if interest rates go down and more people buy houses.
   (d) If interest rates do not go down, the stock market does not go up.
   (e) If more people buy houses and unemployment does not go up, then the stock market goes up.
   (f) If unemployment goes up, more people buy houses only if interest rates go down.

4. With \(I, H, S, \) and \(U\) as above, complete these sentences in words.

   (a) \(S \Rightarrow U\) is false if and only if . . .
   (b) The converse of \(S \Rightarrow U\) is false if and only if . . .
   (c) The contrapositive of \(\sim I \Rightarrow U\) is false if and only if . . .

5. Find the converse and contrapositive of each implication below.

   (a) If my sister visits, then we will eat out.
   (b) If you knock and I am in, then I will answer.
6. Find the converse of each implication below. Determine the truth value of each implication and its converse. Justify your answers. Assume all variables represent real numbers.
   (a) $x^2 = 4$ only if $x = 2$.
   (b) If $2x \leq x$, then $x^2 > 0$.
   (c) If $2$ is a prime number, then $2^2 = 4$ is a prime number.
   (d) If $x$ is an integer, then $\sqrt{x}$ is an integer.
   (e) If every line has a $y$-intercept, then every line contains infinitely many points.
   (f) A line has undefined slope only if it is vertical.
   (g) $x = -5$ only if $x^2 - 25 = 0$.
   (h) $x^2$ is positive only if $x$ is positive.

7. (a) Find a condition on $m$ which is sufficient but not necessary for $\frac{m}{2} \in \mathbb{Z}$.
   (b) Find a condition on $m$ which is necessary but not sufficient for $\frac{m}{2} \in \mathbb{Z}$.
   (c) Find a necessary and sufficient condition on $m$ for $\frac{m}{2} \in \mathbb{Z}$.

8. (a) Find a sufficient condition on $x$ to make $x$ a solution to the inequality $x + \sqrt{x^4 + x + 1} \geq 100$. Is your condition necessary? Explain.
   (b) Find a necessary condition on $c$ for the implication $ac = bc \Rightarrow a = b$ to be true for all $a, b \in \mathbb{R}$. Is your condition sufficient? Explain.
   (c) Find a sufficient condition on $m$ to ensure that the line $y = mx + b$ has an $x$-intercept. Is your condition necessary? Explain.
   (d) Find necessary and sufficient conditions on $a$ and $b$ to ensure that the equation $ax^2 + b = 0$ has a real solution.

9. Suppose $R \subseteq A \times A$.
   (a) Find the contrapositive of the implication
   $[(a, b) \in R \land (b, a) \in R] \Rightarrow a = b$.
   (b) Find the negation of the statement
   $\forall a, b \in A, [(a, b) \in R \land (b, a) \in R] \Rightarrow a = b$.
   (c) Find the negation of the statement
   $\forall a, b, c \in A, [(a, b) \in R \land (b, c) \in R] \Rightarrow (a, c) \in R$.

10. (a) Construct a truth table with columns for the following statements:
    i. $P$
    iv. $\sim P \Rightarrow Q$
    vii. $\sim(P \Rightarrow Q)$
    ii. $Q$
    v. $\sim P \lor Q$
    viii. $P \land Q$
    iii. $P \Rightarrow Q$
    vi. $P \lor Q$

   (b) Identify all pairs of equivalent statements from part (a).

11. Use truth tables to verify the following statements.
   (a) $[(P \Rightarrow S) \land (P \Rightarrow R)] = [P \Rightarrow (S \land R)]$
   (b) $[(P \Rightarrow S) \lor (P \Rightarrow R)] = [P \Rightarrow (S \lor R)]$
   (c) $[(P \Rightarrow S) \lor (Q \Rightarrow S)] \neq [(P \lor Q) \Rightarrow S]$
   (d) $[(P \Rightarrow S) \lor (Q \Rightarrow R)] \neq [(P \lor Q) \Rightarrow (S \lor R)]$