Solutions

Name:
MTH 355 Quiz 3

No notes nor calculators are allowed. THERE ARE MORE QUESTIONS ON THE BACK!

1. (1.5 points) Prove that

\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4} \quad \forall n \in \mathbb{N} \]

By induction on \( n \).

Base case \( n=1 \)

\[ 1 = \frac{1 \cdot 4}{4} \checkmark \]

Inductive step: We assume

\[ 1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} \quad \text{(*)} \]

and prove

\[ 1^3 + 2^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} \]

By adding \( (k+1)^3 \) to \( (*) \) we have

\[ 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \]

\[ = (k+1)^2 \cdot \left[ \frac{k^2}{4} + k+1 \right] \]

\[ = (k+1)^2 \cdot \left( \frac{k^2 + 4k + 4}{4} \right) \]

\[ = \frac{(k+1)^2 \cdot (k+2)^2}{4} \]
2. (1 point) Show that any five integers contain a pair whose difference is divisible by 4.

For an integer there are exactly 4 possibilities of the remainder when divided by 4 (0, 1, 2, and 3). Thus by the pigeonhole principle given 5 integers $x_1, x_2, x_3, x_4$, and $x_5$ there exists $x_i$ and $x_j$ so that

$$x_i = 4q_i + r \quad \text{and} \quad x_j = 4q_j + r$$

with $i \neq j$ and $0 \leq r \leq 3$.

But then $x_i - x_j = 4(q_i - q_j)$ which is divisible by 4.