Solutions

Name:
MTH 355 Midterm

No notes nor calculators are allowed.

1 (5 points).
(a) Prove the distributive property with a truth table

\[ A \land (B \lor C) = (A \land B) \lor (A \land C). \]

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(b) Use part (a) to prove the distributive property

\[ X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z). \]

\[
X \cap (Y \cup Z) \\
= \{ a \mid a \in X \wedge a \in Y \cup Z \} \\
= \{ a \mid a \in X \wedge (a \in Y \vee a \in Z) \} \\
= \{ a \mid (a \in X \wedge a \in Y) \vee (a \in X \wedge a \in Z) \} \\
= \{ a \mid (a \in X \cap Y) \vee (a \in X \cap Z) \} \\
= (X \cap Y) \cup (X \cap Z)
\]
2 (2.5 points). Determine whether each statement below is true or false.

(a) $\forall x \in \mathbb{R} \exists a \in \mathbb{R}$ with $|x| < a$. $\text{True}$

(b) $\exists a \in \mathbb{R}$ such that $\forall x \in \mathbb{N}$, $a < x$. $\text{True}$

(c) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}$ such that $xy = 1$. $\text{False}$

(d) $\forall a \in \mathbb{R}$, $\sqrt{a^2} = a$. $\text{False}$

(e) $\forall a \in [0, \infty)$, $\exists x \in \mathbb{R}$ such that $x^2 = a$ and $-x^2 = a$. $\text{False}$
3 (2.5 points). Prove the following:
Suppose \( a \) is an integer. If \( a^2 \) is an odd integer, then \( a \) is odd.

By contrapositive.

Suppose \( a \) is even. Say \( a = 2n \) where \( n \in \mathbb{Z} \). Then \( a^2 = 4n^2 = 2 \cdot 2n^2 \).
So with \( m = 2n^2 \), we have \( m \in \mathbb{Z} \) and \( a^2 = 2 \cdot m \) showing that \( a^2 \) is even.
4 (2.5 points). Prove that the sum of a rational number and an irrational number is irrational.

By contradiction.
We assume $a$ is rational, $b$ is irrational, and $a+b$ is rational.
We can then take integers $m,n,p,q \in \mathbb{Z}$ with $n,q \neq 0$ such that

$$a = \frac{m}{n} \quad \text{and} \quad a+b = \frac{p}{q}.$$  

But then $b = a+b-a = \frac{np-mq}{nq}.$

Now $np-mq, nq \in \mathbb{Z}$ and $nq \neq 0,$
so $b$ is rational. This contradicts $b$ is irrational.
5 (2.5 points). Prove the following:

\[ 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \forall n \in \mathbb{N}. \]

By induction on \( n \).

**Base case \( n = 1 \)**

\[ 1 = 1^2 \]

**Inductive step.** We assume

\[ 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \quad \text{\( \Box \)} \]

and prove

\[ 1 + 3 + 5 + \cdots + (2(k+1) - 1) = (k+1)^2. \]

We have

\[
\begin{align*}
& 1 + 3 + 5 + \cdots + (2k+1) - 1 \\
= & 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\
= & k^2 + (2k+1) \quad \text{by \( \Box \)} \\
= & (k+1)^2.
\end{align*}
\]
6 (2.5 points). Prove the following:
Suppose \( a, b, c \in \mathbb{Z} \) and \( a \neq 0 \). If \( a \mid b \) and \( a \mid c \), then \( a \mid (sb + tc) \) for any integers \( s \) and \( t \).

We take integers \( m \) and \( n \) such that 
\[
b = m \cdot a \quad \text{and} \quad c = n \cdot a. \]

For integers \( s \) and \( t \) we set 
\[
d = sm + tn \quad \text{so that} \quad d \in \mathbb{Z}
\]

and 
\[
sb + tc = sma + tna \\
= (sm + tn) \cdot a \\
= da.
\]

Thus \( sb + tc \) is a multiple of \( a \), so \( a \mid sb + tc \).
7 (2.5 points). Use the Euclidean Algorithm to find the greatest common divisor of 1012 and 295. Express the greatest common divisor as an integer linear combination of 1012 and 295.

\[1012 = 3 \cdot 295 + 127\]
\[295 = 2 \cdot 127 + 41\]
\[127 = 3 \cdot 41 + 4\]
\[41 = 10 \cdot 4 + 1\]
\[4 = 4 \cdot 1 + 0\]

Thus, \( \gcd(1012, 295) = 1 \) and

\[1 = 41 - 10 \cdot 4\]
\[= 41 - 10(127 - 3 \cdot 41)\]
\[= 31 \cdot 41 - 10 \cdot 127\]
\[= 31(295 - 2 \cdot 127) - 10 \cdot 127\]
\[= 31 \cdot 295 - 72 \cdot 127\]
\[= 31 \cdot 295 - 72(1012 - 3 \cdot 295)\]
\[= 31 \cdot 295 - 72 \cdot 1012 + 216 \cdot 295\]
\[= -72 \cdot 1012 + 247 \cdot 295\]