Problem I.

i) Show that in the finite complement topology on \( \mathbb{R} \), every subset of \( \mathbb{R} \) is compact.

ii) Show that a subset \( C \subset \mathbb{R}^n \) is compact if and only if every function \( f: C \to \mathbb{R} \) is bounded.

Problem II.

i) Let \( \tau_1 \) and \( \tau_2 \) be two topologies on the same space \( X \). Suppose that \( \tau_2 \) is finer than \( \tau_1 \). If \( (X, \tau_1) \) is compact, does it follow that \( (X, \tau_2) \) is compact? Conversely, if \( (X, \tau_2) \) is compact, does it follow that \( (X, \tau_1) \) is compact?

ii) Let \( Y \subset X \) be equipped with the subspace topology. Show that \( Y \) is compact in the subspace topology if and only if any cover of \( Y \) with open sets in \( X \) possesses a finite subcover of \( Y \).

Problem III.

i) Does the conclusion of Lebesgue lemma hold true if the underlying space \( X = \mathbb{R}^2 \) is the plane? Justify your answer!

ii) Let \( (X, d_1) \) be a compact metric space and \( (Y, d_2) \) a metric space. Suppose that \( f: X \to Y \) is continuous. Use Lebesgue lemma to show that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( d_1(x, y) < \delta \) then \( d_2(f(x), f(y)) < \epsilon \), that is, \( f \) is uniformly continuous.