RATIONAL ROTATION NUMBERS FOR HOMEOMORPHISMS WITH TWO BREAK-TYPE SINGULARITIES

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1. Introduction

This work deals with the family of homeomorphisms of the circle $T_a : S^1 \to S^1$, where $S^1 = [0, 1], a \in [0, 1]$. Each $T_a$ is defined by its lift $f_a : \mathbb{R} \to \mathbb{R}$ such that \{f_a(x)\} = T x. We consider simplest family $f_a = f + a$, where $0 \leq f(0) < 1$.

Suppose that $T = \{f\}$ has two break-type singularities at points $x_{cr}^{(1)} \neq x_{cr}^{(2)}$, i.e., the following conditions hold:

1) $T \in C^{2+\epsilon}(S^1 \setminus \{x_{cr}^{(1)}, x_{cr}^{(2)}\})$,
2) There exists $T'(x_{cr}^{(i)})$ and $T'(x_{cr}^{(i)})$, $i = 1, 2$, and

$$c_i = \frac{T'(x_{cr}^{(i)})}{T'(x_{cr}^{(i)})} \neq 1.$$  

Denote by $\rho(T_a) = \rho(f_a) = \lim_{n \to \infty} f_a^n(x_0)/n$ the rotation number of the homeomorphism $\rho(T_a)$.

**Theorem 1.** Suppose $c_1c_2 \neq 1$ and $c_1 \neq c_2$. Then the Lebesgue measure of the set \{a | $\rho(T_a) \in [0, 1] \setminus \mathbb{Q}$\} is zero.

This theorem for one break-type singularity was proven in [1].

2. Notations and A priori Estimates

Let us denote an interval with endpoints $\alpha$ and $\beta$ as follows

$$I(\alpha, \beta) = \begin{cases} 
(\alpha, \beta), & \text{if } \alpha \leq \beta, \\
(\beta, \alpha), & \text{if } \alpha > \beta. 
\end{cases}$$

We also put $I[\alpha, \beta] = \overline{I(\alpha, \beta)}$. A good description for Farey numbers is given in [1]. We shall need only the following properties of Farey interval $A = (p_1/q_1, p_2/q_2) \subset (0, 1)$:

1) $p_2q_1 - p_1q_2 = 1$.
2) All rationals inside $A$ have the form $(kp_1 + lp_2)/(kq_1 + lq_2)$. The rational number with smallest denominator is $(p_1 + p_2)/(q_1 + q_2)$.

*Этот текст получен из файла 2.TEX*
Consider homeomorphism $T$ with two break-type singularities and corresponding lifting $f$. Choose arbitrary point $x_0$ on the circle and consider the trajectory 
\[ \{x_i = T^i x_0, 0 \leq i \leq q_1 + q_2 \} \]. Denote by $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ the closed intervals 
$[x_0, x_{q_1}]$ and $[x_{q_2}, x_0]$, respectively. Denote by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ the images of these intervals under the action of $T$:
\[
\Delta_i^{(1)} = T^i \Delta_0^{(1)}, \quad \Delta_j^{(2)} = T^j \Delta_0^{(2)}.
\]

The following Proposition was proved in [1] and the proof works in our situation without any change.

**Proposition 1.** Suppose $\rho(T) \in (p_1/q_1, p_2/q_2)$. The trajectory \( \{x_i = T^i x_0, 0 \leq i \leq q_1 + q_2 \} \) forms a partition of the circle consisting of intervals $\Delta_i^{(1)}$, $0 \leq i \leq q_2$ and $\Delta_j^{(2)}$, $0 \leq j \leq q_1$.

Denote this partition by $\xi(A, x_0)$. Denote
\[
v = |\ln c_1^2| + |\ln c_2^2| + \text{Var}_{S_1} \ln f'.
\]

Consider now trajectory $y_i = T^i y_0$ of point $y_0$, such that $y_i \neq x_i^{(1)}$ and $y_i \neq x_i^{(2)}$ for any $0 \leq i \leq q_2$. Denote $q = \max(q_1, q_2)$ and $p = \max(p_1, p_2)$.

**Proposition 2.** Suppose $\rho(T) \in I((p_1 + p_2)/(q_1 + q_2), p/q)$ or $\rho(T) = p/q$. Then
\[
e^{-v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^v. \tag{1}
\]

**Proof:** We follow the lines of proof given in [1] with corrections due to the second singularity point. To be definite, assume that $q_2 > q_1$. Then $q = q_2$, $p = p_2$. Let us consider the case $\rho(T) \neq p_2/q_2$ (the same arguments apply to case $\rho(T) = p_2/q_2$ because there exists a periodic trajectory $x_0, \ldots, x_{q_2-1}$, such that $x_0, \ldots, x_{q_2-1}$ form a proper partition of the circle). Consider partition $\xi((p_1 + p_2)/(q_1 + q_2), p_2/q_2, x_0)$, where the trajectory $x_i = T^i x_0$ of $x_0$ do not touch critical points. Notice that $x_{q_1+q_2} \in \Delta_0^{(1)}$, since $\rho(T) \in ((p_1 + p_2)/(q_1 + q_2), p_2/q_2)$. Let us show that
\[
\left| \sum_{i=0}^{q_2-1} \ln f'(y_i) - \sum_{i=0}^{q_2-1} \ln f'(x_i) \right| \leq \text{Var}_{S_1} \ln f'. \tag{2}
\]

Indeed, assume that $y_0 \in \Delta_j^{(2)}$, $0 \leq j \leq q_1$. Let $x_{k+j}$ correspond to $y_k$ for $0 \leq k < q_1 - j$ and let $x_{k+j-q_1}$ correspond to $y_k$ for $q_1 - j \leq k < q_2$. We have $[x_{k+j}, y_k] \subset \Delta_j^{(2)}$, $0 \leq k < q_1 - j$ for $0 \leq k < q_1 - j$ and $[y_k, x_{k+j-q_1}] \subset \Delta_j^{(2)}$, $q_1 - j \leq k < q_2$, i.e., the intervals do not intersect each other. Hence, we get estimate (2). The same arguments apply to the case $y_0 \in \Delta_i^{(1)}$, $0 \leq i \leq q_2$.\]
Notice now that

$$\int_{s_1=[0,1]} \prod_{i=0}^{q_2-1} f'(y_i) dy_0 = \int_0^1 \frac{\partial}{\partial y_0} f^{q_2}(y_0) dy_0 = f^{q_2}(1) - f^{q_2}(0) = 1.$$ 

Notice that $g = \partial f^{q_2}/\partial y_0$ is not equal 1. It follows that $g$ takes values above and below 1. Without loss of generality suppose that $x_{cr}^{(1)} = 0$ and $x_{cr}^{(2)} \in (0,1)$. If $g$ is arbitrary close to 1 in neighbourhood of some point $y_0$, then we are done (this case is shown on Figure 1).

Hence, assume that $g$ never approaches 1, as shown on Figure 2. Notice that $g$ is piecewise-continuous (and even of class $C^{1+\epsilon}$ on each interval of continuity). All disconinuity points are formed by preimages $T^{-i_1}x_{cr}^{(1)}$ and $T^{-j_2}x_{cr}^{(2)}$ of singularities $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$ for $0 \leq i, j \leq q_2$. For some $i, j$ we can have $T^{-i_1}x_{cr}^{(1)} = T^{-j_2}x_{cr}^{(2)}$ and this imply that corresponding singularity of $g$ has break $c_1c_2$. Other singularities have break equal $c_1$ or $c_2$. Notice that no other breaks can happen. Indeed, if for some $0 \leq i_1, j_1, i_2, j_2 \leq q_2$ we have

$$T^{-i_1}x_{cr}^{(1)} = T^{-j_1}x_{cr}^{(2)} \text{ and } T^{-i_2}x_{cr}^{(1)} = T^{-j_2}x_{cr}^{(2)},$$

then point $y^* = T^{-j_2}x_{cr}^{(2)}$ is periodic of period $q^* = (j_1 - j_2) + (i_1 - i_2)|$, since

$$T^{-j_2} \circ T^{j_1} \circ T^{-i_1} \circ T^{i_2} y^* = y^*$$
Hence $\rho(T) = p^*/q^*$ for some $p^*$. But $q^* = |(j_1 - j_2) + (i_1 - i_2)| \leq 2q_2$, and this contradicts property (2) of Farey intervals, since

$$\frac{p^*}{q^*} \in \left(\frac{p_1 + p_2}{q_1 + q_2}, \frac{p_2}{q_2}\right) = B$$

and the rational with smallest possible denominator in $B$ is $(p_1 + 2p_2)/(q_1 + 2q_2)$, but

$$q_1 + 2q_2 \geq 1 + 2q_2 > 2q_2 \geq q^*.$$  

Using the same arguments we arrive at contradiction if $T^{-i_1}x^{(1)}_{cr} = T^{-i_2}x^{(1)}_{cr}$ or $T^{-i_1}x^{(1)}_{cr} = T^{-i_2}x^{(1)}_{cr}$ for some $0 \leq i_1, i_2 \leq q_2$.

The case $\rho(T) = p_2/q_2$ is even easier. If for some $i, j$ we have $T^{-i}x^{(1)}_{cr} = T^{-j}x^{(2)}_{cr}$, then $x^{(1)}_{cr} = T^{i-j}x^{(2)}_{cr}$ and all $q_2$ singularities have break $c_1c_2$. If $T^{-i_1}x^{(1)}_{cr} = T^{-i_2}x^{(2)}_{cr}$ for some $i_1 < i_2$, $i_2 - i_1 < q_2$, $j = 1, 2$, then we have $\rho(T) = p^*/q^*$, where $q^* = i_2 - i_1$. But $q^* < q_2$ and we arrive at contradiction with $\rho(T) = p_2/q_2$.

Hence the following inequality holds for every singular point $y^*$:

$$\left| \ln \sqrt{\frac{g(y^*-)}{g(y^+)}} \right| \leq |\ln c_1| + |\ln c_2| = C_{1,2}. \quad (3)$$
Notice now that since \( g \) take values above and below 1, one can choose a singularity point \( y^* \) such that one of the following conditions holds.

1) \( g(y^*-) < 1, g(y^*+) > 1 \),
2) \( g(y^*-) > 1, g(y^*+) < 1 \).

Consider case 1). Then, by (3)

\[
\frac{g(y^* -)}{g(y^* +)} \geq \frac{1}{e^{2C_{1,2}}}. \tag{4}
\]

Since \( g(y^*+) > 1 \),

\[
g(y^*-) > \frac{1}{e^{2C_{1,2}}},
\]

and since \( g(y^-) < 1 \),

\[
e^{2C_{1,2}} > g(y^*+).
\]

The same argument works for case 2). Hence,

\[
g(y^*-) , g(y^*+) \in (e^{-2C_{1,2}}, e^{2C_{1,2}}).
\]

Hence given \( \varepsilon > 0 \) we can choose \( x_0 \) such that

\[
|\ln g(x_0)| < 2C_{1,2} + \varepsilon = |\ln c_1^2| + |\ln c_2^2| + \varepsilon. \tag{5}
\]

Estimates (2) and (5) imply

\[
\left| \sum_{i=0}^{q-1} \ln f'(y_i) \right| \leq \text{Vars}_1 \ln f' + |\ln c_1^2| + |\ln c_2^2| + \varepsilon. \tag{6}
\]

Since \( \varepsilon \) is arbitrary, inequality (6) holds for \( \varepsilon = 0 \) and taking exponents of both sides of (6) we obtain (1). \( \square \)

Now denote

\[
q = \max(q_1, q_2); \quad q' = \min(q_1, q_2)
\]

\[
p = \max(p_1, p_2); \quad p' = \min(p_1, p_2).
\]

Preposition 2 gives a-priori estimate for derivative of \( f^q \) for \( \rho(T) \in I((p_1 + p_2)/(q_1 + q_2), p/q) \) and \( \rho(T) = p/q \). The following prepositions provide estimates for derivatives of \( T^q \) and \( T^q \) on the whole interval \( [p_1/q_1, p_2/q_2] \). Both prepositions are simple corollaries of Preposition 2. They are analogous to Prepositions 3.3 and 3.4 in [1] and are proven in the same way.

**Preposition 3.** Suppose one of the following conditions hold:

1) \( \rho(T) \in (p_1/q_1, p_2/q_2); \)
2) \( \rho(T) = p'/q'; \)
3) \( \rho(T) = p/q, q \geq 2q'. \)

Then

\[
e^{-\nu} \leq \prod_{i=0}^{q'-1} f'(y_i) \leq e^\nu.
\]

If the following condition 3') holds
3’) \( \rho(T) = p/q \) and \( q < 2q' \), then
\[
e^{-2v} \leq \prod_{i=0}^{q-1} f^i(y_i) \leq e^{2v}.
\] (7)

Proof. Take \( 0 < q_3 = q - lq' < q' \), \( l \in \mathbb{N} \), and, respectively, \( p_3 = p - lp' \). Applying of Preposition 1 to Farey interval \( I(p_3/q_3, p'/q') \) we obtain that the following estimate
\[
e^{-v} \leq \prod_{i=0}^{q'-1} f^i(y_i) \leq e^{v}
\] (8)
holds if \( \rho(T) \in \left((p_3 + p')/(q_3 + q'), p'/q'\right) \) or \( \rho(T) = p'/q' \). Hence, we immediately obtain our statement in case 2). Case 1) is also obtained from (8), since
\[
\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = I\left(\frac{p'}{q'}, \frac{p}{q}\right) \subset I\left(\frac{p_3}{q_3}, \frac{p_3 + p'}{q_3 + q}\right).
\]
Consider now case 3). Notice that if \( q = 2q' \), then by property (1) of Farey intervals we have
\[
1 = |pq' - 2p'q| = |pq' - 2p'q'| = |p - 2q'|q',
\]
and since \( q' \) is integer, we have \( q' = 1 \). Hence, our estimate is trivial. Therefore, assume that \( q > 2q' \). Then \( q_3 = q - lq' \) with \( l \geq 2 \) and hence
\[
\frac{p}{q} \in I\left(\frac{p'}{q'}, \frac{p_3 + p'}{q_3 + q}\right).
\]
It stays to consider case 3’). We have \( q < 2q' \). Apply our statement to Farey interval \( I((p - p')/(q - q'), p'/q') \). Since \( q - q' < q' \) and \( \rho(T) = p/q \in I((p - p')/(q - q'), p'/q') \), we fall in case 1) considered before. Hence, the following estimate hold:
\[
e^{-v} \leq \prod_{i=0}^{q-1} f^i(z_i) \leq e^{v}.
\]
Applying Preposition 1 to Farey interval \( I(p'/q', p/q) \), and remembering that \( \rho(T) = p/q \), we obtain
\[
e^{-v} \leq \prod_{i=0}^{q-1} f^i(z_i) \leq e^{v}.
\]
Hence,
\[
e^{-2v} \leq \prod_{i=0}^{q-1} f^i(z_i) \leq \prod_{i=0}^{q-1} f^i(z_i) \leq e^{2v}
\]
or
\[
e^{-2v} \leq \prod_{i=q-q'}^{q-1} f^i(z_i) \leq e^{2v}.
\]
Choosing $z_{q-q'} = y_0$ we obtain (7). □

**Proposition 4.** Choose $q_3 = q - lq'$ such that $0 < q_3 < q'$, $l \in \mathbb{N}$. 
1) If $\rho(T) \in I(p'/q', (p_1 + p_2)/(q_1 + q_2))$, then
\[
e^{-(l+1)v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{(l+1)v}.
\] (9)

2) If $\rho(T) = p'/q'$, then
\[
e^{-(l+2)v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{(l+2)v}.
\] (10)

3) Denote
\[Y_k = I \left( \frac{(k + 1)p' + p}{(k + 1)q' + q}, \frac{kp' + p}{kq' + q} \right).
\]
If $\rho(T) \in Y_k$ or $\rho(T) = (kp' + p)/(kq' + q)$ for $k \geq 1$, then
\[
e^{-(k+1)v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{(k+1)v}
\] (11)

**Proof.** Consider Farey interval $I(p_3/q_3, p'/q')$. If $\rho(T) \in I(p_3/q_3, p'/q')$, then, by case 1) of Proposition 3 we have estimate
\[
e^{-v} \leq \prod_{i=0}^{q_3-1} f'(z_i) \leq e^v.
\] (12)

Notice also, that $q' + q \geq 2q'$. Hence Farey interval $I(p'/q', (p' + p)/(q' + q))$ and rotation number $\rho(T) \in I[p'/q', (p' + p)/(q' + q)]$ satisfy conditions 1)-3) of Proposition 3. Therefore,
\[
e^{-v} \leq \prod_{i=0}^{q'-1} f'(z_i) \leq e^v.
\] (13)

Since $q = lq' + q_3$, combining (12) and (13) we obtain (9).

If $\rho(T) = p'/q'$, then for Farey interval $I(p_3/q_3, p'/q')$ conditions of cases 3) or 3') of Proposition 3 holds. In worst case 3') we have estimate
\[
e^{-v} \leq \prod_{i=0}^{q_3-1} f'(z_i) \leq e^v.
\]

Combining the last inequality with (13), we get (10).

Now consider case 3), i.e., $\rho(T) \in Y_k$ or $\rho(Tf) = (kp' + p)/(kq' + q)$. Apply Proposition 2 to Farey interval $I(p'/q', (kp' + p)/(kq' + q))$. It follows that
\[
e^{-v} \leq \prod_{i=0}^{kq'+q-1} f'(z_i) \leq e^v
\] for $\rho(T) \in Y_k$ or $\rho(Tf) = \frac{kp' + p}{kq' + q}$. (14)
Application of Preposition 3 to Farey interval $I(p/q, (kq' + p)/(kq' + q))$ yields

$$e^{-v} \leq \prod_{i=0}^{q'-1} f'(z_i) \leq e^{v} \text{ for } \rho(T) \in Y_k \text{ or } \rho(Tf) = \frac{kq' + p}{kq' + q}, \quad (15)$$

since $kq' + q \geq q' + q \geq 2q'$ and estimates for cases 1)–3) of Preposition 3 hold. Combining (14), (15) and $q = lq' + q_3$ we get (11).

As one can see, if $q < lq'$, then we have effective estimates for derivatives of $T^q$ and $T^q$. In fact, this is one of reasons for introducing so-called “good” Farey intervals used by G. Šwiatek [2] and K. Khanin [1] in similiar setting. A Farey interval $A = (p_1/q_1, p_2/q_2)$ is called “good” if

$$q = \max(q_1, q_2) < 2q' = \max(q_1, q_2).$$

It follows from statement 3) of Preposition 4 that if rotation number is not too close to the ends of Farey interval $(p_1/q_1, p_2/q_2)$, then we also have effective estimates for derivatives. In particular, the following corollary holds.

**Corollary 1.** Consider Farey interval $(p_1/q_1, p_2/q_2)$. Suppose $\rho(T) = (p_1 + p_2)/(q_1 + q_2)$. Then

$$e^{-v} \leq \prod_{i=0}^{q_1+q_2-1} f'(y_i) \leq e^{v}, \quad (16)$$

$$e^{-2v} \leq \prod_{i=0}^{q_1-1} f'(y_i) \prod_{i=0}^{q_2-1} f'(y_i) \leq e^{2v} \quad (17)$$

for any trajectory $y_i = T^iy_0, \ i = 0, \ldots, q_1 + q_2 - 1$ such that $y_i \neq x^{(j)}_c, \ j = 1, 2.$

**Corollary 2.** Consider Farey interval $(p_1/q_1, p_2/q_2)$. Suppose

$$\rho(T) \in \left[\frac{p_1 + p_2}{q_1 + q_2}, \frac{p_1 + 2q_2}{q_1 + 2q_2}\right].$$

Then

$$e^{-3v} \leq \prod_{i=0}^{q_1-1} f'(y_i) \prod_{i=0}^{q_2-1} f'(y_i) \leq e^{3v} \quad (18)$$

**Proof.** Let us prove an estimate for $\partial f^q / \partial x$. If $q_1 \leq q_2$, then Preposition 7, gives an estiamate

$$e^{-2v} \leq \prod_{i=0}^{q_1-1} f'(y_i) \leq e^{2v}.$$

If $q_1 > q_2$, then $q' = \min(q_1, q_2) = q_2, q = \max(q_1, q_2) = q_1$ and we can apply Preposition 4 for

$$Y_1 = I\left(\frac{2p' + p}{2q' + q}, \frac{p'_1 + p}{q'_1 + q} \right) = (\frac{p_1 + p_2}{q_1 + q_2}, \frac{p_1 + 2p_2}{q_1 + 2q_2}),$$
and we obtain that
\[ e^{-2v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{2v} \text{ for all } \rho(T) \in Y_1. \]

Application of case 3) of preposition 4 for \( \rho(T) = (p_1 + 2p_2)(q_1 + 2q_2) \) yields estimate
\[ e^{-3v} \leq \prod_{i=0}^{q-1} f'(y_i) \leq e^{3v} \text{ for all } \rho(T)Y_k. \]

Notice that estimate (2) imply the following important Lemma about decay of lengths of intervals \( \Delta_i^{(1)} \) and \( \Delta_j^{(2)} \), that could be proven like Preposition 3.5 in [1].

**Lemma 1.** Take \( \lambda = (1 + e^{-v})^{-1/2} < 1 \). Suppose that \( \rho(T) \in [p_1/q_1, p_2/q_2] \) and the expansion of \( p_1/q_1 \) to the continuous fraction has length \( n \):
\[ \frac{p_1}{q_1} = [k_1, \ldots, k_n], \ k_n \geq 2. \]

Then
\[ |\Delta_i^{(1)}|, |\Delta_j^{(2)}| \leq \text{const } \lambda^n \]
for all \( 0 \leq i \leq q_2, 0 \leq j \leq q_1 \).

3. Periodic Rotation Numbers

Here and further we shall denote by const any universal constant that depends on homeomorphism \( f \) only and do not depend on \( a \). In this section we consider \( f_a \) for some \( a \) such that \( \rho(f_a) = p/q = [k_1, \ldots, k_n], \ k_n \geq 2 \). For sake of shortness we shall omit index \( a \) till the end of the section.

It is well-known that for \( f = f_a \) there exists a periodic trajectory of period \( q \). Denote by \( \Delta_0 = [y_1, y_2] \) an interval of this trajectory, containing point \( x_{cr}^{(1)} \). Denote \( \Delta_i = f^i\Delta_0 \). For some \( r \) we have \( \Delta_r \ni x_{cr}^{(2)} \). Let us introduce relative coordinates of \( x_{cr}^{(1)}, x_{cr}^{(2)} \) and \( f^{-r}x_{cr}^{(2)} \) on intervals \( \Delta_0, \Delta_r \) and \( \Delta_0, \) resp. Clearly,
\[
\begin{align*}
d_1 &= x_{cr}^{(1)} - y_1 = \frac{x_{cr}^{(1)} - y_1}{|\Delta_0|}, \\
d_2 &= \frac{x_{cr}^{(2)} - f^r y_1}{f^r y_2 - f^r y_1} = \frac{x_{cr}^{(2)} - f^r y_1}{|\Delta_r|}, \\
\tilde{d}_2 &= \frac{f^{-r} x_{cr}^{(2)} - y_1}{y_2 - y_1} = \frac{f^{-r} x_{cr}^{(2)} - y_1}{|\Delta_0|}.
\end{align*}
\]

Let us introduce
\[
\bar{f}(w) = \frac{1}{y_2 - y_1} (f^q(y_1 + w(y_2 - y_1)) - y_1),
\]
where \( w \in [0,1] \). Clearly, \( \bar{f}(w) \) is a rescaled return map \( f^q : \Delta_0 \to \Delta_0 \).
In order to study $\bar{f}(w)$ we need the following notations:

\[
\begin{align*}
    f^l_{c,d}(w) &= \frac{c^2 w}{1 + d(c^2 - 1)}, \\
    f^r_{c,d}(w) &= \frac{w + d(c^2 - 1)}{1 + d(c^2 - 1)}, \\
    F_M(w) &= \frac{w}{M(1 - w) + w}.
\end{align*}
\]

One can see that function

\[
f^r_{c,d}(w) = \begin{cases} 
    f^l_{c,d}(w), & w \in [0, d], \\
    f^r_{c,d}(w), & w \in [d, 1],
\end{cases}
\]

is a continuous piecewise-linear map $[0, 1] \to [0, 1]$ such that

\[
f^r_{c,d}(0) = 0, \quad f^r_{c,d}(1) = 1, \quad \frac{\partial}{\partial w} f^r_{c,d}(w) \bigg|_{w=d-} = \frac{d}{2} c^2.
\]

This map will be used to approximate mapping $f$ on intervals $\Delta_0$ and $\Delta_r$, since by Lemma 1 they are exponentially small.

As was shown in Preposition 4.3 in [1] the mapping $F_M(w)$ is a very good approximation for mapping $f^{r_2-r_1} : \Delta_{r_1} \to \Delta_{r_2}$ if intervals $\Delta_i$ do not cover critical points $x^{(1)}_{c_{r_i}}$ and $x^{(2)}_{c_{r_i}}$ for $i = r_1, \ldots, r_2 - 1$. Denote

\[
M(r_1, r_2) = \exp \left( \sum_{i=r_1}^{r_2-1} \int_{y \in \Delta_i} \frac{f''(y)}{2f'(y)} dy \right).
\]

Also, define

\[
f^{r_2-r_1}(w) = \frac{1}{\Delta_{r_2}} \left( f^{r_2-r_1}((1 - w)f^{r_1}y_1 + w f^{r_1}y_2) - f^{r_2}y_1 \right)
\]

**Lemma 2.** Suppose intervals $\Delta_i$ do not cover critical points $x^{(1)}_{c_{r_i}}$ and $x^{(2)}_{c_{r_i}}$ for $i = r_1, \ldots, r_2 - 1$. Then

\[
\| F_{M(r_1, r_2)} - f^{r_2-r_1} \|_{C^2([0,1])} \leq \text{const} \lambda^n,
\]

where $\lambda$ is a universal constant, given in Lemma 1.

This lemma is proved as Preposition 4.3 in [1]. The only prerequisite required in proof of Preposition 4.3 [1] is an apriori estimates for geometrical decay of size of intervals. In our setting, this estimate is provided with Lemma 1. Hence we omit the proof for sake of saving space.
Let us define
\[ M_1 = M(1, r) = \exp \left( \sum_{i=1}^{r-1} \int_{y \in \Delta_i} \frac{f''(y)}{2f'(y)} \, dy \right), \]
\[ M_2 = M(r + 1, q) = \exp \left( \sum_{i=r+1}^{q-1} \int_{y \in \Delta_i} \frac{f''(y)}{2f'(y)} \, dy \right). \]

Functions \( f^l_{c,d} \), \( f^r_{c,d} \) and \( F_M \) have the following useful properties:
\[ F_N \circ F_M = F_{MN}, \tag{19} \]
\[ f^l_{c,d}(d) = f^r_{c,d}(d) = F_{1/c^2}(d) \tag{20} \]
for all \( M, N, c, d \). Indeed,
\[ F_M(w) = \frac{w}{M(1 - w) + w}, \quad 1 - F_M(w) = \frac{M(1 - w)}{M(1 - w) + w} \tag{21} \]
and
\[ F_N \circ F_M(w) = \frac{w}{MN(1 - w) + w} = F_{MN}(w), \]

as required for (19). To see why (20) holds, one needs very simple algebra:
\[ f^l_{c,d}(d) = f^r_{c,d}(d) = \frac{c^2d}{1 + d(c^2 - 1)} = \frac{c^2d}{(1 - d) + c^2d} \]
\[ = \frac{d}{c^2(1 - d) + d} = F_{1/c^2}(d). \]

Now let us introduce function
\[ G_{c_1, d_1, c_2, d_2, M_1, d_2}(w) : [0, 1] \to [0, 1] \]
by the following rules. If \( 0 \leq d_1 \leq d_2 \), then
\[ G_{c_1, d_1, c_2, d_2, M_1, d_2}(w) = \begin{cases} 
F_{c_1, d_1} \circ f^l_{c_2, d_2} \circ F_{M_1} \circ f^l_{c_1, d_1}(w) & \text{for } 0 \leq w < d_1, \\
F_{c_1, d_1} \circ f^r_{c_2, d_2} \circ F_{M_1} \circ f^r_{c_1, d_1}(w) & \text{for } d_1 \leq w < \tilde{d}_2, \\
F_{c_1, d_1} \circ f^r_{c_2, d_2} \circ F_{M_1} \circ f^r_{c_1, d_1}(w) & \text{for } \tilde{d}_2 \leq w \leq 1. 
\end{cases} \]
If $\tilde{d}_2 < d_1 \leq 1$, we have

$$G_{c_1,d_1,d_2,M_1,d_2}(w) = \begin{cases}
F_{c_1,d_2} \circ f_{c_2,d_2} \circ F_{M_1} \circ f_{c_1,d_1}(w) & \text{for } 0 \leq w < \tilde{d}_2, \\
F_{c_1,d_2} \circ f_{c_2,d_2} \circ F_{M_1} \circ f_{c_1,d_1}(w) & \text{for } \tilde{d}_2 \leq w < d_1, \\
F_{c_1,d_2} \circ f_{c_2,d_2} \circ F_{M_1} \circ f_{c_1,d_1}(w) & \text{for } d_1 \leq w \leq 1.
\end{cases}$$

\textbf{Lemma 3.}

$$\| \tilde{f}(w) - G_{c_1,d_1,d_2,M_1,d_2} \|_{C^2([0,1] \setminus \{d_1, \tilde{d}_2\})} \leq \text{const } \lambda^{\text{ne}}. \quad (22)$$

\textbf{Proof.} Clearly,

$$\tilde{f}(w) = f^{r+1,q} \circ f^{r+1,q} \circ f^{1,r} \circ f^{0,1}.$$ 

We can use Lemma 1 to approximate $f^{0,1} \circ f^{r} \circ f^{1,r}$ with $f^{c_1,d_1}$ and $f^{r+1,q}$ with $f^{c_2,d_2}$. Lemma 2 imply that $f^{1,r}$ is close to $F_{M}(w)$ and $f^{r+1,q}$ is close to $F_{M}(w)$. To obtain formula for $G$ we only need to notice that $M_1 M_2 \approx c_1 c_2$. More precisely,

$$\ln M_1 + \ln M_2 = \sum_{i=0}^{q-1} \int_{\Delta_i} f''(y) dy - \int_{\Delta_0} f''(y) dy - \int_{\Delta_q} f''(y) dy - \int_{\Delta_1} f''(y) dy \leq \text{const } \lambda^n,$$

as required. \hfill \Box

By Lemma 1

$$\left| \int_{\Delta_0} f''(y) dy + \int_{\Delta_1} f''(y) dy \right| \leq \text{const } \lambda^n,$$

Let us introduce

$$\hat{d}_2(\tilde{d}_2) = \begin{cases}
F_{M_1}(f_{c_1,d_1}(\tilde{d}_2)) & 0 \leq \tilde{d}_2 \leq d_1, \\
F_{M_1}(f_{c_1,d_1}(d_2)) & d_1 \leq \tilde{d}_2 \leq 1.
\end{cases} \quad (23)$$

\textbf{Lemma 4.} We have

$$\| G_{c_1,d_1,c_2,d_2,M_1,d_2} - G_{c_1,d_1,c_2,d_2,M_1,d_2} \|_{C^2([0,1] \setminus \{d_1, \tilde{d}_2\})} \leq \text{const } \lambda^{\text{ne}} \quad (24)$$

\textbf{Proof.} Notice that

$$|\hat{d}_2(\tilde{d}_2) - d_2| \leq C \lambda^{\text{ne}}.$$

Both functions are continuous on the same intervals. On each of them they are just fractional-linear functions with exponentially close parameters. Moreover, all parameters are bounded away from 0 and infinity. Hence, these functions are close with both derivatives. \hfill \Box

Therefore, the following preposition holds

\textbf{Preposition 5.}

$$\| G_{c_1,d_1,c_2,d_2,M_1,d_2} - f \|_{C^2([0,1] \setminus \{d_1, \tilde{d}_2\})} \leq \text{const } \lambda^{\text{ne}}$$
For any monotonous function \(g : [0, 1] \to [0, 1]\), satisfying \(g(0) = 0\) and \(g(1) = 1\) let us introduce relation \(R(g, d)\), given by formula
\[
R(g, d) = \frac{1 - g(d)}{g(d)} : \frac{1 - d}{d}.
\]
This relation satisfy the following important equality:
\[
R(F_M, d) = M. \tag{25}
\]
Indeed, by (21)
\[
R(F_M, d) = \frac{M(1 - d)}{M(1 - d) + d} : \frac{1 - d}{d} = M.
\]

**Lemma 5.** We have
\[
|\frac{R(G_{c_1, d_1, c_2, d_2, M_1, d_2}, d_1)}{R(f, d_1)} - 1| \rightarrow 0
\]
as \(n \to \infty\) for any fixed \(f\) (and hence fixed \(c_1, c_2\)).

*Proof.* For sake of shortness let us denote
\[
G(w) = G_{c_1, d_1, c_2, d_2, M_1, d_2}(w).
\]
Notice now, that
\[
\frac{R(G, d_1)}{R(f, d_1)} = \frac{1 - G(d_1)}{1 - f(d_1)} \frac{f(d_1)}{G(d_1)}.
\]
Fix some \(\varepsilon > 0\). There exists \(\delta > 0\) such that for all \(d_1 > \varepsilon\) we have \(G(d_1) > G(\varepsilon) > \delta > 0\). Let us consider fraction \(\frac{f(d_1)}{G(d_1)}\). Clearly,
\[
|\frac{f(d_1)}{G(d_1)} - 1| = \left|\frac{f(d_1) - G(d_1)}{G(d_1)}\right|
\]
Since \(\|\tilde{f}(d_1) - G(d_1)\| \leq \lambda^n\), we have
\[
\left|\frac{f(d_1) - G(d_1)}{G(d_1)}\right| \leq \frac{\lambda^n}{\delta} \leq \varepsilon
\]
for all \(n\) large enough. Hence, the only “dangerous” case is \(d_1 < \varepsilon\).

The following three situation can occur (cf 3):
1) \(d_1 < \tilde{d}_2\),
2) \(d_1 > \tilde{d}_2\) and \(\tilde{d}_2 < d_1 - \tilde{d}_2\),
3) \(d_1 > \tilde{d}_2\) and \(d_2 > d_1 - \tilde{d}_2\).

In case 1) we have
\[
\frac{\tilde{f}(d_1) - G(d_1)}{G(d_1)} = \frac{\tilde{f}(d_1)/d_1 - G(d_1)/d_1}{G(d_1)/d_1} = \frac{\tilde{f}'(\xi) - G'(\zeta)}{G'(\zeta)},
\]
where $\xi, \zeta \in (0, d_1)$. Notice that second derivatives of $\tilde{f}$ and $G$ are bounded by $C_2$ and first derivatives are bounded away from $0$ and $\infty$. Hence, $G'(\zeta) > C_1 > 0$ and

$$|\tilde{f}'(\xi) - G'(\zeta)| = |\tilde{f}'(0) - G'(0) + \tilde{f}''(\xi_1)\xi - G''(\zeta_1)\zeta| \leq |\tilde{f}'(0) - G'(0)| + 2C_2\varepsilon.$$ 

Hence, for all $n$ large enough we have

$$\frac{\tilde{f}(d_1) - G(d_1)}{G(d_1)} \leq \frac{\tilde{f}'(0) - G'(0)}{C_1} + \frac{2C_2\varepsilon}{C_1} \leq C_4\varepsilon,$$ 

where $C_4 > 1$ is independent of $\varepsilon$ (here we used that $|f'(0) - G'(0)| < \text{const } \lambda^n$).

Hence if $d_1 < d_2$, then for any fixed $\varepsilon$ for all $n$ large enough we have

$$\left|\frac{\tilde{f}(d_1) - G(d_1)}{G(d_1)}\right| \leq C_4\varepsilon.$$

Therefore, this fraction converges to $0$ as $n \to \infty$.

If case 2) or 3) holds we use Lagrange theorem twice for $\tilde{f}$ and for $G$:

$$\tilde{f}(d_1) = \tilde{f}(d_2) + \tilde{f}'(\xi_2)(d_1 - \tilde{d}_2) = \tilde{f}'(\xi_1)d_2 + \tilde{f}(\xi_2)(d_1 - \tilde{d}_2),$$

$$G(d_1) = G'(\zeta_1)d_2 + G'(\zeta_2)(d_1 - \tilde{d}_2),$$

where $\xi_1, \zeta_1 \in (0, \tilde{d}_2), \xi_2, \zeta_2 \in (\tilde{d}_2, d_1)$. Again, it is enough to consider case $d_1 < \varepsilon$ and show that the following inequality holds:

$$\left|\frac{\tilde{f}(d_1) - G(d_1)}{G(d_1)}\right| = \left|\frac{\tilde{f}'(\xi_1)d_2 + \tilde{f}(\xi_2)(d_1 - \tilde{d}_2)}{G'(\zeta_1)d_2 + G'(\zeta_2)(d_1 - \tilde{d}_2)}\right| = \left|\frac{P}{Q}\right| \leq \text{const } \varepsilon$$
Suppose \( \tilde{d}_2 \leq d_1 - \tilde{d}_2 \), i.e., case 2) holds. Let us multiply \( P \) and \( Q \) by \( 1/(d_1 - \tilde{d}_2) \). In this case
\[
\left| \frac{P}{d_1 - d_2} \right| = \left| \frac{f'(\xi_1) - G'(\xi_1)}{d_1 - d_2} \right| \tilde{d}_2 + \left| \frac{G'(\zeta_2) - G'(\zeta_2)}{d_1 - d_2} \right| \\
\leq \left| \frac{f'(\xi_1) - G'(\xi_1)}{d_1 - d_2} \right| + \left| \frac{f'(\xi_2) - G'(\zeta_2)}{d_1 - d_2} \right| \\
= \left| \frac{f'(0) - G'(0)}{d_1 - d_2} \right| + \left| \frac{f''(\xi_{11})\xi_1 - G''(\zeta_{11})\zeta_1}{d_1 - d_2} \right| \\
+ \left| \frac{f''(\tilde{d}_2) - G'(\tilde{d}_2)}{d_1 - d_2} \right| + \left| \frac{f''(\xi_{21})\xi_2 - G''(\zeta_{21})\zeta_2}{d_1 - d_2} \right| \\
\leq 2\lambda^n + 4C_2\varepsilon,
\]
where \( \xi_{11} \in (0, \xi_1), \zeta_{11} \in (0, \zeta_1), \xi_{21} \in (\tilde{d}_2, \xi_2) \) and \( \zeta_{21} \in (\tilde{d}_2, \zeta_2) \). We also have
\[
\left| \frac{Q}{d_1 - d_2} \right| = G'(\zeta_1) - \frac{d_2}{d_1 - d_2} + G'(\zeta_2) \geq G'(\zeta_2) > C_1.
\]
Hence,
\[
\left| \frac{P}{Q} \right| \leq \frac{2\lambda^n + 4C_2\varepsilon}{C_1} \leq 4C_4\varepsilon
\]
for all \( n \) large enough.

Case 3) is considered in the same way with multiplying \( P \) and \( Q \) by \( 1/\tilde{d}_2 \).

Let us denote
\[
R(d_1, \tilde{d}_2) = R(G_{c_1,d_1,c_2,\tilde{d}_2,M_1,\tilde{d}_2(\tilde{d}_2)}, \tilde{d}_2).
\]

**Lemma 6.** We have \( R(d_1, d_1) = 1/(c_1c_2) \) and \( R(d_1, 0) = R(d_1, 1) = c_2/c_1 \). For any fixed \( d_1 \) the function \( g(\tilde{d}_2) = R(d_1, \tilde{d}_2) \) is monotonous for \( \tilde{d}_2 \in [0, d_1] \) and for

**Figure 4**
$d_2 \in [d_1, 1]$. Hence the ratio $R(d_1, d_2)$ takes values between $1/(c_1 c_2)$ and $c_2/c_1$ (cf. 4).

Proof. Notice that by (19)–(20) and (23),

$$G_{c_1, d_1, c_2, d_1, M_1, d_1} (d_1) = F_{c_1 c_2, M_1} \circ f_{c_2, d_2}^l (d_1) \circ F_{c_1, d_1} (d_1)$$

$$= F_{c_1 c_2, M_1} \circ F_{c_2, c_2} \circ F_{M_1} \circ f_{c_1, d_1} (d_1) = F_{1/(c_1 c_2)} (d_1).$$

Using (25), we obtain

$$R(d_1, d_1) = R(G_{c_1, d_1, c_2, d_1, M_1, d_1} (d_1), d_1) = R(F_{1/(c_1 c_2)}, d_1) = \frac{1}{c_1 c_2}$$
as required.

If $d_2 > d_1$, then

$$G_{c_1, d_1, c_2, d_2, M_1, d_2} (d_1) = F_{c_1 c_2, M_1} \circ f_{c_2, d_2}^l (d_2) \circ F_{M_1} \circ f_{c_1, d_1} (d_1).$$

Suppose $0 < d_1 < 1$. Notice that $\tilde{d}_2 (\tilde{d}_2) = 1$ if $\tilde{d}_2 = 1$ and hence

$$G_{c_1, d_1, c_2, d_2, M_1, d_2} (d_2) = G_{c_1, d_1, c_2, 1, M_1, 1}$$

when $\tilde{d}_2 = 1$. Using (19) and (20), we obtain

$$G_{c_1, d_1, c_2, 1, M_1, 1} (d_1) = F_{c_1 c_2, M_1} \circ f_{c_2, 1}^l \circ F_{M_1} \circ f_{c_1, d_1} (d_1)$$

$$= F_{c_1 c_2, M_1} \circ F_{M_1} \circ f_{c_1, d_1} (d_1)$$

$$= F_{c_1 c_2} \circ f_{c_1, d_1} (d_1) = F_{c_1 c_2} \circ F_{1/(c_1 c_2)} (d_1) = F_{c_2/c_1} (d_1).$$

Using (25) we obtain

$$R(d_1, 1) = R(F_{c_2/c_1}, d_1) = \frac{c_2}{c_1}.$$

If $d_2 < d_1$, then

$$G_{c_1, d_1, c_2, d_2, M_1, d_2} (d_1) = F_{c_1 c_2, M_1} \circ f_{c_2, d_2}^r \circ F_{M_1} \circ f_{c_1, d_1} (d_1).$$

For $d_2 = 0$ we have

$$G_{c_1, d_1, c_2, 0, M_1, 0} (d_1) = F_{c_1 c_2, M_1} \circ f_{c_2, 0}^r \circ F_{M_1} \circ f_{c_1, d_1} (d_1)$$

$$= F_{c_1 c_2} \circ F_{M_1} \circ f_{c_1, d_1} (d_1) = F_{c_1 c_2} f_{c_1, d_1} (d_1) = F_{c_2/c_1} (d_1),$$

and hence,

$$R(d_1, 0) = R(d_1, 1) = \frac{c_2}{c_1}.$$

Finally, let us study the derivative of $R(d_1, \tilde{d}_2)$ w.r.t. $\tilde{d}_2$. Using chain rule we obtain

$$\frac{\partial}{\partial \tilde{d}_2} R(d_1, \tilde{d}_2) = \frac{d_1}{1 - d_1 (G(d_1))^2} \frac{\partial}{\partial \tilde{d}_2} G(d_1).$$
Now notice that if \( d_1 < \tilde{d}_2 \),
\[
\frac{\partial}{\partial \tilde{d}_2} G(d_1) = \frac{\partial}{\partial \tilde{d}_2} F_{c_1c_2/M_1} \circ f^i_{c_2, \tilde{d}_2(\tilde{d}_2)}(F_{M_1} \circ f^i_{c_1, d_1}(d_1)) =
\]
\[
= F_{c_1c_2/M_1} (f^i_{c_2, \tilde{d}_2(\tilde{d}_2)}(w)) \frac{\partial}{\partial \tilde{d}_2} f^i_{c_2, \tilde{d}_2(\tilde{d}_2)}(w),
\]
where \( w = F_{M_1}(f^i_{c_1, d_1}(d_1)) \). Notice now that
\[
\frac{\partial}{\partial \tilde{d}_2} f^i_{c_2, \tilde{d}_2(\tilde{d}_2)}(w) = \frac{\partial}{\partial \tilde{d}_2} \left( \frac{c_2^2 w}{1 + \tilde{d}_2(\tilde{d}_2)(c_2^2 - 1)} \right) = -\frac{c_2^2 w(c_2^2 - 1)}{(1 + \tilde{d}_2(\tilde{d}_2)(c_2^2 - 1))^2} \frac{\partial}{\partial \tilde{d}_2} \tilde{d}_2(\tilde{d}_2).
\]
Hence the sign of the derivative \( \partial R(d_1, \tilde{d}_2) / \partial \tilde{d}_2 \) coincide with the sign of \(- (c_2^2 - 1)\) when \( \tilde{d}_2 > d_1 \). Proceeding in the same way we obtain that for \( \tilde{d}_2 < d_1 \) the sign of the derivative \( \partial R(d_1, \tilde{d}_2) / \partial \tilde{d}_2 \) is the same as the sign of derivative
\[
\frac{\partial}{\partial \tilde{d}_2} f^r_{c_2, \tilde{d}_2(\tilde{d}_2)} = \frac{(c_2^2 - 1)(1 - w)}{(1 + \tilde{d}_2(\tilde{d}_2)(c_2^2 - 1))^2} \frac{\partial}{\partial \tilde{d}_2} \tilde{d}_2(\tilde{d}_2),
\]
i.e., the sign of \( \partial R(d_1, \tilde{d}_2) / \partial \tilde{d}_2 \) is the same as the sign of \((c_2^2 - 1)\).

Hence, if \( c_2 < 1 \), then \( g(\tilde{d}_2) = R(d_1, \tilde{d}_2) \) is an increasing function for \( \tilde{d}_2 \in [0, d_1] \) and \( g(\tilde{d}_2) \) is a decreasing function for \( \tilde{d}_2 \in [d_1, 1] \). Finally, \( g(0) = c_2/c_1 \), \( g(d_1) = 1/c_1c_2 \), \( g(1) = c_2/c_1 \).

If \( c_2 > 1 \), then \( g(\tilde{d}_2) = R(d_1, \tilde{d}_2) \) is a decreasing function for \( \tilde{d}_2 \in [0, d_1] \) and \( g(\tilde{d}_2) \) is an increasing function for \( \tilde{d}_2 \in [d_1, 1] \). Again, \( g(0) = c_2/c_1 \), \( g(d_1) = 1/c_1c_2 \), \( g(1) = c_2/c_1 \).

\[\square\]

4. CRITICAL PERIODIC TRAJECTORY

Suppose now that \( d_1 = 0 \) or \( d_1 = 1 \). One can easily check that
\[
G_{c_1, 0, c_2, \tilde{d}_2, M_1, \tilde{d}_2(\tilde{d}_2)} = G_{c_1, 1, c_2, \tilde{d}_2, M_1, \tilde{d}_2(\tilde{d}_2)} = G
\]
and
\[
G(w) = \begin{cases} F_{c_1c_2/M_1} \circ f^i_{c_2, \tilde{d}_2} \circ F_{M_1}(w), & 0 \leq w < \tilde{d}_2, \\ F_{c_1c_2/M_1} \circ f^r_{c_2, \tilde{d}_2} \circ F_{M_1}(w), & \tilde{d}_2 \leq w < 1, \end{cases}
\]
Notice that
\[
(F_a(w))'_{w=0} = \frac{1}{a} \quad \text{and} \quad (F_a(w))'_{w=1} = a.
\]
Let us introduce \( m_- = G'(1) \) and \( m_+ = G''(0) \). Since
\[
F'_{c_1c_2/M_1}(0) = f^i_{c_2, \tilde{d}_2}(0) = F_{M_1}(0) = 0 \quad \text{and} \quad F'_{c_1c_2/M_1}(1) = f^i_{c_2, \tilde{d}_2}(1) = F_{M_1}(1) = 1,
\]
we have

\[
    m_- = G'(1) = \frac{c_1 c_2}{M_1} \frac{1}{1 + \tilde{d}_2(d_2)(c_2^2 - 1)} M_1 = \frac{c_1 c_2}{1 + \tilde{d}_2(d_2)(c_2^2 - 1)},
\]

(26)

\[
    m_+ = G'(0) = \frac{M_1}{c_1 c_2} \frac{c_2^2}{1 + \tilde{d}_2(d_2)(c_2^2 - 1)} = \frac{c_2}{c_1 + \tilde{d}_2(d_2)(c_2^2 - 1)}.
\]

(27)
Notice that, as one could expect

\[ \frac{m_-}{m_+} = a_1^2. \]
Let us also consider \( G(\tilde{d}_2) \). Using (19) and (20), we obtain

\[
G(\tilde{d}_2) = \frac{1}{c_1 c_2} f_{c_2}^{i^2} \circ F_{M_1}(\tilde{d}_2) = \frac{1}{c_1 c_2} f_{c_2}^{i_1} \circ F_{M_1}(\tilde{d}_2) = \frac{1}{c_1 c_2} f_{c_2}^{i_1} \circ F_{M_1}(\tilde{d}_2) = \frac{1}{c_1 c_2} f_{c_2}^{i_2}(\tilde{d}_2) = \frac{1}{c_1 c_2} \circ F_{M_1}(\tilde{d}_2) = F_{c_1/c_2}(\tilde{d}_2),
\]

i.e.,

\[
G(\tilde{d}_2) = \frac{\tilde{d}_2}{(c_1/c_2)(1 - \tilde{d}_2) + \tilde{d}_2}.
\]  

(28)

It follows from Lemma 6 and (26),(27) that

\[
R(d_1, \tilde{d}_2) \in I[\frac{1}{c_1 c_2}, \frac{c_2}{c_1}],
\]

\[
m_+ \in I[\frac{1}{c_1 c_2}, \frac{c_2}{c_1}],
\]

\[
m_- \in I[\frac{1}{c_1 c_2}, \frac{c_1}{c_2}].
\]

Now we would like to enumerate critical points in such a way, that segment \( I[\frac{1}{c_1 c_2}, \frac{c_2}{c_1}] \) have not touched point 1. As one can see, only the following two cases are possible.

A) If \( c_1 c_2 > 1 \), then we have to enumerate \( x_{cr}^{(i)} \) in such a way, that \( c_2/c_1 < 1 \).

B) If \( c_1 c_2 < 1 \), then we have to enumerate \( x_{cr}^{(i)} \) in such a way, that \( c_2/c_1 > 1 \).

Let us study the graph of \( G(w) \). In both cases it would be a graph of piecewise-fractional-linear function.

In case A) \( m_- = G'(1) > \text{const} > 1 \) and \( m_+ = G'(0) < \text{const} < 1 \). Using (28), we obtain that for \( \tilde{d} \in [\delta, 1 - \delta] \) we have

\[
G(\tilde{d}_2) \leq \frac{\tilde{d}_2}{1 + \varepsilon}
\]

for some \( \varepsilon \), since

\[
\frac{c_1}{c_2}(1 - \tilde{d}_2) + \tilde{d}_2 \in [1, \frac{c_1}{c_2}].
\]

Hence, the graph of \( G(w) \) is completely below the diagonal (cf Figures 5, 6).

In case B) \( m_- = G'(1) < \text{const} < 1 \) and \( m_+ = G'(1) > \text{const} > 1 \). Using (28), we obtain that for \( \tilde{d} \in [\delta, 1 - \delta] \) we have

\[
G(\tilde{d}_2) \geq \frac{\tilde{d}_2}{1 - \varepsilon}
\]

for some \( \varepsilon \), since

\[
\frac{c_1}{c_2}(1 - \tilde{d}_2) + \tilde{d}_2 \in [\frac{c_1}{c_2}, 1].
\]

Hence, the graph of \( G(w) \) is completely above the diagonal (cf Figures 7, 8).

Let us denote by \( \hat{T}(p/q) = [a_0(p/q), a_1(p/q)] \) a closed interval of values \( a \) such that \( \rho(f_a) = p/q \).
Theorem 2. Suppose that critical points \( x_{cr}^{(i)} \) are enumerated in such a way that
A) for \( c_1c_2 < 1 \) we have \( c_2 < c_1 \) and
B) for \( c_1c_2 > 1 \) we have \( c_2 > c_1 \).

In case A) for all \( n > \text{const} \) there exists a unique critical periodic trajectory
for \( a = a_0(p/q) \). This is a trajectory of the critical point \( x_{cr}^{(1)} \).
In case B) for all \( n > \text{const} \) there exists a unique critical periodic trajectory
for \( a = a_1(p/q) \). This is a trajectory of the critical point \( x_{cr}^{(1)} \).

Proof. Without loss of generality, consider case A). If \( d_2 \) is bounded away from 0
and 1, then the result follows from Preposition 5, since the whole graph of \( \tilde{f} \) would
lie below the diagonal. If \( d_2 \) is close to 0, then \( \tilde{f}(d_1) = G'(0)d_1 + o(d_1) < d_1 \),
since \( G'(0) = m_+ < \text{const} < 1 \). The same argument works for \( d_2 \) close to 1.
Using estimates of Preposition 5, we obtain that the whole graph of \( \tilde{f} \) lie below
the diagonal. \( \square \)

5. NON-LINEARITY OF \( f^q \)

Notice that it is enough to consider the case \( c_1c_2 < 1, c_2/c_1 < 1 \). Consider
interval \( I(p/q) = (a_0, a_1) \). Suppose that \( p/q \) is obtained from Farey interval
\( (p_1/q_1, p_2/q_2) \). It follows from Theorem 2 that \( T_{a_0}^q x_{cr}^{(1)} = x_{cr}^{(1)} \) and
\[ T_{a_0}^q \tilde{x} < \tilde{x} \] for any point \( \tilde{x} \in (T_{a_0}^{q_{x_{cr}^{(1)}}} x_{cr}^{(1)}) \).
\[ (29) \]

Denote now by \( R \) the ratio
\[ \frac{x_{cr}^{(1)} - \tilde{x}}{\tilde{x} - T_{a_0}^{q_{x_{cr}^{(1)}}} x_{cr}^{(1)}} = R. \]

It follows from the following lemma that there exists a unique \( \tilde{a} \in I(p/q) \) such
that \( T_{\tilde{a}}^q \tilde{x} = \tilde{x} \).

Lemma 7. Suppose \( \rho(f_{a_1}) < p/q < \rho(f_{a_2}) \). Then for any \( x \in [0, 1] \) there exists
a unique \( a(x) \in [a_1, a_2] \) such that \( \rho(f_{a(x)}) = p/q \) and \( f^q(x) = x + p \).

Proof. Note that there exists \( K \) such that for all \( k > K \) we have
\[ f_{a_1}^{kq}(x) - x - kp < 0 \] and \( f_{a_2}^{kq}(x) - x - kp > 0 \).

Since \( f_{a}^{kq}(x) \) is a strictly monotonous function of \( a \), there exists a unique \( a = a(x), \)
\( a(x) \in [a_1, a_2] \) such that we have \( f_{a}^{kq}(x) - x - kp = 0 \).

Now note that if \( f_a^{2q}(x) - p > x \) for all \( x \in \mathbb{R} \) then, by \( f_a(x) - 1 = f_a(x - 1) \),
we have
\[ f_{a_2}^{2q}(x) - 2p = f_a^{2q}(f_a^{q}(x) - p) - p > f_a^{q}(x) - p > x \]
and by induction we obtain that \( f_a^{lq}(x) - lp > x \) for all \( l \geq 1 \). Therefore, we
arrive to contradiction with \( f_{a_1}^{kq}(x) - kp = x \). The same arguments works for
\( f_{a_2}^{q}(x) - p < x \). Therefore, \( f_{a}^{q}(x) - p = x. \) \( \square \)
Let us show that at moment $\bar{a}$ the relative coordinate of $x_{cr}^{(1)}$ on critical trajectory passing through $\bar{x}$ is bounded away from 0 and 1 by constant.

**Proposition 6.**

$$\frac{1}{2} \frac{e^{-2v}}{1 + e^{4v}} \frac{R}{R + 1} \leq \frac{x_{cr}^{(1)} - \bar{x}}{T_{a_{0}}^{q_{0}} \bar{x} - x_{cr}} \leq \frac{R}{R + e^{-2v}}. \quad (30)$$

**Proof.** Notice that $p/q$ is obtained from some Farey interval $A = (p_{1}/q_{1}, p_{2}/q_{2})$, i.e., $p/q = (p_{1} + p_{2})/(q_{1} + q_{2})$. Proposition 1, monotonicity of $T_{a}$ and (29) imply that the following order of points takes place:

$$T_{a_{0}}^{q_{2}, x_{cr}^{(1)}} < T_{a_{0}}^{q_{1}, \bar{x}} < x_{cr}^{(1)} < T_{a_{0}}^{q_{1}, \bar{x}} < T_{a_{0}}^{q_{1}, x_{cr}^{(1)}}$$

and

$$T_{a_{0}}^{q_{0}, \bar{x}} < T_{a_{0}}^{q_{0}, \bar{x}}.$$

Since $\rho(T_{a_{0}}) = p/q$, it follows from (17) that

$$e^{-2v} \leq (\frac{\partial}{\partial x} T_{a_{0}}^{q_{1}})(x_{0}), (\frac{\partial}{\partial x} T_{a}^{q_{2}})(x_{0}) \leq e^{2v} \quad (31)$$

for each $a \in [a_{0}, a_{1}]$ and for each point $x_{0}$, such that the derivative of $T_{a}$ w.r.t. $x_{0}$ is well-defined.

Let us prove upper bound in (30). Notice that

$$\frac{x_{cr}^{(1)} - \bar{x}}{T_{a_{0}}^{q_{1}} \bar{x} - x_{cr}} \leq \frac{x_{cr}^{(1)} - \bar{x}}{x_{cr}^{(1)} - \bar{x} + T_{a_{0}}^{q_{1}, \bar{x}} - x_{cr}^{(1)}}.$$

Using (31), we obtain

$$T_{a_{0}}^{q_{1}, \bar{x}} - x_{cr}^{(1)} = \int_{T_{a_{0}}^{q_{2}, x_{cr}^{(1)}}}^{x_{cr}^{(1)}} \frac{\partial}{\partial x} T_{a_{0}}^{q_{1}}(x) dx$$

$$\geq e^{-2v} (\bar{x} - T_{a_{0}}^{q_{2}, x_{cr}^{(1)}}).$$

But

$$\bar{x} - T_{a_{0}}^{q_{2}} = \frac{1}{R} (x_{cr}^{(1)} - \bar{x}).$$

Hence

$$\frac{x_{cr}^{(1)} - \bar{x}}{x_{cr}^{(1)} - \bar{x} + T_{a_{0}}^{q_{1}, \bar{x}} - x_{cr}^{(1)}} \leq \frac{x_{cr}^{(1)} - \bar{x}}{x_{cr}^{(1)} - \bar{x} + e^{-2v} (x_{cr}^{(1)} - \bar{x})} = \frac{R}{R + e^{-2v}},$$

as required.

In order to prove the lower bound in (30) we need an estimate

$$T_{a_{0}}^{q_{1}, \bar{x}} - T_{a_{0}}^{q_{1}, \bar{x}} \leq e^{2v} (T_{a_{0}}^{q_{1}, \bar{x}} - T_{a_{0}}^{q_{1}, \bar{x}}). \quad (32)$$
To see why it is true, notice that by (49) (proven in Appendix),
\[
\frac{\partial}{\partial a} T^q_a \bar{x} = \frac{\partial}{\partial a} T^{q-q_1}_a \circ T^q_a \bar{x} \prod_{j=q_1}^{q-1} f'(\bar{x}_j(a)) \frac{\partial}{\partial a} T^{q_1}_a \bar{x} + \left( \frac{\partial}{\partial a} T^{q-q_1}_a \right) (T^q_a \bar{x}),
\]
where \( \bar{x}_j(a) = T^j_a \bar{x} \). Since the second summand in r.h.s. is non-negative,
\[
\frac{\partial}{\partial a} T^q_a \bar{x} \geq \prod_{j=q_1}^{q-1} f'(\bar{x}_j(a)) \frac{\partial}{\partial a} T^{q_1}_a \bar{x}
\]
or
\[
\frac{\partial}{\partial a} T^q_a \bar{x} \geq \prod_{j=0}^{q_1-1} f'(z_j(a)) \frac{\partial}{\partial a} T^{q_1}_a \bar{x},
\]
where \( z_j(a) = T^j_a z_0(a) \), \( z_0(a) = \bar{x}_{q_1}(a) \). But (31) imply that
\[
e^{-2v} \leq \prod_{j=0}^{q_1-1} f'(z_j(a)) \leq e^{2v}
\]
for all \( a \in I(p/q) \) such that \( \bar{x}_i(a) \neq x^{(j)}_c \) for all \( 0 \leq i \leq q-1, \ j = 0, 1 \). Therefore
\[
\frac{\partial}{\partial a} T^q_a \bar{x} \leq e^{2v} \frac{\partial}{\partial a} T^{q_1}_a \bar{x}
\]
(33)
for almost all \( a \in I[p/q] \). To prove (32) it stays to notice that
\[
T^q_a \bar{x} - T^{q_1}_a \bar{x} = \int_{a_0}^{\bar{a}} \frac{\partial}{\partial a} T^q_a (\bar{x}) da
\]
\[
\leq e^{2v} \int_{a_0}^{\bar{a}} \frac{\partial}{\partial a} T^{q_1}_a (\bar{x}) da = T^{q_1}_a \bar{x} - T^{q_1}_a \bar{x}.
\]
Notice that
\[
T^{q_1}_a \bar{x} - \bar{x} = T^q_a \bar{x} - T^{q_1}_a \bar{x} + T^{q_1}_a \bar{x} - \bar{x}
\]
(34)
It follows from (32) that
\[
T^q_a \bar{x} - T^{q_1}_a \bar{x} \leq e^{2v}(T^q_a \bar{x} - T^{q_1}_a \bar{x}) = e^{2v}(\bar{x} - T^{q_1}_a \bar{x}).
\]
Notice that
\[
T^{q_1}_a (T^q_a \bar{x}, \bar{x}) \subset (x^{(1)}_c, T^{q_1}_a \bar{x}) \subset (\bar{x}, T^{q_1}_a \bar{x})
\]
Using (31), we obtain
\[
e^{-2v}|(T^q_{a_0} \bar{x}, \bar{x})| \leq |(x^{(1)}_c, T^{q_1}_a \bar{x})| \leq |(\bar{x}, T^{q_1}_a \bar{x})|.
\]
To sum up
\[
T^{q_1}_a \bar{x} - T^{q_1}_a \bar{x} \leq e^{4v}(T^{q_1}_a \bar{x} - \bar{x}).
\]
Hence, (34) yields
\[
T^{q_1}_a \bar{x} - \bar{x} \leq (1 + e^{4v})(T^{q_1}_a \bar{x} - \bar{x})
\]
Therefore,
\[
\frac{x_{\text{cr}}^{(1)} - \bar{x}}{T_{q_a}^a \bar{x} - \bar{x}} \geq \frac{1}{1 + e^{\nu \left( T_{q_0}^q \bar{x} - \bar{x} \right)}} \quad (35)
\]

Now notice that
\[
\frac{x_{\text{cr}}^{(1)} - \bar{x}}{T_{q_0}^q \bar{x} - \bar{x}} \geq \frac{x_{\text{cr}}^{(1)} - \bar{x}}{T_{q_0}^q x_{\text{cr}}^{(1)} - T_{q_0}^q x_{\text{cr}}^{(1)}}.
\]

The r.h.s. fraction could be estimated by the following lemma.

**Lemma 8.** Given \( a, b, c > 0 \) such that
\[
\frac{a}{b} \geq \delta \quad \text{and} \quad \frac{a}{c} \geq \gamma,
\]

the following inequality holds:
\[
\frac{a + a}{b + c} \geq \min(\delta, \gamma), \quad \text{i.e.,} \quad \frac{a}{b + c} \geq \frac{1}{2} \min(\delta, \gamma). \quad (36)
\]

**Proof.** Consider points \( B = (b, a), C = (c, a), \) and \( A = (b + c, a + a) \) on the plain \( xOy \) (cf Figure 9). The slope of ray \( OA \) lies between slopes of rays \( OB \) and \( OC \). Hence,
\[
\frac{a + a}{b + c} \geq \min\left(\frac{a}{b}, \frac{a}{c}\right) \geq \min(\delta, \gamma)
\]
as required. \( \square \)

Let us use (36) with \( a = x_{\text{cr}}^{(1)} - \bar{x}, \) \( b = T_{q_a}^a x_{\text{cr}}^{(1)} - x_{\text{cr}}^{(1)} \) and \( c = x_{\text{cr}}^{(1)} - T_{q_0}^q x_{\text{cr}}^{(1)} \). We have
\[
\frac{a}{c} = \frac{x_{\text{cr}}^{(1)} - \bar{x}}{x_{\text{cr}}^{(1)} - T_{q_0}^q x_{\text{cr}}^{(1)}} = \frac{R}{R + 1}
\]
and
\[
\frac{a}{b} = \frac{x_{\text{cr}}^{(1)} - \bar{x}}{T_{q_0}^q x_{\text{cr}}^{(1)} - x_{\text{cr}}^{(1)}}.
\]
\[ \begin{array}{cccc}
\bar{x}_c^2 & x_c & \bar{x}_c & \bar{x}_c^1 \\
y_0 & y_1 & \bar{x}_c & y_2 \\
w = 0 & & \bar{x}_c & y_3 \\
w = 1 & & \bar{x}_c & \bar{x}_c^1 \\
\end{array} \]

**Figure 10.** Intervals \( \Delta_c^{(1)} = [x_c^2, \bar{x}_c] \), \( \Delta_c^{(2)} = [\bar{x}_c, \bar{x}_c^1] \), \( \Delta_c^{(3)} = [\bar{x}_c^1, \bar{x}_c] \), where \( \bar{x}_c^2 = Tf_{a_1}^q x_c \), \( \bar{x}_c^1 = Tf_{a_1}^q x_c \), and \( \bar{x}_c = Tf_{a_1}^q x_c \). Since

\[
T_{a_0}^q(T_{a_0}^q x_{cr}^{(1)}, x_{cr}^{(1)}) = (x_{cr}^{(1)}, T_{a_0}^q x_{cr}^{(1)}),
\]

we can apply (31) to obtain

\[
e^{-2v} R \leq \frac{x_{cr}^{(1)} - \bar{x}}{T_{a_0}^q x_{cr}^{(1)} - x_{cr}^{(1)}} \leq e^{2v} \frac{R}{R + 1}
\]

Therefore,

\[
\frac{x_{cr}^{(1)} - \bar{x}}{T_{a_0}^q x_{cr}^{(1)} - T_{a_0}^q x_{cr}^{(1)}} \geq \frac{e^{-2v} R}{2} \frac{R}{R + 1}
\]

Combining our estimates, we obtain

\[
\frac{x_{cr}^{(1)} - \bar{x}}{T_{a}^q - \bar{x}} \geq \frac{1}{2} \frac{e^{-2v} R}{1 + e^{2v}} \frac{R}{R + 1},
\]

as required. \( \square \)

Now take

\[
\bar{x} = \frac{T_{a_0}^q x_{cr}^{(1)} + x_{cr}^{(1)}}{2}.
\]

By Lemma 7 there exists a unique \( \bar{a} \), corresponding to passing of periodic trajectory through \( \bar{x} \):

\[
\rho(T_{\bar{a}}) = \frac{p}{q}, \quad T_{\bar{a}}^q \bar{x} = \bar{x}.
\]

Denote \( y_0 = T_{a_0}^q y_1, y_1 = \bar{x}, y_2 = T_{a_0}^q y_1, y_3 = T_{\bar{a}}^q y_2 \). Clearly, \( x_{cr}^{(1)} \in (y_0, y_1) \).

It follows from (30), that relative coordinate \( d_1(\bar{a}) \) of \( x_{cr}^{(1)} \) is bounded away from 0 and 1 for all \( n \) large enough:

\[
\frac{1}{2} \frac{e^{-2v}}{1 + e^{2v}} \frac{1}{2} \leq d_1(\bar{a}) = \frac{x_{cr}^{(1)} - y_1}{y_2 - y_1} \leq \frac{1}{1 + e^{-2v}}.
\]

Denote \( \bar{x}_c^1 = T_{a_1}^q x_c \), \( \bar{x}_c^2 = T_{a_1}^q x_c \), \( \bar{x}_c = T_{a_1}^q x_c \). Evidently, \( \bar{x}_c^2 \in (y_0, y_1), \bar{x}_c \in (y_2, y_1), \bar{x}_c^1 \in (y_1, y_0) \). Notice that \( \bar{x}_c > x_{cr}^{(1)} \). Indeed, \( T_{a_0}^q x_{cr}^{(1)} > T_{a}^q x_{cr}^{(1)} = x_{cr}^{(1)} \).
Points $x_c^2, x_c^{(1)}, x_c^1$ form three successive intervals. Denote this intervals by $\Delta_c^{(1)}, \Delta_c^{(2)}, \Delta_c^{(3)}$, respectively. We have $\Delta_c^{(1)} = [x_c^2, x_c^{(1)}], \Delta_c^{(2)} = [x_c^{(1)}, x_c^1], \Delta_c^{(3)} = [x_c^1, x_c^1]$ (see Figure 10). We shall show that the length of each of these intervals is of the same order.

**Proposition 7.** There exists a constant $C_4$ such that for all $n$ large enough

$$\frac{\max(l(\Delta_c^{(1)}), l(\Delta_c^{(2)}), l(\Delta_c^{(3)}))}{\min(l(\Delta_c^{(1)}), l(\Delta_c^{(2)}), l(\Delta_c^{(3)}))} \leq C_4.$$  

**Proof.** Notice that (37) and Lemmas 5 and 6 imply that $\Delta_c^{(2)}/(x_c^{(1)} - y_0)$ and $\Delta_c^{(2)}/(y_1 - x_c^1)$ are bounded away from 0 and 1 by universal constants. Hence it is sufficient to show that

$$(y_2 - y_1) \leq \text{const} (y_1 - y_2), \ (y_0 - y_1) \leq (y_1 - y_2).$$

But $[y_3, y_2] = T f^{q_2}[y_2, y_1], [y_1, y_0] = T f^{q_1}[y_2, y_1]$ and these estimates follow from Corollary 1. \qed

6. **Proof of the Theorem 1**

Let $A = (p_1/q_1, p_2/q_2)$ be a “good” Farey interval of rank $n$, i.e. a Farey interval such that $q = \max(q_1, q_2) < 2q' = 2 \min(q_1, q_2)$. Denote by $a^{(1)} = a_0(p_1/q_1), a^{(2)} = a_0(p_2/q_2)$ those ends of the intervals $I(p_1/q_1), I(p_2/q_2)$ which correspond to the passage of the periodic trajectory through the singular point. By Theorem 2, these ends are left. Let $I_c(A)$ be the interval $(a^{(1)}, a^{(2)})$ and let $\bar{I}_c(A) = [a^{(1)}, a^{(2)}]$ be its closure (see Figure 11). Denote $x^m = T f^m_a(x_c)$ where $x_c = 0$ is a critical point and $m \geq 0$.

**Proposition 8.** There exists $0 < \gamma < 1$ such that for all $n$ large enough

$$\frac{l(I(p_1/q_1))}{l(I_c(A))} \geq 1 - \gamma$$

(38)

This statement implies theorem 1.
Proof of Theorem 1. Suppose that $\rho = \rho(a)$ is irrational and $a$ is a condensation point for set $\{a \mid \rho(a) \in [0, 1) \setminus \mathbb{Q}\}$. Then there exists an infinite sequence of “good” Farey intervals $A_i$, containing $\rho$. Since $a \in I_c(A_i)$ and $l(I_c(A_i)) \to 0$ as $i \to \infty$, using (38) we arrive at contradiction with assumption that $a$ is a condensation point. \hfill \Box

Let us denote

$$M_{1,2} = \max_{a \in I_c(A)} \left( \frac{\partial}{\partial a} x^{q,a}(a) \right)$$

$$m_{1,2} = \min_{a \in I_c(A)} \left( \frac{\partial}{\partial a} x^{q,a}(a) \right)$$

Proof of Proposition 8. First notice, that the we have the following ordering:

$$a_0 \left( \frac{p_1}{q_1} \right) < \tilde{a} \left( \frac{p_1}{q_1} \right) < a_1 \left( \frac{p_1}{q_1} \right) < a_0 \left( \frac{p_2}{q_2} \right) < a_1 \left( \frac{p_2}{q_2} \right)$$

(see Figure 11).

Let us denote

$$\alpha = a_0 \left( \frac{p_2}{q_2} \right) - \tilde{a} \left( \frac{p_1}{q_1} \right), \quad \beta = \tilde{a} \left( \frac{p_1}{q_1} \right) - a_0 \left( \frac{p_1}{q_1} \right).$$

For any function $f(z)$ such that $f'(z) > 0$ we have the following estimates for any $z_1 < z_2$:

$$\frac{f(z_2) - f(z_1)}{\max_{z \in [z_1, z_2]} f'(z)} \leq z_2 - z_1 \leq \frac{f(z_2) - f(z_1)}{\min_{z \in [z_1, z_2]} f'(z)}.$$

It follows that

$$\beta \geq \frac{x^{q_1}(\tilde{a} \left( \frac{p_1}{q_1} \right)) - x^{q_1}(a_0 \left( \frac{p_1}{q_1} \right))}{M_1}. \quad (39)$$

Using the equality

$$x^{q_1}(a_0 \left( \frac{p_1}{q_1} \right)) = x_c,$$

and notation

$$v = x^{q_1}(\tilde{a} \left( \frac{p_1}{q_1} \right)) - x_c \quad (\text{see Figure 6})$$
we can rewrite (39) as follows:

\[
\beta \geq \frac{x^{q_1}(\hat{a} \left( \frac{p_1}{q_1} \right)) - x_c}{m_1} = \frac{v}{M_1}.
\]  

(40)

Also, we have

\[
\frac{x^{q_1}(a_0 \left( \frac{p_2}{q_2} \right)) - x^{q_1}(\hat{a} \left( \frac{p_1}{q_1} \right))}{M_1} \leq \frac{x^{q_1}(a_0 \left( \frac{p_2}{q_2} \right)) - x^{q_1}(\hat{a} \left( \frac{p_1}{q_1} \right))}{m_1},
\]

and after notation

\[
u = x^{q_1}(a_0 \left( \frac{p_2}{q_2} \right)) - x^{q_1}(\hat{a} \left( \frac{p_1}{q_1} \right)) \text{ (see Figure 6),}
\]

we obtain

\[
\frac{u}{M_1} \leq \alpha \leq \frac{u}{m_1}.
\]  

(41)

Now, let us write

\[
\alpha \leq \frac{x^{q_2}(a_0 \left( \frac{p_2}{q_2} \right)) - x^{q_2}(\hat{a} \left( \frac{p_1}{q_1} \right))}{m_2}.
\]

Since

\[
x^{q_2}(a_0 \left( \frac{p_2}{q_2} \right)) = x_c,
\]

this estimate with notation

\[
w = x^{(1)}_{cr} - x^{q_2}(a_0 \left( \frac{p_2}{q_2} \right)) \text{ (see Figure 6)}
\]

yields

\[
\alpha \leq \frac{w}{m_2},
\]  

(42)

Now let us notice that the following ordering takes place:

\[
x^{q_2}(\hat{a} \left( \frac{p_1}{q_1} \right)) < x_c < x^{q_2}(\hat{a} \left( \frac{p_1}{q_1} \right)) < x^{q_2}(a_0 \left( \frac{p_2}{q_2} \right)).
\]

Suppose that \(p_1/q_1\) is obtained from some Farey interval \((p_2/q_2, p_0/q_0)\), i.e. \(q_1 = q_0 + q_1, p_1 = p_0 + p_2\). In this case we can use Lemma 7, where

\[
\tilde{a} = \hat{a} \left( \frac{p_1}{q_1} \right), p/q = p_1/q_1, \Delta_c^{(1)} = w, \Delta_c^{(2)} = v.
\]

It follows from Lemma 7, that for some \(C\) we have

\[
C \leq \frac{w}{v} \leq 1/C.
\]  

(43)
Combining estimates (40), (41), (42) and (43), we get the following inequalities

$$\frac{l(I(p_1/q_1))}{l(I_c(A))} \geq \frac{\beta}{\beta + \alpha} = \frac{1}{1 + \alpha/\beta}$$

$$\geq \frac{1}{1 + \frac{u M_1}{m_1 v}} = \frac{1}{1 + \frac{M_1 u w}{m_1 w v}}$$

$$\geq \frac{1}{1 + \frac{M_1 \alpha M_1 w}{m_1 \alpha m_2 v}} = \frac{1}{1 + \frac{M_1 M_1 w}{m_1 m_2 v}} \geq \frac{1}{1 + C_1} < 1 - \gamma,$$

where we assumed that

$$\frac{M_1}{m_1}, \frac{M_1}{m_2} \leq \text{const} < \infty.$$

These inequalities are proven in the following Preposition 9. \(\square\)

**Preposition 9.** Let \(A\) be a “good” Farey interval. Then

$$\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{m_1}{M_1}, \frac{m_2}{M_1} \geq \text{const} > 0.$$

**Proof.** We use the following preposition (see [2] and [3, Preposition 6.3]). Denote

$$\Delta_i = \left(T f_{a(i)}^i(x_c), T f_{a(i)}^{i+1}(x_c)\right).$$

**Preposition 10.** The system of intervals \(\Delta_i, 1 \leq i \leq q = \max(q_1, q_2),\) covers any point of the circle at most twice.

**Corollary 3.** \(\sum_{i=1}^{q} l(\Delta_i) \leq 2.\)

Let \(\bar{q} \leq q = \max(q_1, q_2), a \in I(A).\) It follows from (52) that

$$\frac{\partial}{\partial a} x^{\bar{q}}(a) = \sum_{i=0}^{\bar{q}-1} \prod_{j=i+1}^{\bar{q}-1} f'_a(x^j(a)),$$  \(44\)

where

$$x^j(a) = T f_a^j(x_c).$$

Consider one of the summands in (44),

$$B^q_i(a) = \prod_{j=i+1}^{\bar{q}-1} f'_a(x^j(a))$$

**Lemma 9.** There exists constant \(C\) such that for any \(a_1, a_2 \in I(A)\) we have

$$\frac{1}{C} \leq \frac{B^q_i(a_1)}{B^q_i(a_2)} \leq C.$$
Proof. It is easy to see that \( x_c \notin \Delta_j, 1 \leq j \leq q \). Hence,

\[
\ln B_i^q(a_1) - \ln B_i^q(a_2) \leq \left| \sum_{j=i}^{q-1} (\ln f'(x_j(a_1)) - \ln f'(x_j(a_2))) \right| \\
\leq \text{const} \max_{y \in s^1} \left| \frac{f''(y)}{f'(y)} \right| \sum_{j=i}^{q} l(\Delta_j) \leq 2 \max_{y \in s^1} \left| \frac{f''(y)}{f'(y)} \right| = \ln C. 
\]

Here we used Corollary 3 and \((\ln f'_a(y))' = f''_a(y)/f'_a(y)\). \(\square\)

Therefore, Lemma 9 implies that \( m_1/M_1, m_2/M_2 \geq 1/C \). Now let us compare \( \frac{\partial x^{q_1}(a)}{\partial a} \) with \( \frac{\partial x^{q_2}(a)}{\partial a} \). Assume that \( q = \max(q_1, q_2) = q_2 \). Using (49) for \( f(a) = f_{a}^{q_2-q_1}(x_c), g(a) = f_{a}^{q_1}(x_c) \), we obtain

\[
\frac{\partial}{\partial a} x^{q_2}(a) = \left( \prod_{j=q_1}^{q_2-1} f_a'(x_j^j(a)) \right) \frac{\partial}{\partial a} x^{q_1}(a) + \frac{\partial}{\partial a} \left| f_{a}^{q_2-q_1}(x_0) \right|_{x_0=x^{q_1}(a)}. \tag{45}
\]

Applying Corollary 1 to the Farey interval \((p_1/q_1, (p_2 - p_1)/(q_2 - q_1))\), we get

\[
e^{-2v} \leq \prod_{j=q_1}^{q_2-1} f_a'(x_j^j(a)) \leq e^{2v}, \ a \in I(A). \]

It follows from (45), that

\[
\frac{\partial x^{q_2}(a)/\partial a}{\partial x^{q_1}(a)/\partial a} \geq \left( \prod_{j=q_1}^{q_2-1} f_a'(x_j^j(a)) \right) \geq e^{-2v}
\]

and \( m_2/M_1 \geq e^{-2v}/C \). The following estimates could be obtained like Lemma 9.

\[
\frac{1}{C} \leq \frac{(\partial/\partial a)I f_{a}^{q_2-q_1}(x_0)|_{x_0=x^{q_1}(a)}}{(\partial/\partial a)x^{q_2-q_1}(a)} \leq C, \ a \in I(A). \tag{46}
\]

As one can see, in order to prove (46), one have to consider

\[
\hat{B}_i^q(a) = \prod_{j=i+1}^{q-1} f'(x_j^j(a)), 
\]

where \( x_j^j(a) = T_{a}^{q-1} x_j^j(a) \).

Lemma 10. There exists constant \( C \) such that for any \( a_1, a_2 \in I(A) \) we have

\[
\frac{1}{C} \leq \frac{\hat{B}_i^q(a_1)}{\hat{B}_i^q(a_2)} \leq C.
\]
Proof.

\[
|\ln \tilde{B}_i^q(a_1) - \ln \tilde{B}_i^q(a_2)| \leq \left| \sum_{j=i}^{q} (\ln f'(x_0^j(a_1)) - \ln f'(x^j(a_2))) \right|
\]

\[
\leq \left| \sum_{j=i}^{q} (\ln f'(T_{a_1}^j x^j(a_1)) - \ln f'(x^j(a_1))) \right|
\]

\[
+ \left| \sum_{j=i}^{q} (\ln f'(x^j(a_1)) - \ln f'(x^j(a_2))) \right|
\].

The second sum in r.h.s is estimated as in Lemma 9, and the first sum is bounded by constant, since the total length of intervals \([x^j(a_1), T_{a_1}^j x^j(a_1)], j = 0, \ldots, q\), do not exceed 1, i.e., the finite length of the circle. \(\square\)

Using (49) for \(f(a, \cdot) = f_{a}^{2q_1-q_2}(\cdot), g(a) = f_{a}^{2q_2-q_1}(x_c)\), we obtain

\[
\frac{\partial}{\partial a} x^{q_1}(a) = \left( \prod_{j=q_2-q_1}^{q_1-1} f_0'(x^j(a)) \right) \frac{\partial}{\partial a} x^{q_2-q_1}(a) + \frac{\partial}{\partial a} T f_{a}^{2q_2-q_1}(x_0) \bigg|_{x_0 = x^{q_2-q_1}(a)}. \tag{47}
\]

We have

\[
\frac{(\partial/\partial a)x^{q_1}(a)}{(\partial/\partial a)x^{q_2-q_1}(a)} \geq \prod_{j=q_2-q_1}^{q_1-1} f_0'(x^j(a)) \geq e^{-3v}, \ a \in I(A), \tag{48}
\]

where we applied [3, Preposition 3.4, p. 65] to the Farey interval \([(2p_1-p_2)/(2q_1-q_2), (p_2-p_1)/(q_2-q_1))\]. Using (46) and (48) we obtain

\[
\frac{\partial}{\partial a} T f_{a}^{2q_2-q_1}(x_0) \bigg|_{x_0 = x^{q_1}(a)} \leq C \frac{\partial}{\partial a} x^{q_2-q_1}(a) \leq C e^{3v} \frac{\partial}{\partial a} x^{q_1}(a).
\]

Substituting last estimate to (45) we obtain

\[
\frac{\partial}{\partial a} x^{q_2}(a) \leq (e^{2v} + e^{3v}C) \frac{\partial}{\partial a} x^{q_1}(a).
\]

Let a be the value such that \(\frac{\partial x^{q_1}(a)}{\partial a} \bigg|_{a=a} \) is minimal. Combining our estimates, we obtain

\[
\frac{m_1}{M_2} = \frac{(\partial/\partial a)x^{q_1}(a)}{(\partial/\partial a)x^{q_2}(a)} \cdot \frac{(\partial/\partial a)x^{q_2}(a)}{M_2} \geq \left( \frac{1}{e^{2v} + Ce^{3v}} \right) \frac{1}{C} \ \square
\]
Consider function $f(x, y) \in C^1(\mathbb{R} \times \mathbb{R})$ and $g(y) \in C^1(\mathbb{R})$. Denote

$$f_x(x, y) = \frac{\partial}{\partial x} f(z, y) \bigg|_{z=x} \quad \text{and} \quad f_y(x, y) = \frac{\partial}{\partial y} f(x, a) \bigg|_{a=y}. $$

Note that,

$$\frac{\partial}{\partial y} f(g(y), y) = f_x(g(y), y)g'(y) + f_y(g(y), y). \quad (49)$$

Denote by $f^n(x, y)$ the $n$th iteration of function $f$ with respect to variable $x$:

$$f^n(x, y) = f(f(\ldots f(f(x, y), y), \ldots, y), y).$$

**Lemma 11.** For any $n \in \mathbb{N}$

$$\frac{\partial}{\partial y} f^n(g(y), y) = \sum_{i=1}^{n} f_y(x_i(y), y) \prod_{j=i+1}^{n} f_x(x_j(y), y) + \frac{\partial}{\partial y} f(g(y), y) \left( \prod_{j=1}^{n} f_x(x_j(y), y) \right), \quad (50)$$

where $x_j(y) = f^{j-1}(g(y), y)$, $x_1(y) = g(y)$.

Here and later the product with low limit higher than upper limit is equal to 1.

**Proof.** It follows from (49) that the statement is true for $n = 1$. By induction, suppose that (50) is true for $n = k - 1$. To obtain this formula for $n = k$ we substitute $g(y) \mapsto f(g(y), y)$, $x_j(y) \mapsto x_{j+1}(y)$ in (50) for $n = k - 1$.

$$\frac{\partial}{\partial y} f^k(g(y), y) = \frac{\partial}{\partial y} f^{k-1}(f(g(y), y), y)$$

$$= \sum_{i=1}^{k-1} f_y(x_{i+1}(y), y) \prod_{j=i+1}^{k-1} f_x(x_{j+1}(y), y) + \frac{\partial}{\partial y} f(g(y), y) \left( \prod_{j=1}^{k-1} f_x(x_{j+1}(y), y) \right)$$

$$= \sum_{i=2}^{k} f_y(x_i(y), y) \prod_{j=i+1}^{k} f_x(x_j(y), y)$$

$$+ (f_x(g(y), y)g'(y) + f_y(g(y), y)) \left( \prod_{j=2}^{k} f_x(x_j(y), y) \right)$$

$$= \sum_{i=1}^{k} f_y(x_i(y), y) \prod_{j=i+1}^{k} f_x(x_j(y), y) + g'(y) \prod_{j=1}^{k} f_x(x_j(y), y).$$

Here we use (49). \qed

**Lemma 12.** For any $k \in \mathbb{N}$

$$\frac{\partial}{\partial y} f^k(x, y) = \sum_{i=0}^{k-1} f_y(x_i(y), y) \prod_{j=i+1}^{k-1} f_x(x_j(y), y). \quad (51)$$

where $x_j(y) = f^j(x, y)$, $x_0(y) = x$. 


\textbf{Proof.} Fix $x$ and put $g(y) = f(x, y)$, $n = k - 1$ in (50).

It follows from (51) that

$$\frac{\partial}{\partial a} f^n_a(x) = \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} f'_a(x_j(a)), \quad (52)$$

where

$$x_j(a) = f'_a(x).$$

\section*{References}


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