OREGON STATE UNIVERSITY

DEPARTMENT OF MATHEMATICS

CALCULUS STUDY GUIDE

MTH 252

2010-2011

By Betty Fein and Peter Willard-Argyres
## LESSON SUMMARY

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SUGGESTED PROBLEMS

This short list of problems should not be viewed as sufficient! Please read the note on page one about Homework.

The suggested exercises listed below are relatively straightforward applications of the basic computational skills, while the suggested problems on the next page typically involve both computational skill and conceptual reasoning.

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§5.2: 1–4, 7, 14
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§7.7: 1a, 2, 4, 5, 7, 9, 17, 21, 28, 30
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(Turn the page for more problems.)
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§8.1: 8, 9, 11–14, 16–28.
§8.2: 17, 18, 19–37, 39–41.
§8.4: 1, 5, 9, 10, 12, 13, 16, 19, 23, 27
§8.5: 6–28, 31, 32.
Preface

This is a Study Guide for MTH 252, Integral Calculus. It is intended for use with the fourth edition of Calculus by Deborah Hughes Hallett, Andrew Gleason, William McCallum, et al. (either the Single Variable or the combined Single and Multivariable versions), but could also be adapted for use with other texts.

The Study Guide summarizes the key concepts of integral calculus, but is not intended as a substitute for either the textbook or the lectures.

Acknowledgments

This study guide borrows heavily from previous study guides, with material written by Bob Burton, Christine Escher, and Dennis Garity. The implementation of a new textbook was accompanied by extensive discussion, involving among others Dianne Hart, John Lee, Corinne Manogue, Lea Murphy, Scott Peterson, and Brian Smith. The teaching assistants, Aniruth “Ake” Phon-on, Dan Rockwell, Nicholas Stanford, and Aaron Wood, had the dubious distinction of debugging many of the laboratory activities. To all of these people we are extremely grateful.

The laboratory activities are based on activities developed by the members of the Calculus Consortium for this text, and are included here with the kind permission of their publisher, John Wiley & Sons, Inc.
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Introduction

WARNING:

This course may be different from any previous math course you have taken. This is especially true for those of you who have seen calculus before, perhaps in high school. In this course, you will be expected to understand and use mathematical concepts and reasoning, not merely perform calculations.

Prerequisites

It is expected that you are comfortable with differential calculus. This includes familiarity with material from previous courses, such as basic algebraic manipulations, including those involving fractions and exponents, as well as trigonometry. The most common difficulty for students in this course is with “old” concepts such as algebra; if your skills are rusty, practice now, and get help early.

Homework

The only way to master the material in this course is to solve problems. There are several categories of problems in the text. Exercises at the end of each section are fairly straightforward applications of the basic computational skills. Problems at the end of each section usually involve both computational skill and conceptual reasoning; few of these problems can be solved by simply looking up the appropriate formula. Check your Understanding questions at the end of each chapter check your conceptual understanding of the material; many of these questions are True/False.

Your instructor will most likely assign a few problems each week. Do not expect to succeed in this course if you only attempt the assigned problems. It is your responsibility to attempt most of the Exercises, and many more Problems than assigned, on your own, and to seek help as needed. A short list of recommended problems appears on the first page of this guide; again, this is intended as a starting place, and most definitely not as an indication that this is a sufficient number of problems to attempt.

Time Management

It is not unreasonable to spend two to three hours going over the material and assignments per lecture. On the other hand, if you spend 30 minutes on a problem without making forward progress, seek help! Your instructor and GTA both hold regular office hours; use them!
Lectures
Most instructors have their own insights into mathematics, as well as their own way of presenting the material. You are responsible for everything presented in class, whether or not it perfectly matches either the text or the study guide. It is usually difficult to meet this responsibility without regular class attendance.

Other Resources
The Mathematics Learning Center (MLC) provides drop-in help for all lower division mathematics courses. The MLC is located on the ground floor of Kidder Hall (Kidder 108), and is normally open M–F from 9 AM to 6 PM (Fridays to 5 PM), from the second week of term through Dead Week (Week 10). The MLC also provides evening tutoring in the Valley Library, normally Su–Th from 6–9 PM. Current hours can be found at http://www.math.oregonstate.edu/?q=mlc.

Labs
Most weeks there will be a group activity during recitation, which may well take up the entire period. These activities are an essential component of the course.

Skills Test
Towards the end of the term, your instructor will most likely give you a Skills Test on basic integration techniques. This test is closed-book, with neither notes nor calculators allowed. While different instructors may use slightly different grading schemes, this test is usually offered on an “all or nothing” basis. Typically, a score of 80% or better earns full credit, with lower scores earning no credit. You will normally have 2 chances to pass this test.

The questions on the skills test will be similar in difficulty to the Exercises which test basic integration skills, such as (but not limited to) those in §7.1 and §7.2. Do not expect to pass this test by (only) studying for it at the last minute. Instead, make sure to practice your integration (and algebra!) skills throughout the course.
1 The Definite Integral

These lessons introduce integration as an *accumulation*, determining total change from incremental change.
Lesson 1: Measuring Distance

Let’s take a car trip with constant velocity. Here is a familiar formula:

\[
\text{Distance} = \text{Rate} \times \text{Time}
\]

This means the distance traveled (Distance) at a \textbf{constant} velocity (Rate), is that rate multiplied by the time elapsed (Time). If a car travels along I-5 at a speed \(^1\) of 65 mph for 1 hour and 30 minutes, the distance traveled is 65 mi/hr \times 1.5 \text{ hr} = 90 \text{ mi}. However, the formula does not work if velocity is variable. (Why?)

How can you find distance traveled at variable velocity? The key: use what you know, \(D = RT\), but in a clever way which stands at the heart of the second main branch of calculus, the integral calculus.

If velocity is variable over the one and one-half hour trip, then it will be very nearly constant over any short time interval during the trip. So you can estimate the distance traveled as follows:

1. Break up the 1.5 hour trip into many short subtrips of, say, 10 seconds each.
2. Glance at the speedometer at some instant during each 10-second subtrip and record your velocity.
3. Use what you know, \(d = rt\), and the velocity estimate in Step 2 to approximate the distance traveled during each 10-second interval.
4. Add up all the distances of the “short” subtrips.

The sum in Step 4, which you will soon call a Riemann sum, should be a good estimate of the total distance of your trip. You can expect to get an even better estimate by repeating the process above using 1-second subintervals instead of 10-second subintervals. (Why should you expect this will be better?)

Taking a limit of the sums formed in Step 4 as the subtrip-length gets shorter and shorter gives you the exact distance traveled during the trip.

There is a geometric view of the procedure above. If velocity is constant, the graph of velocity versus time is a horizontal line segment and the “area” between that segment and the time axis is \(D = RT\); so the “area” \(RT\) under the constant velocity graph is numerically equal to the distance traveled. The summation process above leads to the same conclusion, even when velocity is variable: The “area” under the velocity versus time curve is numerically equal to the distance traveled.

\(^1\) \text{Speed} is always positive (or zero), as is the (total) distance traveled, but \textbf{velocity} depends on the direction traveled. The net displacement — the distance between the starting and ending points — can be zero even if the total distance traveled is not!
Lesson 2: Definite Integrals

The procedure in Lesson 1 of breaking up a variable-velocity trip into many short subtrips, at nearly constant velocity, adding the distances of the short trips, and finally taking a limit as the time intervals of all the subtrips shrinks to zero to get the total distance traveled can be used to solve a vast array of problems having nothing to do with motion. Such problems share a common feature: The process of interest takes place at a variable rate that is known. The problem is to determine the total accumulation (of distance, mass, area, volume, population, ...) that results from the variable rate process. Lesson 2 sets the stage for handling such problems in a unified way.

Very often the rate is a time rate of change; so we will use $t$ for the independent variable, but you will soon meet applications in which the independent variable has other units. Let $f(t)$ be the rate of a continuously evolving process at time $t$. ($f(t)$ could be velocity in the trip context or $f(t)$ could be the tons of carbon dioxide per hour produced by a coal fired plant used to generate electricity.) Suppose the period of interest is $a \leq t \leq b$. Then no matter what $f$ signifies or what units $t$ has, the four-step process in Lesson 1 can be expressed as:

1. Subdivide $a \leq t \leq b$ into nonoverlapping subintervals of equal length
   \[
   \Delta t = \frac{b - a}{n}
   \]
   by means of a subdivision by equally spaced points $a = t_0 < t_1 < \ldots < t_n = b$.

2. Sample $f$ at any convenient point $c_i$ in the subinterval $t_{i-1} \leq t \leq t_i$ to get $f(c_i)$.

3. $f(c_i)\Delta t$ (use what you know about constant rate processes) approximates the accumulation of $f$ over the interval $t_{i-1} \leq t \leq t_i$.

4. Form the Riemann sum
   \[
   f(c_1)\Delta t + f(c_2)\Delta t + \cdots + f(c_n)\Delta t = \sum_{i=1}^{n} f(c_i)\Delta t
   \]
   which estimates the total accumulation of $f$ over $a \leq t \leq b$.

The exact accumulation of $f$ over the interval $a \leq t \leq b$ is

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta t.
\]

This limit is called the definite integral of $f$ from $a$ to $b$ and is denoted by

\[
\int_{a}^{b} f(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta t.
\]
Just as in Lesson 1, the limit of the Riemann sums has a geometric interpretation: If $f(t) \geq 0$, then $\int_a^b f(t) \, dt$ is numerically equal to the “area” between the graph of $f$ and the $t$-axis for $a \leq t \leq b$. If $f(t)$ varies in sign, then $\int_a^b f(t) \, dt$ is the signed area between the graph of $f$ and the $t$-axis for $a \leq t \leq b$. Signed area means that “areas” above the $t$-axis are counted as positive and “areas” below are counted as negative. If $f(t)$ is the velocity of an object at time $t$, then $\int_a^b f(t) \, dt$ is the net displacement of the object over the time interval $a \leq t \leq b$. 
Lesson 3: First Fundamental Theorem

The bottom line of Lesson 2 is this: If the rate of accumulation \( f \) of a process is known, then the total accumulation of \( f \) over an interval \( a \leq t \leq b \) is the definite integral of \( f \) from \( a \) to \( b \):

\[
\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta t.
\]

If \( f(t) \) is the velocity of a moving object at time \( t \), then \( \int_a^b f(t) \, dt \) is the displacement of the object over the time interval \( a \leq t \leq b \). If \( f(t) \) is the rate of change with respect to time of biomass of an elk population at time \( t \), then \( \int_a^b f(t) \, dt \) is net change in biomass of the population over the time interval \( a \leq t \leq b \).

By now you know that Riemann sums are tedious to calculate, even for small \( n \). Harder still is a direct evaluation of the limit that gives the definite integral, \( \int_a^b f(t) \, dt \). What you need is an effective means for evaluating definite integrals. Newton and Leibniz discovered such a means, the first fundamental theorem of calculus.

The fundamental theorem is easy to understand in terms of motion. If \( F(t) \) is the position of a moving object at time \( t \), then its velocity \( f(t) \) at time \( t \) is

\[
f(t) = F'(t)
\]

or in Leibniz notation \( f = \frac{dF}{dt} \). On the one hand, the net displacement of the object from time \( a \) to time \( b \) is \( \int_a^b f(t) \, dt \). On the other hand, it is \( F(b) - F(a) \). (Why?) So

**FTC I:** \( \int_a^b f(t) \, dt = F(b) - F(a) \) where \( F'(t) = f(t) \).

Here is an important observation: Given any function \( f \) and another function \( F \) such that \( F' = f \), you can think of \( F \) as the position of an object. Then \( f \) is its velocity. So FTC I is true regardless of the actual physical meanings of \( F \) and \( f \) so long as \( F' = f \). This is the content of the fundamental theorem of calculus. Here is another important point: The letter used to denote the independent variable can be replaced by any other convenient letter. So FTC I can also be expressed as

\[
\int_a^b f(x) \, dx = F(b) - F(a) \text{ where } F'(x) = f(x).
\]

**Example:** \( \int_0^{\pi/2} \cos \theta \, d\theta = \sin \frac{\pi}{2} - \sin 0 = 1 \) because \( \frac{d}{d\theta} \sin \theta = \cos \theta \).

If you are not impressed try to evaluate \( \int_0^{\pi/2} \cos \theta \, d\theta \) directly as a limit of Riemann sums!
Lesson 4: Properties of Integrals

Each of the following properties of the definite integral has a simple algebraic and geometric justification. You should strive to understand both. Each property holds if the functions \( f(x) \) and \( g(x) \) that appear are piecewise continuous over the intervals of integration. So the rules are applicable to any functions you are likely to meet in this course.

1. \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \) for any \( a \) and \( b \)

2. \( \int_c^a f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \) for any \( a, b, \) and \( c \)

3. \((The Integral is Linear)\n\[ \int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \]
\[ \int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx \) for any constant \( c \)

4. \((The Integral is Monotone)\nIf \( f(x) \leq g(x) \) for \( a \leq x \leq b \), then \( \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \)

The first property results from a natural extension of the (original) definition,

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \frac{b-a}{n},
\]

where it was assumed that the lower limit was less than the upper limit of integration; that is, \( a < b \) so that \( \Delta x > 0 \). The definition makes perfectly good sense and is meaningful in applications also when the lower limit is greater than the upper limit, in which case \( \Delta x < 0 \). Then

\[
\int_b^a f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \frac{a-b}{n} = -\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \frac{b-a}{n} = -\int_a^b f(x) \, dx
\]

which is the first property. It is natural to define \( \int_a^a f(x) \, dx = 0 \). (Why?) Then \( \int_a^b f(x) \, dx \) is defined for any choice of limits \( a \) and \( b \).

Symmetry, especially even and odd symmetry, is very useful in evaluating certain integrals. Each of the results that follow have simple area interpretations. (What are they?)

1. \((Even Symmetry)\) The graph of \( y = f(x) \) for \( -a \leq x \leq a \) has even symmetry if the part of the graph to the left of the \( y \)-axis is the mirror image of the part to the right of the \( y \)-axis; equivalently, \( f(-x) = f(x) \) for all \( x \) in \([-a,a]\) and we call \( f \) an even function. If \( f \) is even on \(-a \leq x \leq a\), then

\[
\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx.
\]

2. \((Odd Symmetry)\) The graph of \( y = f(x) \) for \(-a \leq x \leq a\) has odd symmetry if the part of the graph to the left of the \( y \)-axis is the reflection through the origin of the part to the right of the \( y \)-axis; equivalently, \( f(-x) = -f(x) \) for all \( x \) in \([-a,a]\) and we call \( f \) an odd function. If \( f \) is odd on \(-a \leq x \leq a\), then

\[
\int_{-a}^a f(x) \, dx = 0.
\]
2 Constructing Antiderivatives

These lessons treat integration as the opposite of differentiation.
Lesson 5: Graphs and Tables

If a function $F$ has derivative $f$ on some interval, we call $F$ an antiderivative of $f$. Then $F'(x) = f(x)$ for all $x$ in the interval. With this language, the first fundamental theorem of calculus can be expressed as follows:

If $F$ is an antiderivative of a continuous function $f$ on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

So finding antiderivatives gives a convenient way to evaluate many integrals but that is a story for later this term.

In this lesson your goal is to better understand the relationship between a function and its antiderivative, especially as concerns graphs and tables. You may recall from differential calculus that you have already done this, but from a different point of view. There you started with the graph (or table) of a function, say $F$, and used the graph (or table) of $F$ to sketch the graph of (or tabulate) $F' = f$. Now you will turn the tables and start with the graph (or table) of a function $f$ and use this data to sketch the graph of (or tabulate) an antiderivative $F$ of $f$.

Key points you need to keep in mind as you do this are: If $F$ is an antiderivative of $f$, so that $F' = f$, then

- $F$ is increasing on any interval on which $f > 0$.
- $F$ is decreasing on any interval on which $f < 0$.
- $F$ has critical points when $f$ vanishes (crosses the $x$-axis).
- Concavity of $F$ can be inferred from $f'$, assuming this derivative exists. (How?)
- The signed area between the graph of $f$ and the $x$-axis for $a \leq x \leq b$ is $F(b) - F(a)$. (Why?) So if you know $F(a)$ and the signed area you can find $F(b)$. 
Lesson 6: Antiderivatives

Since $F$ is an antiderivative of $f$ if $F' = f$, the process of antidifferentiation, finding an antiderivative of a given function, amounts to doing differentiation backwards. Consequently, every differentiation rule you know when read backwards turns into a corresponding antidifferentiation rule. There is really nothing new to learn except for some new notation.

The differentiation rule
\[
\frac{d}{dx} \sin x = \cos x
\]
tells you that an antiderivative of $\cos x$ is $\sin x$. Likewise,
\[
\frac{d}{dx} (\sin x + 7) = \cos x
\]
tells you that $\sin x + 7$ is also an antiderivative of $\cos x$. This makes good sense graphically. The graph of $\sin x + 7$ is the translation of the graph of $\sin x$ vertically upward by 7 units. The two graphs clearly have the same slope at corresponding points. That (same) slope is the derivative, $\cos x$, for each graph.

Since
\[
\frac{d}{dx} (\sin x + C) = \cos x
\]
for any choice of the constant $C$, $\cos x$ has infinitely many antiderivatives. In this context, $C$ is usually referred to as an arbitrary constant. All antiderivatives of $\cos x$ must be vertical shifts of the graph of $\sin x$. (Why?) The same reasoning applies generally:

If the function $f(x)$ has a particular antiderivative $F(x)$, then the most general antiderivative of $f(x)$ is $F(x) + C$, where $C$ is an arbitrary constant.

Another way to express this is:

Any two antiderivatives of $f(x)$ differ by a constant.

The collection of all antiderivatives of $f(x)$ is denoted by $\int f(x) \, dx$, which is called an indefinite integral: If $F(x)$ is a particular antiderivative of $f(x)$, then
\[
\int f(x) \, dx = F(x) + C
\]
where $C$ is an arbitrary constant.

Example: $\int \cos \theta \, d\theta = \sin \theta + C$.

Example: $\int_{0}^{\pi/2} \cos \theta \, d\theta = \sin \frac{\pi}{2} - \sin 0 = 1$ because $\sin \theta$ is an antiderivative of $\cos \theta$.

IMPORTANT: $\int_{a}^{b} f(x) \, dx$ is a number. $\int f(x) \, dx$ is a family of functions.
Differentiation is a linear process:
\[
\frac{d}{dx} \left( f(x) + g(x) \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]
\[
\frac{d}{dx} (kf(x)) = k \frac{d}{dx} f(x)
\]
Consequently, antidifferentiation (differentiation done backwards) is also a linear process:
\[
\int \left( f(x) + g(x) \right) \, dx = \int f(x) \, dx + \int g(x) \, dx
\]
\[
\int kf(x) \, dx = k \int f(x) \, dx
\]
Here are a few familiar differentiation rules expressed as indefinite integrals:
\[
\frac{d}{du} \left( u^{n+1} \right) = (n + 1)u^n \quad \iff \quad \int u^n \, du = \frac{u^{n+1}}{n + 1} + C \quad (n \neq -1)
\]
\[
\frac{d}{du} e^u = e^u \quad \iff \quad \int e^u \, du = e^u + C
\]
\[
\frac{d}{du} \sin u = \cos u \quad \iff \quad \int \cos u \, du = \sin u + C
\]
\[
\frac{d}{du} \cos u = -\sin u \quad \iff \quad \int \sin u \, du = -\cos u + C
\]
\[
\frac{d}{du} \ln u = \frac{1}{u} \quad \iff \quad \int \frac{1}{u} \, du = \ln u + C \quad (u > 0)
\]
A nice summary of both sides of this table is obtained by rewriting it using differentials:
\[
d \left( u^{n+1} \right) = (n + 1)u^n \, du
\]
\[
d (e^u) = e^u \, du
\]
\[
d(\sin u) = \cos u \, du
\]
\[
d(\cos u) = -\sin u \, du
\]
\[
d(\ln u) = \frac{1}{u} \, du
\]
To obtain the derivative rules, “divide” both sides by \(du\); to obtain the integral rules, integrate both sides and read the equality from right to left, noting that FTC I says
\[
\int df = f + C
\]
Use of basic antiderivatives like those in the table above (and others coming later) will enable you to find many antiderivatives easily, virtually by inspection.

Example: For \(u > 0\)
\[
\int \left( 3e^u - 2\cos u + \frac{1}{u} \right) \, du = 3 \int e^u \, du - 2 \int \cos u \, du + \int \frac{1}{u} \, du
\]
\[
= 3e^u - 2\sin u + \ln u + C
\]
Finally, note that since
\[ d\left(\ln(-u)\right) = \frac{1}{(-u)} d(-u) = \frac{1}{u} du \]
the last differential identity holds for \( u < 0 \) as well as for \( u > 0 \). Thus, the last rule in the table can be extended to
\[ \int \frac{1}{u} du = \ln |u| + C \]
which holds both for \( u > 0 \) and for \( u < 0 \).
Lesson 7: Differential Equations

A differential equation (DE) is just an equation for an unknown function that involves one or more of its derivatives.

Differential equations occur in many mathematical models of physical, biological, economic, etc. processes because what is known experimentally or from exact laws is often the rate at which the processes change. In mathematical terms, we are given the rate of change of a function and seek to recover the function from its rate of change. So, we have an antidifferentiation problem! A differential equation gives the rate of change of a process but you need side conditions, such as initial conditions or boundary conditions, to nail down the process exactly.

Example: (A motion problem, of course!) You throw a ball straight up in the air. The essential conclusion of Galileo’s inclined plane experiments is that the rate of change of the velocity \( v = v(t) \) of the ball (its acceleration) is constant. Nowadays, we denote that constant acceleration near the surface of the earth by \( g \). (Why \( g \)?) So, the velocity \( v \) changes in time according to the DE

\[
\frac{dv}{dt} = g
\]

For the record

\[
g = -9.8 \text{ m/s}^2 \quad \text{in the mks-system, and} \\
g = -32 \text{ ft/s}^2 \quad \text{in the English system},
\]

where the minus sign indicates that the acceleration is downward.

Why doesn’t the DE

\[
\frac{dv}{dt} = g
\]

completely determine the ball’s velocity? The DE tells how the velocity changes but this rate of change is the same for a ball thrown straight up at 2m/s as it is for a ball thrown up at 200m/s. Apparently, the velocity of the ball is quite different for these two different initial conditions.

The motion of the first ball is determined by the initial value problem (IVP)

\[
\frac{dv}{dt} = g \\
v(0) = 2
\]

consisting of the DE together with the Initial Condition(s), where we start measuring time \( t \) in seconds at the instant the ball is thrown. What IVP determines the motion of the second ball?

You will meet other differential equations and initial value problems as we go along. In general, if \( y \) represents an unknown function whose rate of change, say \( f(x) \) is known, then \( y \) satisfies the DE

\[
\frac{dy}{dx} = f(x).
\]
If you know a particular antiderivative $F(x)$ of $f(x)$, then $y = F(x) + C$ for some constant $C$ because $y$ is one of the antiderivatives of $f$. You will need a side condition to determine $C$. If there are no side conditions, then $C$ remains arbitrary and $y = F(x) + C$ is the general solution to the DE.
Lesson 8: Second Fundamental Theorem

It turns out that finding antiderivatives is more challenging than may first meet the eye. Even relatively simple functions don’t have elementary antiderivatives. For example, you know that \( \cos \theta \) has antiderivative \( \sin \theta \). But change the problem a little and ask for an antiderivative for \( \cos(\theta^2) \) and the search is much harder! In fact there is no “elementary” function that is an antiderivative of \( \cos(\theta^2) \). An elementary function is one that can be expressed as an algebraic combination of any of the functions you have met in calculus to date (plus a few other functions you may not yet know).

Given this you might well ask does \( \cos(\theta^2) \) even have an antiderivative. The answer is yes and that is part of the second fundamental theorem of calculus:

Any function that is continuous on an interval has an antiderivative on that interval.

Assuming this much of the FTC II, we know that \( \cos(u^2) \) has an antiderivative \( F(u) \); let’s see what \( F(u) \) might be. We can assume \( F(0) = 0 \). (Why?) Then by FTC I,

\[
\int_0^\theta \cos(u^2) \, du = F(\theta) - F(0) = F(\theta).
\]

In this formula, \( \theta \) is regarded as a constant, but the formula holds for any value of \( \theta \). So one antiderivative of \( \cos(u^2) \) is

\[
F(\theta) = \int_0^\theta \cos(u^2) \, du.
\]

where the upper limit \( \theta \) is now regarded as variable. So geometrically speaking \( F(\theta) \) is the signed area between the graph of \( \cos(u^2) \) and the \( u \)-axis for \( 0 \leq u \leq \theta \). (What is the geometric interpretation when \( \theta \leq u \leq 0 \)?) The formula for \( F(\theta) \) may look imposing, but don’t be intimidated. For each upper limit \( \theta \) there is a definite signed area \( F(\theta) \), and the signed area function is an antiderivative of the integrand \( \cos(u^2) \):

\[
\frac{d}{d\theta} \int_0^\theta \cos(u^2) \, du = \cos(\theta^2).
\]

Exactly the same reasoning can be repeated with \( \cos(u^2) \) replaced by any continuous function \( f(u) \). Thus

**FTC II:** \( \frac{d}{dx} \int_0^x f(u) \, du = f(x) \)

on any interval on which \( f(u) \) is continuous. In other words,

\[
F(x) = \int_0^x f(u) \, du
\]

is an antiderivative of \( f(x) \).

In words, in FTC II says:

The derivative of a definite integral with respect to its upper limit equals the integrand evaluated at the upper limit.
Just for the record, if \( F(\theta) = \int_0^\theta \cos(u^2) \, du \), then

\[
\begin{align*}
F(0) &= 0, \\
F'(\theta) &= \cos(\theta^2) \quad (\text{FTCII}), \\
F''(\theta) &= -2\theta \sin(\theta^2),
\end{align*}
\]

and you can use this information to sketch the graph of \( F(\theta) \), say over \( 0 \leq \theta \leq \sqrt{\pi} \). (How do you use the information to sketch the graph?) You can also use Riemann sums to approximate \( F(\theta) \) for various values of \( \theta \) and thereby improve the accuracy of the sketch.
3 Integration Techniques

These lessons develop the basic tools needed to compute many integrals. Some of the rules involve several steps of algebra, but don’t lose sight of the fact that calculus is done in context! Apply these rules to story problems! Compare the graph of the function you are integrating with the graph of the (indefinite) integral! Your instructor may use such techniques to motivate some of the rules, but you are strongly encouraged to do this on your own as well.
Lesson 9: Substitution

One of the most useful methods of evaluating integrals involves making a substitution in the integral that (hopefully) simplifies the task of antidifferentiation. Substitutions that do this are often motivated by educated guesses about what the antiderivative might be. With a little practice, you can often do the substitution mentally and find antiderivatives virtually by inspection. When the going gets tough, substitutions can help you discover antiderivatives that are not at all apparent.

Here are two antidifferentiation problems:

(a) \( \int \cos(\theta^2) \, d\theta = ? \)

(b) \( \int 2\theta \cos(\theta^2) \, d\theta = ? \)

Hopefully you recognize the integral in (a). There is no elementary antiderivative for this integrand — see Lesson 8. The integral in (b) is easy to evaluate — by educated guessing. Does the integrand look familiar from differential calculus? Hopefully you said yes. It is just a chain-rule derivative:

\[
\frac{d}{d\theta} \sin(\theta^2) = \cos(\theta^2) \frac{d}{d\theta}(\theta^2) = 2\theta \cos(\theta^2).
\]

Hence,

\[
\int 2\theta \cos(\theta^2) \, d\theta = \sin(\theta^2) + C.
\]

A key observation related to guessing the antiderivative of \( \cos(\theta^2) \) was the appearance of \( \theta^2 \) in the composite function \( \cos(\theta^2) \) and also the appearance of its derivative, \( 2\theta \), in the integrand.

Let’s try a related problem and use a substitution to assist with antidifferentiation:

(c) \( \int \theta \cos(\theta^2) \, d\theta = ? \)

Now the integrand isn’t quite a chain-rule derivative but it is except for a constant factor. The appearance of \( \theta^2 \) in the composite function \( \cos(\theta^2) \) and of the factor \( \theta \) proportional to the derivative of \( \theta^2 \) suggest the substitution (AKA change of variable)

\[
w = \theta^2.
\]

Then \( dw = 2\theta \, d\theta \) and the integral expressed in terms of the new variable is

\[
\int \theta \cos(\theta^2) \, d\theta = \int \cos(\theta^2) \, \theta \, d\theta = \int \cos w \left( \frac{1}{2} \, dw \right)
= \frac{1}{2} \int \cos w \, dw = \frac{1}{2} \sin w + C = \frac{1}{2} \sin(\theta^2) + C
\]

**IMPORTANT:** Did you notice a subtle change of point of view? It is best to think of the integrand as a differential, here \( \theta \cos(\theta^2) \, d\theta \), which the substitution enables you to express in terms of a differential of a function of \( w \), here \( \cos w \, dw \).
Effective use of educated guessing and substitutions takes practice! You will need to solve lots of antidifferentiation problems to become adept at it — almost certainly more problems than are assigned in the suggested problems.

**Skills Test:** Later in the term you will take a Skills Test over techniques of integration. This would be a good time to ask your instructor about how that test will be graded in your class and what will be expected of you. You will not be able to use any technology during the skills test.

Substitutions can also be used with definite integrals. Just remember that you must change the limits of integration according to the substitution.

**Example:** Evaluate \( \int_{0}^{4} \frac{x}{1 + x^2} \, dx \).

**Solution:** Since the differential of \( 1 + x^2 \) is proportional to \( x \, dx \), the substitution

\[
w = 1 + x^2
\]

is suggested. Then \( dw = 2x \, dx \) and

\[
\int_{0}^{4} \frac{x}{1 + x^2} \, dx = \int_{x=0}^{x=4} \frac{x}{1 + x^2} \, dx = \int_{w=1+0^2}^{w=1+4^2} \frac{1}{w} \, \left( \frac{1}{2} \, dw \right) = \frac{1}{2} \ln w \bigg|_{1}^{17} = \frac{1}{2} \ln 17.
\]

Substitutions are also useful in contexts other than the “backward” chain rule integrands evaluated above. For example the integral \( \int x \sqrt{x + 1} \, dx \) yields to the rationalizing substitution \( u^2 = x + 1 \) where \( 2w \, dw = dx \). Try it! Then see if you can explain what motivated the substitution.

**Advice:** Guessing an effective substitution is not always easy. Don’t be afraid to experiment. If you make a substitution and the new integrand looks worse, then try something else. What would be an effective substitution for \( \int \frac{x^2}{1 + x^4} \, dx \)?
Lesson 10: Integration by Parts

Integration by parts can be viewed as a “backward” product rule. It again turns out to be useful to work with differentials, just as for substitutions. First notice that

$$\int dw = w + C.$$ 

The product rule in differential form is

$$d(uv) = u \, dv + v \, du$$

where \(u\) and \(v\) are differentiable functions. Integrate to obtain

$$\int d(uv) = \int u \, dv + \int v \, du,$$

$$uv = \int u \, dv + \int v \, du.$$ 

No constant of integration is needed on the left because it is accounted for by the indefinite integrals on the right. (Why?)

Thus

$$\int u \, dv = uv - \int v \, du$$

which is the formula for **integration by parts**. You may well ask why this is a useful formula for integration. The answer is that many integrands that come up in practice are expressible as a product \(u \, dv\) for which \(v \, du\) is easier to antidifferentiate. Here is a basic hint about effective use of integration by parts:

To evaluate \(\int df\) try to express \(df\) as \(u \, dv\) where \(du\) is simpler than \(u\) and \(\int dv\) is an easy integral to evaluate (or at least easier to evaluate than \(\int df\)).

**Example:** Evaluate \(\int p^2 \ln p \, dp\). There are two reasonable first guesses for \(u\), \(u = p^2\) and \(u = \ln p\). If you make the first choice, then \(dv = \ln p \, dp\). If you don’t know an easy antiderivative for \(\ln p\) this route loses some appeal. The second choice gives \(u = \ln p\) and \(dv = p^2 \, dp\). Then \(du = \frac{1}{p} \, dp\), \(v = \frac{p^3}{3}\), and

$$\int p^2 \ln p \, dp = \int u \, dv = uv - \int v \, du$$

$$= \frac{p^3}{3} \ln p - \int \frac{p^3}{3} \frac{1}{p} \, dp = \frac{p^3}{3} \ln p - \frac{1}{9} p^3 + C.$$ 

Can you think of a good choice of parts for \(\int \frac{x^2}{\sqrt{1-x^2}} \, dx\)?

Integration by parts also works for definite integrals. Just take the definite integral of \(d(uv) = u \, dv + v \, du\) to get

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$
Lesson 11: Integral Tables

Since integrals can be challenging to evaluate, much time can sometimes be saved by taking advantage of antiderivatives that have already been discovered. Such antiderivatives can be found in integral tables. An integral table is a carefully organized collection of indefinite integrals and sometimes a few definite integrals.

The table contains standard forms, which are template integrals that you must match to integrals you seek to evaluate.

- Matching is sometimes easy — your integral appears exactly as an entry in the table.

- Matching is sometimes harder — you may need to apply some algebraic manipulations (such as long division, factoring, completing the square, etc.) to put your integral in a form that matches a table entry.

- Matching is sometimes challenging — you may need to apply one or more changes of variable and/or integration by parts or another techniques of integration in order to transform your integral into one that appears in the table.

A short table of integrals sufficient for the purposes of this course can be found on the inside covers of most calculus texts.

A computer algebra system (CAS) that does symbolic (exact) integration can be thought of as an electronic integral table.

- Its main advantage over an integral table is that it attempts to do the table lookup for you.

- Its main disadvantage is that it may report an antiderivative that is correct but in such a cumbersome form as to be nearly useless for many purposes.

Many graphing calculators include a modest CAS that will enable you to evaluate some integrals. Much more powerful CAS integrators can be found in the software packages Maple and Mathematica. A link to the Mathematica online integrator is:

http://www.integrals.com

Follow this link and evaluate a few integrals of your choice or Mathematica’s choice. You may find the link to How to Enter Input helpful.
Lesson 12: Algebraic Identities and Trig Substitutions

Partial Fractions

The integral
\[ I = \int \frac{2x^3 - 5x^2 + 6x - 5}{x^2 - 3x + 2} \, dx \]
looks imposing but an algebraic identity gives
\[ \frac{2x^3 - 5x^2 + 6x - 5}{x^2 - 3x + 2} = 2x + 1 + \frac{2}{x - 1} + \frac{3}{x - 2} \quad (1) \]
Now integration is easy, by inspection!

\[ I = \int \left( 2x + 1 + \frac{2}{x - 1} + \frac{3}{x - 2} \right) \, dx = x^2 + x + 2 \ln |x - 1| + 3 \ln |x - 2| + C \]

The identity (1) is a particular partial fractions expansion. Any rational function has a partial fractions expansion. The expansion expresses the rational function as a sum of terms each of which can be integrated explicitly by known techniques. Thus, use of partial fractions enables you to integrate any rational function. Incidentally, partial fractions have important applications in other areas as well but that is a story for another day.

The form of a partial fractions expansion is determined by the zeros (roots) of its denominator. The form is particularly easy to write down when the denominator of the rational function factors into simple linear factors. The resulting integration of the rational function is easy, much as in the example above. You will concentrate mainly on this case in this lesson. The rest of the story, again mostly for another day, involves denominators that have irreducible quadratic factors and repeated roots.

Trigonometric Substitutions

Many integrals that come up in applications involve either a sum or difference of squares raised to a power which is an integer multiple of 1/2. This happens because the integrand involves some distance(s) and the Pythagorean theorem expresses the distance between two points as \( c = \sqrt{a^2 + b^2} \).

Trigonometric substitutions are often chosen to transform such an integrand into one you can integrate. The strategy is to choose the trigonometric substitution so a sum or difference of squares that appears under a radical sign becomes a perfect square. Such substitutions are based on the Pythagorean theorem in its trigonometric form \( \sin^2 \theta + \cos^2 \theta = 1 \) or one of its algebraic variants. Two frequently used trigonometric substitutions follow:

The Sine Substitution:

If the integrand involves \( \sqrt{a^2 - x^2} \), perhaps raised to some power, let \( x = a \sin \theta \), or equivalently \( \theta = \arcsin \left( \frac{x}{a} \right) \)
The substitution is easier to carry out if you draw a right triangle with acute angle $\theta$, opposite side $x$, and hypotenuse $a$.

The Tangent Substitution:

If the integrand involves $\sqrt{x^2 + a^2}$, perhaps raised to some power, let $x = a \tan \theta$, or equivalently let $\theta = \arctan \left( \frac{x}{a} \right)$.

Again, drawing the corresponding right triangle can be helpful.

Leveraging What You Know: Every technique or trick of the trade you learn in one context may be just as useful in another. Consider the following examples:

$$(c) \int \frac{1}{x^2 - 4x + 7} \, dx = ?$$

$$(d) \int \frac{1}{\sqrt{7 + 4x - x^2}} \, dx = ?$$

What do you know that would turn the quadratic terms in the integrals above into a sum or difference of squares, thus making them amenable to a trigonometric substitution?
Lesson 13: Improper Integrals

Improper integrals extend the idea of proper Riemann integrals — the integrals you’ve studied so far — to the case in which the integrand and/or limits of integration become unbounded. Many applications of integration lead to such integrals. In fact, it is fair to say that some of the most significant applications of calculus to problems in mathematics, science, and engineering involve improper integrals.

There are several types of improper integrals, all handled in essentially the same way, as limits of proper integrals. Here we concentrate on one type of improper integral — that in which the integrand is continuous and the interval of integration is of infinite length, from a finite lower limit to infinity. Here are five such integrals:

\[ \int_{1}^{\infty} \frac{1}{x} \, dx \quad \int_{1}^{\infty} \frac{1}{x^2} \, dx \quad \int_{0}^{\infty} \cos \theta \, d\theta \quad \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx \quad \int_{0}^{\infty} e^{-u} \, du \]

Simply use what you know to give a natural definition for such integrals: Let \( a \) be a fixed real number. By definition

\[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \quad \text{provided the limit exists, finite or infinite.} \]

If the limit does not exist, no value is assigned to \( \int_{a}^{\infty} f(x) \, dx \). For example,

\[ \int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx = \lim_{b \to \infty} \ln b = \infty \]

\[ \int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_{1}^{b} = 1 \]

\[ \int_{0}^{\infty} \cos \theta \, d\theta \quad \text{has no value} \]

because \( \int_{0}^{b} \cos x \, dx = [\sin x]_{0}^{b} = \sin b \) does not have a limit as \( b \to \infty \).

In practice, improper integrals are often computed without performing an explicit limit computation. For example,

\[ \int_{0}^{\infty} e^{-u} \, du = \left[ -e^{-u} \right]_{0}^{\infty} = 1 \]

since “\( e^{-\infty} = 1/e^{\infty} = 1/\infty = 0 \)”.

We say the improper integral converges if \( \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \) exists and is finite. The improper integral diverges (to \( \pm \infty \)) if \( \int_{a}^{\infty} f(x) \, dx = \pm \infty \). We say the improper integral \( \int_{a}^{\infty} f(x) \, dx \) does not exist if \( \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \) has no value (finite or infinite).

It is apparent from the definition that whenever \( \int_{a}^{\infty} f(x) \, dx \) has a value (finite or infinite) that value is the signed area between the graph of \( f(x) \) and the \( x \)-axis for \( a \leq x < \infty \). But even though an integral can always be interpreted as a signed area under a curve, that may not be the interpretation you want in a particular context. For example, if \( f(x) \) is the gravitational force exerted on a space shuttle when it is \( x \) miles above the center of the earth, then \( \int_{R}^{\infty} f(x) \, dx \), where \( R \) is the radius of the earth, is the work required to move the shuttle infinitely far from the earth; this is the least amount of energy required to free the shuttle from the gravitational field of the earth, that is, to put it in an unbound orbit.
4 Applications

These lessons show how to use the key idea of integral calculus — Riemann sum and limit passage reasoning — to solve a variety of interesting problems. In each lesson you will learn how to decompose a geometric figure or evolving physical process into many small slices, either spatially or temporally. You can use what you know (just as with $d = rt$ way back in Lesson 1) to “solve” the corresponding small slice problem. Adding results for the slices gives an approximate solution to the problem at hand. Finally, taking a limit as the number of slices becomes infinite gives the exact solution.
Lesson 14: Area and Volume

For the volume problems in this lesson and the next, use what you know means to take advantage of the fact that the volume $V$ of a right cylinder whose base has area $A$ and whose height is $h$ is $V = Ah$. (What is a right cylinder? If you don’t know look it up, perhaps online.)

The Riemann sum and limit passage reasoning led to the definite integral expressed as

$$
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x
$$

where the interval $a \leq x \leq b$ is subdivided into $n$ subintervals of length $\Delta x = (b-a)/n$ and $c_i$ is any conveniently chosen point in the $i$-th subinterval. To concentrate on the main idea when setting up applications of the Riemann sum and limit passage reasoning, it is useful to use the shorthand notation

$$
\int_{a}^{b} f(x) \, dx = \lim \sum f(x) \Delta x,
$$

where on the RHS $x$ stands for a conveniently chosen point in a typical subinterval.

Consider for example finding the volume of a hemisphere of radius 7 cm. The volume of a slice of the hemisphere (parallel to its base) between height $h$ and height $h + \Delta h$ is $\Delta V \approx \pi r^2 \Delta h$, leading to an approximate expression for the volume as

$$
V \approx \sum \pi r^2 \Delta h = \sum \pi (7^2 - h^2) \Delta h
$$

so that

$$
V = \int_{0}^{7} \pi r^2 \, dh = \int_{0}^{7} \pi (7^2 - h^2) \, dh
$$

In practice, one often abbreviates the limit process by working on a small slice of thickness “$dx$”, determining the integrand “$f \, dx$” on each such slice, and then “adding” the resulting pieces by integrating. From this point of view, integrals are sums, and the quantities being added are of the form “$f \, dx$”. From this point of view, the volume $dV$ of a slice of length $dh$ is just

$$
dV = \pi r^2 \, dh = \pi (7^2 - h^2) \, dh
$$

and one then proceeds by integrating both sides of this expression.

Yet another point of view, closely related to antiderivatives and the fundamental theorem of calculus, is illustrated by computing

$$
\frac{\Delta V}{\Delta h} \approx \pi r^2 = \pi (7^2 - h^2),
$$

so that the derivative of volume with respect to height is

$$
\frac{dV}{dh} = \pi (7^2 - h^2).
$$

Integrate this “rate of accumulation of volume” over the extent of the hemisphere to find

$$
V = \int_{0}^{7} \pi (7^2 - h^2) \, dh.
$$
Lesson 15: Applications to Geometry

This lesson simply continues with Riemann sum and limit passage reasoning to find more volumes and to find the length of a curve.

The problem of finding the length $l(C)$ of a curve $C$ in the $xy$-plane given by $y = f(x)$ for $a \leq x \leq b$ can be thought of in two ways using Riemann sum and limit passage reasoning. First, imagine the curve sliced into many short pieces laid end-to-end. Each piece is nearly straight and its length $\Delta s$ can be calculated using the Pythagorean theorem,

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x,$$

$$l(C) = \lim_{\Delta x \to 0} \sum \Delta s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

Second, imagine that a curve $C$ in the $xy$-plane is the path of a moving object. The position $(x, y)$ of the object as it moves along the curve (its trajectory) is a function of time $t$. So $x$ and $y$ are functions of $t$, say for $t_1 \leq t \leq t_2$. The speed (time rate of change of distance traveled) of the moving object at time $t$ is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and the distance traveled (AKA the length of $C$) is the integral of speed (good old $d = rt$ together with Riemann sum and limit passage reasoning). So

$$l(C) = \text{distance traveled} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Such integrals are notoriously difficult to evaluate exactly, and in many cases you cannot find an elementary antiderivative for the integrand. However, such integrals can be evaluated as accurately as desired by numerical methods.
Lesson 16: Density and Center of Mass

Density

Density measures the amount of a quantity (population, weight, mass, traffic, ...) per some natural unit (square mile, meter, square centimeter, mile, ...). For example, population density might be measured in people per square mile, the weight density of an I-beam might be measured in pounds per foot or newtons per meter, the mass density of a meteorite might be measured in kilograms per cubic meter, the mass density of a blueberry pancake (or other laminar object) might be given in grams per square centimeter, and the traffic on I–5 might be measured in vehicles per mile.

If the density of a quantity is constant, the total amount of the quantity is just the product of its density times the number of natural units present. (It’s $d = rt$ again, with $d$ being the total amount of the quantity, $r$ the density, and $t$ the number of natural units.)

If density is variable, then $d = rt$ does not apply globally but (assuming the density varies continuously) it almost does locally, and with increasing accuracy as the local dimensions shrink. Evidently calculating population, weight, mass, traffic ... when density is variable is another problem tailor made for Riemann sum and limit passage reasoning. Such reasoning leads to the conclusion that the total quantity present is the integral of density over the physical extent of the quantity. For example, if $\delta(x)$ is the density of vehicles per mile along I–5 between mile posts $a$ and $b$, then the total number of vehicles between the two mile posts is $\int_a^b \delta(x) \, dx$.

Center of Mass

Center of mass is a key concept in physics related to balance points of objects, and more fundamentally to extending Newton’s laws of motion from idealized point masses to real extended bodies. You probably have thrown a stick into the air, or a handful of small rocks. You see the stick twist and turn and the rocks fly away in an expanding cloud. Your eye senses that the sailing stick and rock cloud follows a well determined arc through the air. You are seeing the path followed by the center of mass of the stick and rock cloud.

Determining balance points goes back to Archimedes when he discovered the principle of the lever: If masses (or weights) $m_1$ and $m_2$ are placed on a homogeneous seesaw at distances $d_1$ and $d_2$ from a fulcrum (pivot), then the masses (weights) will balance when

$$m_1d_1 = m_2d_2$$

In this context, the product of a mass times a distance is called a moment of mass. If an $x$-coordinate system is set up along the seesaw and $m_1$, $m_2$, and the fulcrum have coordinates $x_1$, $x_2$, and $\bar{x}$ respectively, with say $x_1 < \bar{x} < x_2$, then

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

and

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$
If there are more masses on the seesaw, then experiments reveal that
\[
\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.
\]

For a lamina (pancake) in the xy-plane there is also a y-coordinate of the balance point:
\[
\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}.
\]

For a three-dimensional solid there is also a z-coordinate of the balance point. (What is it?)

The balance point \(\bar{x}\) (or \((\bar{x}, \bar{y})\) or \((\bar{x}, \bar{y}, \bar{z})\)) is also called the center of mass of the object. That is the story for point masses. What about real extended masses? You can use what you know. Break up the extended mass into many tiny chunks, regard them as point masses, calculate the center of mass of the system of point masses and pass to the limit to find the mass of the extended body. For a one dimensional object (say a rod) modeled as the segment of the x-axis from \(a \leq x \leq b\) and density \(\delta(x)\), this Riemann sum and limit passage reasoning leads to
\[
\bar{x} = \frac{x\text{-moment of mass}}{\text{mass}} = \frac{\int_a^b x \delta(x) \, dx}{\int_a^b \delta(x) \, dx}.
\]

There are corresponding formulas for \(\bar{y}\) and \(\bar{z}\) in the two- and three-dimensional cases, but the higher dimensional cases are mostly a story for multivariable calculus.

\textit{In all of these examples, a useful strategy is to check whether the units in your solution agree with those in your integral (including the differential!), and whether those units are appropriate for the question asked.}
Lesson 17: Applications to Physics

This lesson uses Riemann sum and limit passage reasoning to calculate the work done by a force and the total force of a liquid on a surface — the force of water on a dam for example.

Suppose you lift a pail with 1 cubic foot of water 20 feet above your car and drop it onto the hood. You are not too happy. The pail of water has stored energy that was turned into kinetic energy during the fall. How much energy did the pail of water contain? This is a story mostly for a course in physics but the short answer is that if a constant force \( F \) acts as it displaces an object through a displacement \( D \) along the line of action of the force, then the work done by the force is

\[
W = FD.
\]

Since the density of water is 62 pounds per cubic foot, the constant gravitational force acting on the pail is \( F = (1 \text{ ft}^3)(62.4 \text{ lbs/ft}^3) = 62.4 \text{ lbs} \), and it acts through a 20 foot displacement. So the work done is

\[
W = (62.4 \text{ lbs})(20 \text{ ft}) = 1,248 \text{ ft-lbs}.
\]

This is the (gravitational potential) energy that was stored in the pail suspended 20 feet above your car.

When the force \( F \) is variable (such as the force required to extend or compress a spring) but still acts along the line of action of the force, then \( W = FD \) no longer applies but the usual Riemann sum and limit passage reasoning does. That reasoning leads to the conclusion that the work done is

\[
W = \int_a^b F(x) \, dx
\]

where the line of action of the force \( F(x) \) is along the \( x \)-axis and the displacement is from position \( a \) to position \( b \).

Pressure is force per unit area. If pressure is constant over a given surface, then the total force on the surface is the product of the constant pressure and the area. When pressure is variable, Riemann sum and limit passage reasoning must be used to calculate the total force. A key experimental fact you need to know to solve pressure problems is:

The pressure at a given depth below the surface of a liquid is the same in all directions.

Suppose a liquid has constant density (mass/volume) \( \delta \). The weight of a column of that liquid directly over small horizontal rectangle of area \( A \) in the liquid at depth \( h \) is \( (\delta Ah)g \). (Why?) The pressure on the rectangle is \( (\delta Ah)g/A = \delta gh \). Conclusion

The pressure at depth \( h \) below the surface of a liquid with density \( \delta \) is \( \delta gh \).

This fact enables you to find the total force on a dam (or any other vertical surface) in a fluid — just split the surface into many thin horizontal strips. The force on the strip at depth \( h \) and with surface area \( \Delta A \) is then the product of the (approximately) constant pressure \( \delta gh \) and the area \( \Delta A \). So \( \Delta F = \delta gh \Delta A \). Now Riemann sum and limit passage reasoning will lead you to an integral for the total force on the dam (or whatever).
Officer Tommy Boy Hopkins and his evil brother Angry Mike are at it again. It seems that a hit and run accident took place the other night at midnight near Galloping Gulch, 40 miles from town. Farmer Bob claims that he witnessed the offending car traveling at approximately 60 miles per hour when his prize-winning Holstein, Bessie, was hit. Tommy Boy had figured his evil brother was up to no good that night and had planted a hidden camera in Angry Mike’s car before Angry Mike left town at 11:00 p.m. Unfortunately, the resolution was so bad that he could only read the speedometer and the dashboard clock on the night in question. Tommy Boy states that not only was Angry Mike out driving that night, but at the time in question, was traveling 60 miles per hour and his car seemed to hit something. In his defense, Mike says that he had not yet arrived at Galloping Gulch but that he does remember hitting a nasty bump around midnight.

Your goal is to figure out which brother is correct. The following questions will help you achieve this goal. You may assume that Angry Mike’s speed never decreased over the course of the entire trip.

<table>
<thead>
<tr>
<th>Time</th>
<th>11:00 pm</th>
<th>11:10 pm</th>
<th>11:20 pm</th>
<th>11:30 pm</th>
<th>11:40 pm</th>
<th>11:50 pm</th>
<th>Midnight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity, mph</td>
<td>0</td>
<td>30</td>
<td>40</td>
<td>45</td>
<td>50</td>
<td>53</td>
<td>60</td>
</tr>
</tbody>
</table>

Use only the data provided above to answer the following questions. Put all your work on a separate sheet of paper.

1. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

2. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

3. Now, assume that you are an investigating officer writing up a report of the incident. Your report should be a paragraph long and contain AT MINIMUM the following:
   - A description of the incident and the available evidence. Please do not copy the description above but rather rephrase in your own words.
   - A description of the calculations that you made in questions (1) and (2) above, including an explanation of how you know that they really do give the maximum and minimum distance traveled by Angry Mike’s car over the time in question.
   - A discussion of whether or not there is sufficient evidence to conclude that Angry Mike reached Galloping Gulch at or before midnight. Remember, use only the data in the above table to support any claims that you make.
Now, consider the following more complete list of data, also compiled that night. Continue to assume that Angry Mike’s speed never decreases over the entire one-hour period. Answer the questions again using the more complete data. As before, put all of your work on a separate sheet of paper.

<table>
<thead>
<tr>
<th>Time</th>
<th>11:00 pm</th>
<th>11:05 pm</th>
<th>11:10 pm</th>
<th>11:15 pm</th>
<th>11:20 pm</th>
<th>11:25 pm</th>
<th>11:30 pm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity, mph</td>
<td>0</td>
<td>25</td>
<td>30</td>
<td>38</td>
<td>40</td>
<td>43</td>
<td>45</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>11:35 pm</th>
<th>11:40 pm</th>
<th>11:45 pm</th>
<th>11:50 pm</th>
<th>11:55 pm</th>
<th>Midnight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity, mph</td>
<td>49</td>
<td>50</td>
<td>51</td>
<td>53</td>
<td>58</td>
<td>60</td>
</tr>
</tbody>
</table>

4. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

5. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

6. Write the same type of paragraph that you did for number 3, basing your conclusions on the more complete data. How do your conclusions change, if at all?

7. Do the following assuming Angry Mike’s velocity never decreases:
   (a) Sketch a graph of Angry Mike’s velocity vs. time that is consistent with the data table above and where Angry Mike moves as slowly as possible at each instant in time.
   (b) Sketch a graph of Angry Mike’s velocity vs. time that is consistent with the data table above and where Angry Mike moves as rapidly as possible at each instant in time.
   (c) What can you conclude about Angry Mike’s whereabouts given the table of data and the fact that Angry Mike’s velocity never decreases?
   (d) What can you conclude about Angry Mike’s whereabouts if it is NOT known that that his velocity never decreases?

8. **FOOD FOR THOUGHT**
   A car initially going 50 ft/sec brakes at a constant rate (constant negative acceleration), coming to a stop in 5 seconds.
   (a) Graph the velocity from $t = 0$ to $t = 5$.
   (b) How far does the car travel?
   (c) How far does the car travel if its initial velocity is doubled, but it brakes at the same constant rate?
MTH 252 — Lab 2
The Fundamental Theorem of Calculus

1. The graph of $f'(x)$ illustrated below is made up of line segments and two semicircles (of radius 1).

   ![Graph of f'(x)](image)

(a) Use the graph of $f'(x)$ and the Fundamental Theorem of Calculus to complete the table below. Give exact values for the missing data. And of course justify your work.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Now, plot the data in the table and sketch the graph of $f(x)$. Be sure your sketch is consistent with the information you can discern from the graph of $f'(x)$.

2. The figure below shows the graph of a function $f(x)$ that has a continuous third derivative. The dashed lines are tangent to the graph of $y = f(x)$ at the points $(0, 2)$ and $(4, 1)$.

![Graph of f'(x)](image)

Based on what is shown, determine, if possible, whether the following integrals are positive, negative, or zero. (Hint! This lab has a title.)

(a) $\int_{-4}^{0} f(x) \, dx$
(b) $\int_{-4}^{1} f'(x) \, dx$
(c) $\int_{-4}^{0} f''(x) \, dx$
(d) $\int_{-4}^{1} f'''(x) \, dx$
3. A bicyclist is pedaling along a straight road for one hour with a velocity \( v \) shown in the figure at the right. She starts out five kilometers from the lake and positive velocities take her toward the lake. [Note: The vertical lines on the graph are at 10 minute (1/6 hour) intervals.]
   (a) Does the cyclist ever turn around? If so, at what time(s)?
   (b) When is she going the fastest? How fast is she going then? Toward the lake or away?
   (c) When is she closest to the lake? Approximately how close to the lake does she get?
   (d) When is she farthest from the lake? Approximately how far from the lake is she then?

4. The graphs in the figures below represent the velocity, \( v \), of a particle moving along the \( x \)-axis for time \( 0 \leq t \leq 5 \). The vertical scales of all graphs are the same. Identify the graph showing which particle:
   (a) Has a constant acceleration.
   (b) Ends up farthest to the left of where it started.
   (c) Ends up the farthest from its starting point.
   (d) Experiences the greatest initial acceleration.
   (e) Has the greatest average velocity.
   (f) Has the greatest average acceleration.

5. For the even function \( f \) graphed at the right:
   (a) Suppose you know \( \int_{-2}^{2} f(x) \, dx \) and \( \int_{0}^{5} f(x) \, dx \). What is \( \int_{2}^{5} f(x) \, dx \)?
   (b) Suppose you know \( \int_{-2}^{5} f(x) \, dx \) and \( \int_{-2}^{0} f(x) \, dx \). What is \( \int_{5}^{2} f(x) \, dx \)?
   (c) Suppose you know \( \int_{2}^{5} f(x) \, dx \) and \( \int_{-2}^{5} f(x) \, dx \). What is \( \int_{0}^{2} f(x) \, dx \)?

6. **Food for thought!**
   How does letting \( f(-4) = 4 \) in Problem 1 change the table and the graph?
1. At the right is the graph of the rate \( r \) (in arrivals per hour) at which patrons arrive at the theater in order to get rush seats for the evening performance. The first people arrive at 8 am and the ticket windows open at 9 am. Suppose that once the windows open, people can be served at an (average) rate of 200 per hour.

Use the graph to find or provide an estimate of:
(a) The length of the line at 9 am when the windows open.
(b) The length of the line at 10 am.
(c) The length of the line at 11 am.
(d) The rate at which the line is growing in length at 10 am.
(e) The time at which the length of the line is maximum.
(f) The length of time a person who arrives at 9 am has to stand in line.
(g) The time at which the line disappears.
(h) Suppose you were given a formula for \( r \) in terms of \( t \). Explain how you would answer the above questions.

2. The rate at which the world’s oil is being consumed is continuously increasing. Suppose the rate of oil consumption (in billions of barrels per year) is given by the function \( r = f(t) \), where \( t \) is measured in years and \( t = 0 \) is the start of 1990.

(a) Write a definite integral which represents the total quantity of oil used between the start of 1990 and the start of 1995.
(b) Suppose \( r = 32e^{0.05t} \). Using a left-hand sum with five subdivisions, find an approximate value for the total quantity of oil used between the start of 1990 and the start of 1995.
(c) Interpret each of the five terms in the sum from part (b) in terms of oil consumption.

3. A bar of metal is cooling from 1000°C to room temperature, 20°C. The temperature, \( H \), of the bar \( t \) minutes after it starts cooling is given, in °C, by

\[
H = 20 + 980e^{-0.1t}.
\]

(a) Find the temperature of the bar at the end of one hour.
(b) Find the average value of the temperature over the first hour.
(c) Is your answer to part (b) greater or smaller than the average of the temperatures at the beginning and the end of the hour? Explain this in terms of the concavity of the graph of \( H \).

---

MTH 252 — Lab 3
Graphs of Antiderivatives

1. Let \( f'(x) = 3 - |2x| \). Sketch the following graphs clearly and accurately and label all important points.
   (a) On one set of axes, sketch a graph of \( f'(x) \) and \( f''(x) \).
   (b) On another set of axes, sketch possible graphs of \( f(x) \), one having \( f(0) = -2 \) and one having \( f(0) = 0 \).

2. Find derivatives and antiderivatives of the following.
   (a) \( e^x \)  
   (b) \( \sin x \)  
   (c) \( x^5 \)  
   (d) \( 5x^4 \)  
   (e) \( 3e^{2x} \)  
   (f) \( \cos 3x + \sin 4x \)  
   (g) \( e^{-x} - \frac{x^3}{x^2} + \frac{5}{\cos^2 2x} \)

3. Explain the difference between an antiderivative and the antiderivative.

4. The Quabbin Reservoir in the western part of Massachusetts provides most of Boston’s water. The graph below represents the flow of water in and out of the Quabbin Reservoir throughout 1993.
   (a) Sketch a possible graph for the quantity of water in the reservoir, as a function of time.
   (b) When, in the course of 1993, was the quantity of water in the reservoir largest? Smallest? Mark and label these points on the graph you drew in part (a).
   (c) When was the quantity of water increasing most rapidly? Decreasing most rapidly? Mark and label these times on both graphs.
   (d) By July 1994 the quantity of water in the reservoir was about the same as in January 1993. Draw plausible graphs for the flow into and the flow out of the reservoir for the first half of 1994. Explain your graphs.

5. The graphs of three functions are given in the figure at the right. Determine which is \( f \), which is \( f' \), and which is \( \int_0^x f(t) \, dt \). Explain your answer.
6. The acceleration, \( a \), of a particle as a function of time is shown in the figure below. Sketch graphs of velocity and position against time. The particle starts at rest at the origin.

![Graph of acceleration vs. time](image)

7. **FOOD FOR THOUGHT**

A particle of mass, \( m \), acted on by a force, \( F \), moves in a straight line. Its acceleration, \( a \), is given by Newton’s Law:

\[
F = ma.
\]

The work, \( W \), done by a constant force when the particle moves through a displacement, \( d \), is

\[
W = Fd.
\]

The velocity, \( v \), of the particle as a function of time, \( t \), is given in the figure below. What is the sign of the work done during each of the time intervals: \([0, t_1], [t_1, t_2], [t_2, t_3], [t_3, t_4], [t_2, t_4]\)?

![Graph of velocity vs. time](image)
1. The graph of \( g(x) \) below is made up of line segments and two semicircles (of radius 1).

![Graph of g(x)](image)

(a) Use the graph of \( g(x) \) and the Fundamental Theorem of Calculus to complete the table below, where \( G(x) = \int_{-4}^{x} g(u) \, du \). Give exact values for the missing data.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G(x) )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Now, plot the data in the table and sketch the graph of \( G(x) \). Be sure your sketch is consistent with the information you can discern from the graph of \( g(x) \).

2. Find the exact value of \( c \) if the area between the graph of \( y = x^2 - c^2 \) and the \( x \)-axis is 36.

3. Let \( F(x) = \int_{2}^{x} \frac{1}{\ln t} \, dt \) for \( x \geq 2 \).

(a) Find \( F'(x) \).

(b) Is \( F \) increasing or decreasing? What can you say about the concavity of its graph?

(c) Sketch a graph of \( F(x) \).

4. A factory is dumping pollutants into a lake continuously at the rate of \( t^2 \) tons per week, where \( t \) is the time in weeks since the factory commenced operations.

(a) After one year of operation, how much pollutant has the factory dumped into the lake?

(b) Assume that natural processes can remove up to 0.15 ton of pollutant per week from the lake and that there was no pollution in the lake when the factory commenced operations one year ago. How many tons of pollutant have now accumulated in the lake? (Note: The amount of pollutant being dumped into the lake is never negative.)

5. Calculate the derivatives:

(a) \( \frac{d}{dt} \int_{t}^{7} \log(x^6) \, dx \)

(b) \( \frac{d}{dx} \int_{\sin x}^{17} \tan^3 t \, dt \)

6. **Food for thought!**

Consider the graph of the equation \( x^2 + y^2 = r^2 \), which is a circle of radius \( r \). Rotate this circle about the \( x \)-axis (think of the points \((\pm r,0)\) as "poles") to produce a sphere of radius \( r \).

(a) Outline an argument using Riemann sums for computing the volume \( V \) of this sphere as a definite integral.
You may find the following trig identities useful:
\[ \sin(2\theta) = 2 \sin \theta \cos \theta \]
\[ \cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta. \]

1. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.
   (a) \[ \int x \sin(x^2) \, dx \]
   (b) \[ \int x^2 \sin x \, dx \]
   (c) \[ \int \frac{x^2}{1 + x^2} \, dx \]
   (d) \[ \int \frac{x}{(1 + x^2)^2} \, dx \]
   (e) \[ \int x^3 e^{x^2} \, dx \]
   (f) \[ \int \frac{\sin x}{2 + \cos x} \, dx \]

2. (a) Find \[ \int \sin \theta \cos \theta \, d\theta. \]
   (b) You probably solved part (a) by making the substitution \( w = \sin \theta \) or \( w = \cos \theta \). (If not, go back and do it that way.) Now find \[ \int \sin \theta \cos \theta \, d\theta \] by making the other substitution.
   (c) There is yet another way of finding this integral which involves the trigonometric identities above. Find \[ \int \sin \theta \cos \theta \, d\theta \] using one of these identities and then the substitution \( w = 2\theta \).
   (d) You should now have three different expressions for the indefinite integral \[ \int \sin \theta \cos \theta \, d\theta. \] Are they really different? Are they all correct? Explain.

3. (a) During the 1970s, ACME Widgets sold at a continuous rate of \( R = R_0 e^{0.15t} \) widgets per year, where \( t \) is time in years since January 1, 1970. Suppose they were selling widgets at a rate of 1000 per year on the first day of the decade. How many widgets did they sell during the decade? How many did they sell if the rate on January 1, 1970 was 150,000,000 widgets per year?
   (b) In the first case above (1000 widgets per year on January 1, 1970), how long did it take for half the widgets in the 1970s to be sold? In the second case (150,000,000 widgets per year on January 1, 1970), when had half the widgets in the 1970s been sold?
   (c) In 1980 ACME began an advertising campaign claiming that half the widgets it had sold in the previous ten years were still in use. Based on your answer to part (b), about how long must a widget last in order to justify this claim?

4. (a) Find the average value of the following functions over one cycle:
   (i) \[ f(\theta) = \cos \theta \]
   (ii) \[ g(\theta) = |\cos \theta| \]
   (iii) \[ k(\theta) = (\cos \theta)^2 \] (You may wish to use the trig identities above.)
   (b) Write the averages you have just found in ascending order. Using words and graphs, explain why the averages come out in the order they do.
5. The curves $y = \sin x$ and $y = \cos x$ cross each other infinitely often. What is the area of the region bounded by these two curves between two consecutive crossings?

6. Let $E(x) = \int \frac{e^x}{e^x + e^{-x}} \, dx$ and $F(x) = \int \frac{e^{-x}}{e^x + e^{-x}} \, dx$.
   
   (a) Calculate $E(x) + F(x)$.
   
   (b) Calculate $E(x) - F(x)$.
   
   (c) Use your results from parts (a) and (b) to calculate $E(x)$ and $F(x)$.

7. **Food for thought!**
   What, if anything, is wrong with the following calculation?
   
   $$\int_{-2}^{2} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_{-2}^{2} = -\frac{1}{2} - \left(-\frac{1}{-2}\right) = -1.$$
The following integrals can be found in most standard integral tables:

\[
\int \frac{bx + c}{x^2 + a^2} \, dx = \frac{b}{a} \ln |x^2 + a^2| + \frac{c}{a} \arctan \frac{x}{a} + C, \quad a \neq 0
\]

\[
\int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} \left( x \sqrt{a^2 + x^2} + a^2 \int \frac{1}{\sqrt{a^2 + x^2}} \, dx \right) + C
\]

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + C
\]

1. Evaluate the following expressions:
   
   (a) \( \int_0^r \frac{2x - 1}{x^2 + 5} \, dx \)
   
   (b) \( \int_0^r \frac{2x - 1}{x^2 + 5} \, dx \) where \( r \) is a constant.
   
   (c) \( \int_0^x \frac{2t - 1}{t^2 + 5} \, dt \) \( \text{(Is } x \text{ a constant or a variable here?)} \)
   
   (d) \( \frac{d}{dx} \int_0^x \frac{2t - 1}{t^2 + 5} \, dt \)

2. (a) Sketch a graph of the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) where \( a \) and \( b \) are positive constants with \( a \geq b > 0 \).
   
   (b) Let \( A \) be the area enclosed by the graph. Explain why \( A = 4B \) where \( B \) is the area bounded by the \( x \)-axis, the \( y \)-axis, and the part of the graph of \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) that lies in the first quadrant.
   
   (c) Express \( B \) as \( B = \alpha \int_r^s \sqrt{\beta - x^2} \, dx \) where \( r, s, \alpha, \) and \( \beta \) are constants that you must determine in terms of known quantities such as \( a \) and \( b \).
   
   (d) Use a table of integrals to find \( B \).
   
   (e) Now, find \( B \) by making a geometric interpretation of the integral in (c) that involves a familiar geometric figure.
   
   (f) Find \( A \), the area of the ellipse.

3. Derive the given formulas:
   
   (a) \( \int x^n \cos ax \, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx \)

   (b) \( \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \)
4. Integrating \( e^{ax} \sin bx \) by parts twice yields a result of the form

\[
\int e^{ax} \sin bx \, dx = e^{ax} (A \sin bx + B \cos bx) + C.
\]

(a) Find the constants \( A \) and \( B \) in terms of \( a \) and \( b \). [Hint: Don’t actually perform the integration by parts.]

(b) Evaluate \( \int e^{ax} \cos bx \, dx \) by modifying the result in part (a). [Again, it is not necessary to perform integration by parts, as the result is of the same form as that in part (a).]

5. You know \( \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \) for \( |x| < 1 \) and \( \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \)

(a) Express each of these differentiation results in terms of indefinite integrals.

(b) Make a substitution (AKA change of variable) to find \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx \) and \( \int \frac{1}{a^2 + x^2} \, dx \) where \( a \) is a given positive constant.

6. Evaluate the following integrals using a table or whatever else seems appropriate.

(a) \( \int \frac{1}{\sqrt{x^2 + 1}} \, dx \) and \( \int \frac{1}{\sqrt{x^2 + 2x + 2}} \, dx \)

(b) \( \int \frac{1}{\sqrt{5 - x^2}} \, dx \) and \( \int \frac{1}{\sqrt{4 - x^2 - 2x}} \, dx \)

(c) \( \int_{\pi/2}^{\infty} (x^3 - 2x^2 + 3) \sin 2x \, dx \)

(d) \( \int_{\pi/2}^{\infty} e^{5x} \sin 3x \, dx \) where \( a \) is a given constant.
MTH 252 — Lab 6b
Techniques of Integration

The list below contains integration problems using the techniques of substitution, integration by parts, method of partial fractions, and trig substitution. Work them in the following order: 1, 2, 3, 6, 9, 10. Then work the remainder in any order.

1. Evaluate \( \int \sin^2 \theta \, d\theta \) using integration by parts letting \( u = \sin \theta \) and \( v' = \sin \theta \). Then use the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to replace \( \cos^2 \theta \). From there, you should be able to finish the problem and find the antiderivative requested.

2. Evaluate \( \int \frac{x^2}{\sqrt{9-x^2}} \, dx \) using the trig substitution \( x = 3 \sin \theta \). You will need the result from Problem 1 above to finish this. Show the triangle that fits this problem. Write your final answer in terms of \( x \).

3. Evaluate \( \int \frac{dx}{x^2\sqrt{x^2+4}} \) using the trig substitution \( x = 2 \tan \theta \). Show the triangle that fits this problem. Write your final answer in terms of \( x \).

4. Evaluate \( \int \frac{x^2}{(4-x^2)^{3/2}} \, dx \) using trig substitution. You may need to use the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to replace \( \sin^2 \theta \) in your integrand.

5. Evaluate \( \int \frac{dx}{x^2+16} \) using trig substitution.

6. Evaluate \( \int e^{2\theta} \sin(3\theta) \, d\theta \). You will need to do integration by parts twice.

7. Evaluate \( \int \arctan(x) \, dx \) using integration by parts.

8. Evaluate \( \int y\sqrt{y+3} \, dy \) first using the substitution \( w = y+3 \). Then try this one using integration by parts, letting \( u = y \).

9. Evaluate \( \int \frac{1+x^2}{x(1+x)^2} \, dx \) using the method of partial fractions.

10. Evaluate \( \int \frac{2s^2+s+23}{(s+4)(s^2+1)} \, ds \) using the method of partial fractions.

11. You can also work Problem 1 above by rewriting the integrand using the trig identity \( \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \) and evaluating without using integration by parts. Try this on your own if you like.
MTH 252 — Lab 7: Integration Practice

Evaluate the following integrals. Assume \( c \) and \( k \) are constants.

You are strongly encouraged to first indicate which technique(s) of integration you expect to use to evaluate the integral. For example, next to the first problem you might write “standard form”, meaning an antiderivative you should know. Other short descriptions you might want to use include “algebraic simplification”, “guess and check”, “substitution”, “integration by parts”, and any other labels you find helpful.

1. \( \int \sin t \, dt \)
2. \( \int e^{5z} \, dz \)
3. \( \int \sin 2\theta \, d\theta \)
4. \( \int \left( x^{3/2} + x^{2/3} \right) \, dx \)
5. \( \int (r + 1)^3 \, dr \)
6. \( \int \frac{x^3 + x + 1}{x^2} \, dx \)
7. \( \int te^t \, dt \)
8. \( \int x^2 e^{-2x} \, dx \)
9. \( \int x \ln x \, dx \)
10. \( \int \left( r + 1 \right)^3 \, dr \)
11. \( \int e^{0.5-0.3t} \, dt \)
12. \( \int x \sqrt{4 - x^2} \, dx \)
13. \( \int \frac{\cos(\sqrt{y})}{\sqrt{y}} \, dy \)
14. \( \int \cos^2 \theta \, d\theta \)
15. \( \int \tan(2x - 6) \, dx \)
16. \( \int \frac{(t + 2)^2}{t^3} \, dt \)
17. \( \int \frac{t + 1}{t^2} \, dt \)
18. \( \int \tan \theta \, d\theta \)
19. \( \int \frac{x}{x^2 + 1} \, dx \)
20. \( \int \sin 5\theta \cos^3 5\theta \, d\theta \)
21. \( \int (t - 10)^{10} \, dt \)
22. \( \int_{1}^{3} x(x^2 + 1)^{70} \, dx \)
23. \( \int \frac{1}{x} \tan(\ln x) \, dx \)
24. \( \int \frac{e^{2y} + 1}{e^{2y}} \, dy \)
25. \( \int \frac{dx}{x \ln x} \)
26. \( \int \frac{x \cos(\sqrt{x^2 + 1})}{\sqrt{x^2 + 1}} \, dx \)
27. \( \int u e^{ku} \, du \)
28. \( \int e^{\sqrt{2}x^3} \, dx \)
29. \( \int (e^x + x)^2 \, dx \)
30. \( \int \frac{5x + 6}{x^2 + 4} \, dx \)
31. \( \int e^{-ct} \sin kt \, dt \)
32. \( \int (x^2 + 5)^3 \, dx \)
33. \( \int \frac{\sin w \cos w}{1 + \cos^2 w} \, dw \)
34. \( \int \frac{x}{\cos^2 x} \, dx \)
35. \( \int \frac{x}{\sqrt{x + 1}} \, dx \)
36. \( \int \frac{e^{2y}}{e^{2y} + 1} \, dy \)
37. \( \int \frac{z}{(z - 5)^3} \, dz \)
38. \( \int \frac{(2x - 1)e^{-x}}{e^x} \, dx \)
39. \( \int \sin x(\sqrt{2 + 3 \cos x}) \, dx \)
40. \( \int (x + \sin x)^3(1 + \cos x) \, dx \)
41. \( \int \frac{1}{x^2 - 4x + 7} \, dx \)
42. \( \int \frac{1}{\sqrt{4x - x^2}} \, dx \)
43. \( \int_0^1 x(1 + x^2)^20 \, dx \)
44. \( \int_0^\pi \sin \theta(\cos \theta + 5)^7 \, d\theta \)
45. \( \int_1^2 \frac{x^2 + 1}{x} \, dx \)
46. \( \int_1^e (\ln x)^2 \, dx \)
47. \( \int_0^{10} ze^{-z} \, dz \)
48. \( \int_1^4 \frac{e^x}{\sqrt{x}} \, dx \)
MTH 252 — Lab 8
Volume

For each problem, you may wish to set up all the integrals before evaluating any of them.

1. Consider the region bounded by \( y = \ln x \), \( x = 3 \), and \( y = 0 \).
   (a) Sketch this region.
   (b) Find the perimeter of this region. (Exact answer possible, but approximation OK.)
   (c) Find the area of this region.
   (d) Find the volume obtained by rotating this region around the \( x \)-axis.
   (e) Find the volume obtained by rotating this region around the \( y \)-axis.
   (f) Find the volume of the object having this region as its base and cross sections perpendicular to the \( y \)-axis that are squares.

2. Consider the region bounded by \( y = \sqrt{x} \) and \( y = \frac{x^4}{4} + \frac{3}{4} \).
   (a) Sketch this region.
   (b) Find the perimeter of this region. (Approximation.)
   (c) Find the area of this region.
   (d) Find the volume obtained by rotating this region around the \( x \)-axis.
   (e) Find the volume obtained by rotating this region around the line \( x = 1 \).
   (f) Find the volume of the object having this region as its base and cross sections perpendicular to the \( x \)-axis that are equilateral triangles.

3. (a) Set up and evaluate an integral giving the volume of a pyramid of height 10 m and square base 8 m by 8 m.
   (b) The pyramid in part (a) is cut off at a height of 6 m. See the figure at the right. Find the volume.

4. Find the volume of the snowman in the figure below. If \( x \) and \( y \) are in meters, and the origin is on the ground, the \( x \)-axis is horizontal and the \( y \)-axis is vertical, then the body is approximated by rotating the curve \( y = 1 - 4x^2 \) about the \( y \)-axis. The neck is a cylinder of radius 0.1 meter and length 0.15 meter; the head is spherical with radius 0.2 meter.
5. The tree in the picture is 300 feet high and has a diameter of 14 feet at the base. The picture below gives some additional information about the tree. Find an approximation of the volume of the tree trunk.

6. (a) Find the volume obtained by rotating the region in Problem 1 around the line \( y = 4 \).
(b) Find the volume obtained by rotating the region in Problem 1 around the line \( y = -4 \).
(c) Find the volume obtained by rotating the region in Problem 2 around the line \( x = 2 \).
(d) Find the volume obtained by rotating the region in Problem 2 around the line \( x = -2 \).
1. Find the total mass of the atmosphere. Assume that the atmosphere extends to $+\infty$ and that its density, $\rho$, is a function of height, $h$, in meters. In particular,

$$\rho(h) = 1.28e^{-0.000124h} \text{ kg/m}^3.$$  

The radius of the earth is approximately 6370 km.

2. Answer one of the following.
   (a) Find the center of mass of an isosceles trapezoid with height $H$ and bases $B_1$ and $B_2$.
   (b) Find the center of mass of a cone of height $H$ and radius $R$.

3. The soot produced by a garbage incinerator spreads out in a circular pattern. The depth, $H(r)$, in millimeters, of the soot deposited each month at a distance $r$ kilometers from the incinerator is given by $H(r) = 0.115e^{-2r}$.
   (a) Write a definite integral giving the total volume of soot deposited within 5 kilometers of the incinerator each month.
   (b) Evaluate the integral you found in part (a), giving your answer in cubic meters.

4. A metal plate, with constant density 2 gm/cm$^2$, has a shape bounded by the two curves $y = x^2$ and $y = \sqrt{x}$, with $0 \leq x \leq 1$, and $x, y$ in cm.
   (a) Find the total mass of the plate.
   (b) Because of the symmetry of the plate about the line $y = x$, we have $\bar{x} = \bar{y}$. Sketch the plate, and decide, on the basis of the shape, whether $\bar{x}$ is less than or greater than $1/2$.
   (c) Find $\bar{x}$ and $\bar{y}$.

5. Water is raised from a well 40 ft deep by a bucket attached to a rope. When the bucket is full, it weighs 30 lb. However, a leak in the bucket causes it to lose water at a rate of 1/4 lb for each foot that the bucket is raised. Neglecting the weight of the rope, find the work done in raising the bucket to the top.

6. A water tank is in the shape of a right circular cone with height 18 ft and radius 12 ft at the top. If it is filled with water to a depth of 15 ft, find the work done in pumping all of the water out of a spigot 2 ft above the top of the tank. (The density of water is $\delta = 62.4$ lb/ft$^3$.)

7. An underground tank filled with gasoline of density 42 lb/ft$^3$ is a hemisphere of radius 5 ft, as in the figure at the right. Use an integral to find the work to pump the gasoline over the top of the tank.