(1) True or False?

(a) If $f \circ g$ is one-to-one, then $g$ is one-to-one.

True.

(b) If $f \circ g$ is onto, then $g$ is onto.

False.

(c) The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n^2$ is injective (1-1).

True.

(d) The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n^2$ is surjective (onto).

False.

(e) The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n^2$ is bijective.

False.

(f) A connected graph on $n$ vertices must have at least $n - 1$ edges.

True.

(g) The sum of all the vertex degrees in a graph is even.

True.

(h) Every Euler cycle is a Hamiltonian cycle.

False.
(i) There exists a graph whose vertices have the following degrees: 2, 2, 1, 1, 1.

True.

(j) There are 60 rearrangements of the word *APPLE*.

True.
(2) How many 5-card deals from a standard deck of 52 contain at least one card from each of the four suits?

A 5-card deal containing at least one card from each suit necessarily contains two cards of one suit and a single card each from the other three suits.

So choose a suit \((C(4, 1) = 4\) choices), then two cards of that suit \((C(13, 2) = 78\) choices), then a card each from three remaining suits \((C(13, 1)^3 = 2,197\) choices). So by the multiplication principle, there are...

\[\ldots C(4, 1)C(13, 2)C(13, 1)^3 = 685,464\] such deals.

(3) How many onto functions are there from a set with 4 elements to a set with 3 elements?

An onto function from a set of 4 elements to a set of 3 elements must map two of the four elements to one of the three elements. There are \(C(4, 2)C(3, 1) = 18\) ways to do this.

Then there remain 2 ways to assign the remaining two elements of the domain onto the remaining two elements of the codomain.

Thus by the multiplication principle, there are \((18)(2) = 36\) such onto functions.
(4) Suppose 8 fish are to distributed in 3 distinct tanks. In how many ways can this be done if...

(a) ...the fish are distinguishable?

Each distinguishable fish can go into any of the 3 tanks. So there are \(3^8 = 6,561\) ways to distribute the fish.

(b) ...the fish are indistinguishable?

Selections of two slots to be \(\mid\)'s (separations symbols) among \(8 + 2 = 10\) slots corresponds (bijectively) to distributions of the indistinguishable fish into 3 tanks.

So there are \(\binom{10}{2} = 45\) such distributions.

(c) ...the fish are indistinguishable and each bucket must get a fish?

Selections of two slots to be \(\mid\)'s (separations symbols) among 7 slots (FFFFFFFFFF) corresponds (bijectively) to distributions of the indistinguishable fish into 3 tanks with at least one fish in each bucket.

So there are \(\binom{7}{2} = 21\) such distributions.

An alternative argument is this: Take three fish and put one in each of the three buckets. The remaining 5 fish can be distributed in \(\binom{7}{2}\) ways following the same logic as in part (b).

(5) Assume \(f : \mathbb{Z} \to A\) given by \(f(x) = 2x - 5\) is onto. Determine the set \(A\).

\(A\) would be the set of odd integers.

(6) Determine if \(g : \mathbb{R} \to (0, 1]\) given by \(f(x) = \frac{1}{1+x^2}\) is an injection (1-1), a surjection (onto), and a bijection.

\(f(1) = f(-1),\) so \(f\) is not 1-1 and therefore not a bijection. Let \(0 < m \leq 1\). Set \(x = \sqrt{\frac{1}{m} - 1}\). Then \(f(x) = m,\) so \(f\) is onto.

(7) Your friend says that his “favorite (simple) graph” has the following degrees: 1, 1, 1, 2, 3, 4, 4, 5. Does such a graph exist? If so, draw it. If not, justify.

Such a graph cannot exist as there are an odd number of odd vertices, resulting in an odd sum of vertex degrees.
(8) Show that if a graph has exactly two vertices \(v, w\) of odd degree than there is a path between \(v\) and \(w\). Hint: Suppose there isn’t a path between \(v\) and \(w\). Consider the connected component of \(v\). This subgraph has a degree sum that is...

Proof: Let \(G\) be a graph with exactly two vertices \(v, w\) of odd degree. Suppose (to yield a contradiction) that there is not path between \(v\) and \(w\). Let \(G_v\) be the connected component of \(G\) containing \(v\). \(G_v\) cannot contain \(w\), or there would be path between \(v\) and \(w\). So \(G_v\) has exactly one odd degree. This is a contradiction as \(G_v\) is a graph and the sum of vertex degrees has to even. This completes the proof □

(9) Is the binary relation \{\((1, 1), (2, 3), (3, 2), (3, 3), (4, 4), (5, 6), (6, 6)\}\) on \(\mathbb{N}\) symmetric?

No.

(10) Is the binary relation \{\((2, 2), (2, 3), (3, 3), (3, 4), (4, 4), (5, 6), (5, 7), (6, 7)\)\} on \(\mathbb{N}\) anti-symmetric?

Yes.
(11) Let \( f : \mathbb{N} \to \mathbb{N} \) be given by \( f(1) = 1 \) and for \( n \geq 2, f(n) = n - 1 \). Prove that \( f \) is a surjection (onto) but not a bijection.

Proof: Let \( n \in \mathbb{N} \). Then \( f(n + 1) = n \). Hence \( f \) is onto. Since \( f(1) = f(2) = 1 \), \( f \) is not \( 1-1 \) and therefore not a bijection □

(12) Let \( f : \mathbb{N} \to \mathbb{N} \) be given by \( f(n) = n + 1 \). Prove that \( f \) is an injection \( (1-1) \) but not a bijection.

Proof: Let \( m, n \in \mathbb{N} \). Assume \( f(m) = f(n) \). Then \( m + 1 = n + 1 \) and so \( m = n \). Thus \( f \) is \( 1-1 \). \( f \) is not onto since \( f(n) = n + 1 > 1 \) and \( 1 \in \mathbb{N} \). So \( f \) is not a bijection □

(13) Let \( X = \{0, 1, 2, 3, 4\} \). Let \( R \) be a relation on \( X \) given by \((a, b) \in R \) if \((b - a) \mod 3 = 0\).

(a) Show \( R \) is an equivalence relation.

Proof: Let \( a \in X \). \( R \) is reflexive as \( a - a = 0 \) implies \((a, a) \in R \). Let \( b \in X \). Assume \((a, b) \in R \). Then \( b - a = 3k \) for some integer \( k \), so it follows that \( a - b = 3(-k) \) and hence \((b, a) \in R \). So \( R \) is symmetric. Let \( c \in X \). Assume \((a, b), (b, c) \in R \). Then \( b - a = 3k_1 \) and \( c - b = 3k_2 \) for some integers \( k_1, k_2 \). Then \( c - a = 3(k_1 + k_2) \) implies that \((a, c) \in R \). So \( R \) is transitive and an equivalence relation □

(b) Find \( A \), the matrix of the relation \( R \). Use the standard numerical order on \( X \) to index the rows and columns.

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]
(14) Consider the binary relation $R$ on $\mathbb{R}$ given by $(x, y) \in R$ if $x^2 = y^2$. Determine if this relation is reflexive, symmetric, transitive, and anti-symmetric. Is $R$ a partial order? Is $R$ an equivalence relation?

This relation is not a partial order since the relation is not anti-symmetric, just consider that $(-1, 1), (1, -1) \in R$. But it is an equivalence relation as it is reflexive, symmetric, and transitive.

(15) Consider the binary relation $R$ on $\mathbb{N}$ given by $(m, n) \in R$ if $m$ divides $n$. Determine if this relation is reflexive, symmetric, transitive, and anti-symmetric. Is $R$ a partial order? Is $R$ an equivalence relation?

This relation is not an equivalence relation since it is not symmetric (for example, 2 divides 6 but not vice versa). But it is a partial order because it is reflexive, anti-symmetric, and transitive.

(16) Consider the binary relation $R$ on the set of all people given by $(p, q) \in R$ if person $p$ is a child of person $q$. Determine if this relation is reflexive, symmetric, transitive, and anti-symmetric. Is $R$ a partial order? Is $R$ an equivalence relation?

The relation is not reflexive (so also not a partial order nor equivalence relation). It is also not symmetric nor transitive. It is, however, vacuously anti-symmetric.

(17) Prove that if $R$ and $S$ are both symmetric binary relations on $X$ then $R \cup S$ is a symmetric binary relation on $X$.

Proof: Let $X$ be a non-empty set. Let $R, S$ be symmetric binary relations on $X$. Let $x, y \in X$. Assume $(x, y) \in R \cup S$. If $(x, y) \in R$ then $(y, x) \in R$. Else $(x, y) \in S$ and then $(y, x) \in S$. Either way $(y, x) \in R \cup S$ and hence $R \cup S$ is symmetric □
(18) Draw a graph such that...

(a) ...there is a Hamiltonian cycle but no Euler cycle.
(b) There is an Euler cycle but no Hamiltonian cycle.
(19) How many non-negative integer solutions are there to \(x_1 + x_2 + x_3 = 20\)?

Corresponds to counting how many choices of 2 slots to be |'s (separation symbols) among \(20 + 2 = 22\) slots. The other twenty slots are filled with +1's. The the left of the first separation symbols is \(x_1\), in between is \(x_2\), and to the right of the second separation symbol is \(x_3\).

So there are \(C(22, 2) = 231\) such solutions.

(20) Apply Dijkstra’s algorithm to find the length of the shortest path between vertices \(u, v\) in the graph. Is the path unique?

Dotted arrows indicate predecessors. In this example, there is a unique path of length 16 between \(u\) and \(v\).
(21) Apply Kruskal’s algorithm to find a minimal spanning tree in the graph.

A minimal spanning tree (produced by Kruskal’s algorithm) is given by its edge set in red:

(the vertex set includes all vertices of the graph)

(22) Review chapters 1 and 2, including the midterm review problems and the midterm!