Math 241 - Calculus for Management and Social Science

Exponential Functions:

A function of the form \( f(x) = b^x \) where \( b > 0 \) is a constant \((b \neq 1)\) is called an exponential function. Here is the graph of \( y = f(x) \) for \( f(x) = 2^x \).

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
-4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\
\hline
-4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\
\end{array}
\]

\( b > 1 \) corresponds to exponential growth and \( 0 < b < 1 \) corresponds to exponential decay. The domain is \((-\infty, \infty)\) and the range is \((0, \infty)\).

Recall the laws of exponents:

1. \( b^x b^y = b^{x+y} \).
2. \( b^{-x} = \frac{1}{b^x} \).
3. \( \frac{b^x}{b^y} = b^{x-y} \).
4. \( (b^x)^y = b^{xy} \).
5. \( (ab)^x = a^x b^x \) and \( \left( \frac{a}{b} \right)^x = \frac{a^x}{b^x} \).
6. \( b^0 = 1 \). The \( y \)-intercept of the graph of an exponential function is \((0,1)\).

Solve the following exponential equations:

1. \[
4^{4-x} = 64.
\]
   By equating exponents, we set \( 4 - x = 3 \), as \( 4^3 = 64 \). Hence, \( x = 1 \).

2. \[
2^x - 2^{-x} = 0 ...
\]
Derivative of an Exponential Function:
Let \( f(x) = b^x \). By the definition of the derivative:

\[
f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h}.
\]

So the derivative of \( f(x) = b^x \) is the product of \( f(x) \) and \( \lim_{h \to 0} \frac{b^h - 1}{h} \).

\( e \approx 2.7 \) is defined such that,

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = 1,
\]

thus for \( f(x) = e^x \), we get \( f'(x) = e^x \). This exponential function is its own derivative.

*Challenge question:* Find \( \lim_{h \to 0} \frac{e^{2h} - 1}{h} \).

Consider the function \( g(x) = e^{-x} \). Then by the general power rule,

\[
g'(x) = \frac{d}{dx} (e^x)^{-1} = -(e^x)^{-2} \cdot e^x = -e^{-x}.
\]

**Applications:**
The chain rule for exponential functions:

\[
\frac{d}{dx} (e^{g(x)}) = e^{g(x)} \cdot g'(x).
\]

In particular,

\[
\frac{d}{dx} (e^{kx}) = ke^{kx}.
\]

An equation that involves a function and its derivatives is known as a **differential equation**.

\( y = f(x) \) satisfies the differential equation \( y' = ky \) if and only if \( y = f(x) = Ce^{kx} \), where \( C \) is a constant. Note that \( k \) is positive for exponential growth and negative for exponential decay.

Suppose \( y' = 2y \) and \((0, 3)\) lies on the graph of \( y = f(x) \). Then \( k = 2 \) and by using \( f(0) = 3 \), we get \( f(x) = 3e^{2x} \).
Suppose the value of a stock portfolio $t$ years after 2000 is modeled by
\[ v(t) = 10,000e^{.08t}. \]

At what rate will the portfolio be growing in 2015?
\[ v'(t) = 10,000e^{.08t} \cdot (.08) = 800e^{.08t}. \] Hence,
\[ v'(15) = 800e^{.12} \approx 2,700. \]

So the portfolio is growing at a rate of 2,700 dollars per year in 2015.

**Logarithmic Functions:**

For \( f(x) = e^x \), we have
\[ f^{-1}(x) = \ln x, \]
the natural logarithm function.

As it is the inverse function, the domain of \( \ln x \) is \((0, \infty)\). Its range is \((−\infty, \infty)\). We have the following properties:

1. For \( x > 0 \),
\[ e^{\ln x} = x. \]

2. 
\[ \ln e^x = x. \]

3. 
\[ \ln (xy) = \ln x + \ln y. \]

4. 
\[ \ln \frac{x}{y} = \ln x - \ln y. \]

5. 
\[ \ln x^r = r \ln x. \]
Notice that 

\[ b^x = (e^{\ln b})^x = e^{(\ln b)x}, \]

so any exponential function can be written in terms of \( e \).

Let’s solve the following exponential and logarithmic equations:

(1) 

\[ e^{x^2} = 25. \]

Taking the natural logarithm of both sides, we get \( x^2 = \ln 25 \). So \( x = \pm \sqrt{\ln 25} \).

(2) 

\[ 2 \ln x + 1 = 9. \]

Firstly, \( 2 \ln x = 8 \) and by dividing by 2, \( \ln x = 4 \). Hence \( x = e^{\ln x} = e^4 \).

(3) 

\[ \ln x + \ln (x - 2) = 0. \]

Noting property (3) from the last page, \( \ln x + \ln (x - 2) = \ln (x^2 - 2x) \).

So \( \ln (x^2 - 2x) = 0 \). Then:

\[ x^2 - 2x = e^0 = 1, \]

\[ x^2 - 2x - 1 = 0, \]

\[ x = \frac{2 \pm \sqrt{4 + 4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}, \]

however noting the domain of the logarithm, only one of these works: \( x = 1 + \sqrt{2} \).

**Derivative of a Logarithmic Function:**

Let’s try to find \( \frac{d}{dx}(\ln x) \). By applying the chain rule for \( x > 0 \),

\[ \frac{d}{dx} (e^{\ln x}) = e^{\ln x} \cdot \frac{d}{dx} (\ln x). \]

But \( e^{\ln x} = x \), and we have that \( \frac{d}{dx}(x) = 1 \). Using this in the above,

\[ 1 = x \cdot \frac{d}{dx} (\ln x) \rightarrow \frac{d}{dx} (\ln x) = \frac{1}{x}. \]
The graph of $y = \ln x$.

First notice that all the tangent lines are positively sloped. Also note that near $x = 0$ the slopes are large and far from $x = 0$ the slopes are small.

So the graph of the derivative should be in the first quadrant, have large values near $x = 0$, and small values far from $x = 0$. Here is the graph of the derivative, $y = \frac{1}{x}$:

Chain rule for taking the derivative of a function of the form $h(x) = \ln f(x)$:

$$\frac{d}{dx} (\ln f(x)) = \frac{1}{f(x)} \cdot f'(x).$$

In Leibniz notation with $u = f(x)$...

$$\frac{d}{dx} (\ln u) = \frac{1}{u} \cdot \frac{du}{dx}.$$
Differentiate:

(1) \[ f(x) = \sqrt{1 + \ln x} \]

Let's do this one using Leibniz notation. Let \( f(u) = \sqrt{u} \) where \( u = 1 + \ln x \). Then

\[
\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{1 + \ln x}}.
\]

(2) \[ g(x) = 1 - x \ln x \]

\[
g'(x) = \frac{d}{dx} (1 - x \ln x) = 0 - \frac{d}{dx} (x \ln x) = - \left( \frac{d}{dx} (x) \cdot \ln x + x \cdot \frac{d}{dx} (\ln x) \right) = - (1 \cdot \ln x + x \cdot \frac{1}{x}) = - \ln(x) - 1.
\]

(3) \[ h(x) = \frac{x}{\ln x} \]

This one involves the quotient rule, derivative of the top times the bottom, minus the top times the derivative of the bottom over the bottom squared!

\[
h'(x) = \frac{1 \cdot \ln(x) - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln(x) - 1}{(\ln x)^2}.
\]

(4) Prove \( k'(x) = \frac{1}{x} \) for \( k(x) = \ln |x| \).

**proof:** If \( x > 0 \) then \( |x| = x \) and \( k'(x) = \frac{1}{x} \).

If \( x < 0 \) then \( |x| = -x \), so \( \ln |x| = \ln (-x) \) and \( k'(x) = \frac{1}{-x} \cdot \frac{d}{dx} (-x) = - \frac{1}{x} \cdot (-1) = \frac{1}{x} \).

\( \square \)
Exercise: Consider the function:

\[ f(x) = \frac{x}{x + \ln x}. \]

Find and justify the coordinates of the minimum point.

Applications

The properties of the logarithm as well as the chain rule, which gives,

\[ \frac{d}{dx} (\ln f(x)) = \frac{1}{f(x)} \cdot f'(x), \]

go a long way to help when differentiating difficult products and quotients as the following examples illustrate.

Let \( f(x) = (x^2 + 1)^2(x + 1)(2x + 1)^3 \). Using the product rule is one way to do this problem. Another is to take the natural logarithm of both sides:

\[ \ln f(x) = \ln \left((x^2 + 1)^2(x + 1)(2x + 1)^3\right) \]

so by the properties of logarithms,

\[ \ln f(x) = 2 \ln (x^2 + 1) + \ln (x + 1) + 3 \ln (2x + 1). \]

Now we can take the derivative of both sides of this equation:

\[ \frac{1}{f(x)} \cdot f'(x) = 2 \cdot \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{x + 1} \cdot 1 + 3 \cdot \frac{1}{2x + 1} \cdot 2. \]

By simplifying and multiplying both sides by \( f(x) \),

\[ f'(x) = f(x) \left( \frac{4x}{x^2 + 1} + \frac{1}{x + 1} + \frac{6}{2x + 1} \right) \]

which gives,

\[ f'(x) = (x^2 + 1)^2(x + 1)(2x + 1)^3 \left( \frac{4x}{x^2 + 1} + \frac{1}{x + 1} + \frac{6}{2x + 1} \right), \]

\[ f'(x) = 4x(x^2 + 1)(x + 1)(2x + 1)^3 + (x^2 + 1)^2(2x + 1)^3 + 6(x^2 + 1)^2(x + 1)(2x + 1)^2. \]
Let
\[ g(x) = \frac{\sqrt{e^x + 3(x^5 - 1)^4}}{(3x^2 - 1)^2}. \]

Then,
\[ \ln g(x) = \ln \left( \frac{\sqrt{e^x + 3(x^5 - 1)^4}}{(3x^2 - 1)^2} \right), \]
and therefore,
\[ \ln g(x) = \frac{1}{2} \ln (e^x + 3) + 4 \ln (x^5 - 1) - 2 \ln (3x^2 - 1). \]

By taking the derivative of both sides,
\[ \frac{1}{g(x)} \cdot g'(x) = \frac{1}{2} \cdot \frac{1}{e^x + 3} \cdot e^x + 4 \cdot \frac{1}{x^5 - 1} \cdot 5x^4 - 2 \cdot \frac{1}{3x^2 - 1} \cdot 6x \ldots \text{so we get}, \]
\[ g'(x) = \frac{\sqrt{e^x + 3(x^5 - 1)^4}}{(3x^2 - 1)^2} \left( \frac{e^x}{2(e^x + 3)} + \frac{20x^4}{x^5 - 1} - \frac{12x}{3x^2 - 1} \right). \]