Models:

In this chapter we often use $t$ instead of $x$ as the independent variable. This reflects that the most common applications involve exponential growth and decay with the variable being time.

Consider the differential equation

$$y' = ky,$$

where $k$ is a constant. This equation describes exponential growth ($k > 0$) and exponential decay ($k < 0$). The solutions of this equation can be uniquely described by

$$y = Ce^{kt},$$

where $C$ is a constant.

In 1993 the world population of humans was roughly 5.5 billion. Now the population is roughly 7 billion. Assuming growth is exponential, let’s estimate in which year there will be 10 billion humans.

Let $t$ be years since 1993. Let $P(t) = Ce^{kt}$ be population. We know that $C = P(0) = 5.5$ and $P(17) = 6.8$.

So,

$$6.8 = 5.5e^{17k} \rightarrow k = \frac{1}{17} \ln \left( \frac{6.8}{5.5} \right).$$

Below is the graph of $P = f(t)$, with

$$f(t) = 5.5e^{\frac{t}{17} \ln \left( \frac{6.8}{5.5} \right)}.$$
So let's solve the following equation and compare it with the graph on the previous page.

\[
10 = 5.5e^{\frac{t}{17} \ln \frac{6.8}{5.5}},
\]

\[
\ln \frac{10}{5.5} = \frac{t}{17} \ln \frac{6.8}{5.5},
\]

\[
t = \frac{\ln \frac{10}{5.5}}{\ln \frac{6.8}{5.5}} \approx 48 \text{ years after 1993}.
\]

If human growth continues exponentially, then there should be 10 billion humans in 2041.

Radioactive carbon 14 has a half-life of about 5730 years. Let's find the decay constant. Let \( A(t) \) be the amount of radioactive carbon 14 after \( t \) years.

\[
\frac{1}{2} A(0) = A(0)e^{5730k},
\]

\[
k = \frac{1}{5730} \ln \frac{1}{2} \approx -.000121.
\]

Hence

\[
A(t) = A(0)e^{-0.000121t}.
\]

What percent of the original carbon 14 would you expect to find in a 4500 year old artifact?

\[
A(4500) = A(0)e^{-0.000121(4500)} \approx .58A(0).
\]

So approximately 58 percent of the original carbon 14 should still be present.

*Compound interest* is interest on initial principal and previously accrued interest. Each period adds the interest on the cumulative total to recompute the total.

Let's deduce some formulae. Let \( P \) stand for the initial principal. Assume the interest is compounded \( n \) times per year. For example use \( n = 12 \) for monthly compounding.

After the initial compounding period, the interest added is \( I = P(r)(\frac{1}{n}) \) by the simple interest formula. So the new principal is \( P + I = P(1 + \frac{r}{n}) \). After two periods, the new principal is

\[
P(1 + \frac{r}{n}) + P(1 + \frac{r}{n})\frac{r}{n} = P(1 + \frac{r}{n})(1 + \frac{r}{n}) = P(1 + \frac{r}{n})^2.
\]
After $t$ years of a principal of $P$ at a per annum interest rate of $r$ compounded $n$ times per year the amount (of the loan or investment) is

$$A = P(1 + \frac{r}{n})^{nt}.$$  

We provide a proof by induction on $nt$. We have seen that this formula holds for $nt = 1, 2$. Assume this formula holds for $nt$. Then after $nt + 1$ periods the amount is

$$P(1 + \frac{r}{n})^{nt} + P(1 + \frac{r}{n})^{nt} \frac{r}{n} = P(1 + \frac{r}{n})^{nt}(1 + \frac{r}{n}) = P(1 + \frac{r}{n})^{nt+1} \square$$

Example: Investment 1 is compounded quarterly (4 times a year) at a rate of 6 percent, while investment 2 is compounded monthly at a rate of 5 percent. If you have 10,000 dollars to invest in one or the other, which would give you a larger return in 5 years?

Let's calculate the amount of investment 1 over 5 years, which is

$$(10,000)(1 + \frac{.06}{4})^{20} = 13,468.60.$$ 

The amount of investment 2 over 5 years is

$$(10,000)(1 + \frac{.05}{12})^{60} = 12,833.60.$$ 

So over 5 years investment 1 is better.

**Exercise:** Find the doubling time for investment 1 above.

Now consider the following algebra.

$$P(1 + \frac{r}{n})^{nt} = P(1 + \frac{1}{\frac{r}{n}})^{nt},$$

$$= P\left[(1 + \frac{1}{\frac{r}{n}})^{\frac{n}{r}}\right]^{rt},$$

$$= P\left[(1 + \frac{1}{h})^h\right]^{rt}.$$ 

Note $h = \frac{n}{r}$ gets larger as the number of times a year the interest is compounded gets larger. In fact, the function $(1 + \frac{1}{h})^h$ has a horizontal asymptote at $e \approx 2.718$.

Hence amount of an investment (or loan) that has continuously compounded interest is calculated via

$$A = Pe^{rt}.$$
Consider a continuously compounding investment of $10,000 dollars at 6 percent. 

\[ A(t) = 10,000e^{0.06t}. \]

At what rate is the investment growing after 10 years? Well, we must calculate \( A'(10). \)

\[ A'(t) = 600e^{0.06t}, \]

\[ A'(10) = 600e^{0.6} \approx 1093 \text{ dollars per year}. \]

If we solve for \( P \) and get

\[ P = Ae^{-rt}, \]

then we can get obtain what is the called the present value \( P \) of an amount \( A \) to be received in \( t \) years.

Find the present value of 50,000 to be received in 4 years if it can be invested at a continuous rate of 3 percent.

\[ P = 50,000e^{-0.03(4)} \approx 44,346 \text{ dollars}. \]

Relative Rate of Change:
Using the chain rule, we get the logarithmic derivative of a function \( f(t) \) defined by

\[ \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}. \]

It computes the percentage increase or decrease, in that it computes the rate of change in \( f(t) \) relative to how large \( f(t) \) is. Often it is called the relative rate of change.

Suppose \( R(x) = 400x - \frac{1}{10}x^2 \) is a revenue function for a sporting event, where \( x \) is the ticket price. Find the percentage change in revenue at \( x = 200 \) dollars.

\[ \frac{R'(x)}{R(x)} = \frac{400 - \frac{1}{5}x}{400x - \frac{1}{10}x^2} \to \frac{R'(200)}{R(200)} = \frac{360}{76,000} \approx .0047, \]

so revenue is increasing at approximately half a percent.
Given a demand equation \( p = f(x) \), one can solve for \( x \) to get a function of price for the quantity demanded. That is, \( x = f(p) \). Elasticity of demand is taken to be

\[
E(p) = -\frac{\text{relative rate of change in quantity demand}}{\text{relative rate of change in price}} = -\frac{f'(p)}{f(p)} = -\frac{pf'(p)}{f(p)}.
\]

If \( E(p) > 1 \), then demand is elastic and price changes affect demand significantly, while if \( E(p) < 1 \), then demand is inelastic and price changes affect demand less so. For example, restaurant meals are elastic, while gasoline is inelastic.

What if we express revenue as a function of price? Then \( R(p) = p \cdot f(p) \), and by the product rule,

\[
R'(p) = f(p) + pf'(p) = f(p) \left( 1 + \frac{pf'(p)}{f(p)} \right) = f(p)(1 - E(p)).
\]

★ When \( E(p) > 1 \), revenue is decreasing with respect to price as \( R'(p) < 0 \).

★ When \( E(p) < 1 \), revenue is increasing with respect to price as \( R'(p) > 0 \).

A bus system charges 1 dollar per person and has 10,000 riders daily. The demand equation for price \( p \) and daily ticket sales \( x \) is given by \( x = f(p) = 2000\sqrt{26 - p} \). Is demand elastic or inelastic at 1 dollar? Should the price be increased?

\[
f'(p) = \frac{-1000}{\sqrt{26 - p}},
\]

\[
E(1) = -\frac{(1) \cdot f'(1)}{f(1)} = -\frac{-200}{10000} = .02 < 1.
\]

So demand is inelastic and the price should be increased to increase revenue.

More models:

A nice model for many situations can be given by

\[
f(t) = M \left( 1 - e^{kt} \right).
\]

For example, if this model is used for a skydiver’s velocity, then \( M \) is terminal velocity (about 120 miles per hour on earth) and \( k \) is a negative constant. The idea is that as \( t \) gets big, \( e^{kt} \) gets small \((k < 0)\), so \( f(t) \) approaches \( M \). This is also a good model for the diffusion of a news story through society \((M \text{ is the population})\) or even the growth of a population as it nears a carrying capacity \((M \text{ is the carrying capacity})\).
For example, suppose a news story occurs and is reported. After 2 hours, suppose 1 percent of the population has heard the news. How many hours until half the population has heard the news? Use the model on the previous page with $M = P$ for population.

\[
\frac{1}{100} P = P(1 - e^{2k}),
\]

\[
\frac{1}{100} = 1 - e^{2k},
\]

\[
e^{2k} = .99 \rightarrow k = \frac{1}{2} \ln .99 \approx -.005.
\]

Here is the graph of percentage of population that has heard the news story, equal to $100 (1 - e^{-0.005t})$, where $t$ is hours after initial reporting.

We can solve for the number of hours it takes for half the population to have heard the news:

\[
\frac{1}{2} P = P \left(1 - e^{-0.005t}\right),
\]

\[
\frac{1}{2} = 1 - e^{-0.005t},
\]

\[
t = \frac{1}{-.005} \ln \frac{1}{2} \approx 140 \text{ hours (or almost six days)}.
\]

This checks out with the graph above.
The equation for logistic growth looks like:

$$y = \frac{M}{1 + Be^{-kt}},$$

where $B > 0$, $M > 0$ and $k > 0$ are constants. $y = M$ is the horizontal asymptote. It comes from solving the differential equation:

$$y' = \frac{k}{M}y(M - y).$$

So the growth of $y$ depends on both $y$ and $M - y$, so that as $y$ approaches its maximum capacity $M$ the exponential growth slows asymptotically.

Let's look at an example. Suppose that initially there are 10 fruit flies in a test tube that can hold no more than 50 flies. After 2 days, there are 25 flies in the tube. How long does it take to reach 45 flies in the tube?

We have that $M = 50$, and letting $P(t)$ be the fly population at $t$ days, we have that $P(0) = 10$, $P(2) = 25$.

So,

$$10 = \frac{50}{1 + B} \rightarrow B = 4.$$

Thus,

$$25 = \frac{50}{1 + 4e^{-2k}} \rightarrow 4e^{-2k} = 1 \rightarrow k = \frac{\ln 4}{2} = \ln 2.$$

So, we want to solve the following equation:

$$45 = \frac{50}{1 + 4e^{-t\ln 2}}.$$

I leave this to you, but you should get $t \approx 5.2$ days.