4.8

So far we have studied the slope of a curve at a point and its applications. This is one of the fundamental problems in calculus. The other fundamental problem in calculus is to find the area of a region under a curve. It turns out these problems are intimately connected.

So far we have developed rules for finding the derivative of a function. Now it is necessary to do the reverse. Given the derivative of a function, determine the function. With this in mind we have the following definition:

\[ F(x) \text{ is an antiderivative of } f(x) \text{ if } F'(x) = f(x). \]

Let's find an antiderivative of \( f(x) = x \).

We know that \( \frac{d}{dx}(x^2) = 2x \), which is almost the same as \( x \) except for the 2.

Any ideas?

How about \( F(x) = \frac{1}{2}x^2 \)? Well, \( F'(x) = x \). So \( F(x) = \frac{1}{2}x^2 \) is an antiderivative of \( x \).

How about \( F(x) = \frac{1}{2}x^2 + 1 \)? Of course this one is also an antiderivative of \( x \).

In fact, for any constant \( C \), \( F(x) = \frac{1}{2}x^2 + C \) is an antiderivative of \( x \).

If \( F(x) \) and \( G(x) \) are both antiderivatives of \( f(x) \) then \( F(x) - G(x) = C \), where \( C \) is a constant. The graphs differ by a vertical shift. The set of all antiderivatives of \( f(x) \) can be described by \( F(x) + C \), where \( C \) is a constant and \( F(x) \) is a particular antiderivative. That motivates our next definition.

The indefinite integral of \( f(x) \) is the set of all antiderivatives, denoted in the following way:

\[ \int f(x) \, dx = F(x) + C. \]

\( \int \) is the integral sign. \( C \) is a constant and \( F'(x) = f(x) \).
Consider the following derivative for $n \neq -1$.

$$\frac{d}{dx} \left( \frac{1}{n+1} x^{n+1} \right) = \frac{1}{n+1} \cdot \frac{d}{dx} (x^{n+1}) = \frac{1}{n+1} \cdot (n+1) x^n = x^n.$$  

So $F(x) = \frac{1}{n+1} x^{n+1}$ is an antiderivative of $f(x) = x^n$ for $n \neq -1$. Thus, for $n \neq -1$, and arbitrary constant $C$,

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

Here are a few examples:

$$\int x^3 \, dx = \frac{1}{4} x^4 + C.$$  

$$\int x^{-2} \, dx = -x^{-1} + C.$$  

$$\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C.$$  

What about $n = -1$?

That is, what is an antiderivative of $f(x) = x^{-1} = \frac{1}{x}$?

We know that $F'(x) = f(x)$ for $F(x) = \ln x$. But since the domain of $F(x) = \ln x$ does not match the domain of $f(x) = \frac{1}{x}$ we must use $G(x) = \ln |x|$ as the antiderivative since $G'(x) = f(x)$ and the domain matches. Hence,

$$\int \frac{1}{x} \, dx = \ln |x| + C, \text{ where } x \neq 0.$$  

Consider the following derivative for $k \neq 0$.

$$\frac{d}{dx} \left( \frac{1}{k} e^{kx} \right) = \frac{1}{k} \cdot \frac{d}{dx} (e^{kx}) = \frac{1}{k} \cdot k e^{kx} = e^{kx}.$$  

Hence, for a constant $k \neq 0$,

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C.$$
We also have the following nice rules, just as for derivatives,

\[ \int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx. \]

\[ \int kf(x)dx = k \int f(x)dx, \] where \( k \) is a constant.

Example:

\[ \int \left( 6x - 1 + \frac{3}{x} \right) dx = 6 \int xdx - \int 1dx + 3 \int \frac{1}{x}dx = \]

\[ = 6 \cdot \frac{1}{2}x^2 - x + 3 \cdot \ln |x| + C = 3x^2 - x + 3 \ln |x| + C. \]

Find a function \( F(x) \) that satisfies \( F'(x) = 4e^{-0.8x} \) and \( F(0) = 0 \). Lets begin with the indefinite integral:

\[ F(x) = \int 4e^{-0.8x}dx = 4 \int e^{-0.8x}dx = 4 \cdot \frac{1}{(-0.8)}e^{-0.8x} + C = -5e^{-0.8x} + C. \]

To determine \( C \) use that \( 0 = F(0) = -5 + C \). Hence \( C = 5 \) and \( F(x) = -5e^{-0.8x} + 5. \)
More indefinite integrals (verify by taking the derivative of the right-hand sides):

\[
\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C,
\]

\[
\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C,
\]

\[
\int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + C,
\]

\[
\int \sec(ax) \tan(ax) \, dx = \frac{1}{a} \sec(ax) + C,
\]

\[
\int \csc^2(ax) \, dx = -\frac{1}{a} \cot(ax) + C,
\]

\[
\int \csc(ax) \cot(ax) \, dx = -\frac{1}{a} \csc(ax) + C,
\]

\[
\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C,
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C,
\]

\[
\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C,
\]

Verify the following integral:

\[
\int \frac{dx}{x^2 + 2x + 2} = \tan^{-1} (x + 1) + C.
\]
A payload is dropped from a hot air balloon that is at an elevation of 500 meters but descending at 10 meters per second. Using constant acceleration due to gravity, $a = -9.8\text{ m/s}^2$, find the velocity and position functions for the payload. How many seconds does it take to hit the ground? At what speed does it hit? Draw the graph of the velocity function. What happens to the graph eventually?

In general, let $a = -g$. Then $v = -gt + v_0$ where $v_0$ is initial velocity. Then $s = -\frac{1}{2}gt^2 + v_0 t + s_0$ where $s_0$ is initial position.

Suppose you are on a planet that experiences $g = 20$ meters per square second. If an object is dropped from a height $h$ and takes 4 seconds to hit the ground, what was height $h$? At what speed does it hit the ground?