Math 252  Integration Techniques and Numerical Methods  Activity 6

This activity is worth 10 points of course credit. See tentative calendar for due dates. Late activities are accepted at the discretion of your recitation instructor and a penalty may be imposed.

(1) Below are several integrals. On each, circle what you think is the optimal integration technique and solve the integral.

   Note: The grader may just grade a random sampling of these...

(a) \( \int \frac{2}{x^2 - 1} \, dx \)

   Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int \frac{2}{x^2 - 1} \, dx = \]

(b) \( \int \frac{2x}{x^2 - 1} \, dx \)

   Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int \frac{2x}{x^2 - 1} \, dx = \]
(c) \[ \int \frac{2x^2}{\sqrt{1-x^2}} \, dx \]

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int \frac{2x^2}{\sqrt{1-x^2}} \, dx = \]

(d) \[ \int \frac{2}{(9+x^2)^2} \, dx \]

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int \frac{2}{(9+x^2)^2} \, dx = \]
(e) \( \int x^2 \ln(x) \, dx \)

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[
\int x^2 \ln(x) \, dx = 
\]


(f) \( \int \frac{\ln(x)}{x} \, dx \)

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[
\int \frac{\ln(x)}{x} \, dx = 
\]
(g) \[ \int e^x \cos(x) \, dx \]

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int e^x \cos(x) \, dx = \]

(h) \[ \int \frac{2}{x^3 - x^2 + x - 1} \, dx \]

Basic U-substitution  Integration by Parts  Trig. Substitution  Partial Fractions

\[ \int \frac{2}{x^3 - x^2 + x - 1} \, dx = \]
(2) Follow these steps to integrate $\int \frac{x^4 + 1}{x^3 - x} \, dx$

(a) Using polynomial long division express $\frac{x^4 + 1}{x^3 - x}$ in the form $\ell(x) + \frac{p(x)}{x^3 - x}$ where $\ell(x)$ is a linear function and $p(x)$ is a polynomial of degree at most 2.

\[
\frac{x^4 + 1}{x^3 - x} = \]

(b) Perform a partial fraction decomposition on $\frac{p(x)}{x^3 - x}$.

\[
\frac{p(x)}{x^3 - x} = \]

(c) Now evaluate the integral!

\[
\int \frac{x^4 + 1}{x^3 - x} \, dx = \]
(3) Let \( f(x) \) be continuous on \([a, b]\). In this problem we will derive the Trapezoid Rule. We will use a regular partition of \([a, b]\) into \(n\) subintervals, but instead of rectangles as in Riemann sums, we will use trapezoids (so linear approximations instead of constant approximations) to estimate the net area under the curve.

(a) Let \( \Delta x = (b-a)/n \) and \( x_i = a + i\Delta x \) for \( i = 0, 1, 2, ..., n \). Draw a trapezoid in the first quadrant with vertices at \((x_{i-1}, 0), (x_i, 0), (x_i, f(x_i)), \) and \((x_{i-1}, f(x_{i-1}))\). Determine the net area of this trapezoid, which we will denote \( A_i \), and express it in terms of \( \Delta x \).

\[
A_i =
\]

(b) Now write down the Trapezoid Rule.

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^{n} \]

(c) Use the Trapezoid Rule to approximate \( \int_0^4 x^3 \, dx \) with \( n = 8 \).

\[
\int_0^4 x^3 \, dx \approx
\]

(d) Notice that we can do this integral by hand. Determine the error in the above approximation in absolute and percent terms.
(4) By using “little parabolic arcs” that are interpolated to pass through the graph of the function at each of the endpoints along 2 successive subintervals, one can derive Simpson’s Rule for an even number of subintervals:

\[
\int_{a}^{b} f(x) \, dx \approx \frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n) \right).
\]

(a) Use Simpson’s rule to approximate \( \int_{0}^{4} x^3 \, dx \) with \( n = 8 \).

\[
\int_{0}^{4} x^3 \, dx \approx \]

(b) Determine the error in the above approximation in absolute and percent terms. Do you think this is a coincidence?
(5) The antiderivative of $y = e^{-x^2/2}$ (graphed below) is not expressible in terms of elementary functions. But we can use numerical methods to estimate a definite integral such as $\int_0^1 e^{-x^2/2} \, dx$.

(a) Estimate $\int_0^1 e^{-x^2/2} \, dx$ using the Trapezoid Rule with $n = 50$. Report your estimate to 6 decimal places. You should use technology to evaluate the sum (I would suggest www.wolframalpha.com where if you type ”sum of f(i) from i=1 to i=50” the value of $\sum_{i=1}^{50} f(i)$ is computed for you!)

$$\int_0^1 e^{-x^2/2} \, dx \approx$$
(b) Estimate $\int_0^1 e^{-x^2/2} \, dx$ using Simpson’s Rule with $n = 10$. Report your estimate to 6 decimal places. You are encouraged to use technology to help in the calculation.

$$\int_0^1 e^{-x^2/2} \, dx \approx$$
(c) Consider the following facts (we skip their justifications):

- The absolute error in using the Trapezoid Rule to approximate \( \int_{a}^{b} f(x) \, dx \)
  can be no more than \( \frac{K(b - a)^3}{12n^2} \) where \( f(x) \) satisfies \( |f''(x)| \leq K \) on \([a, b]\).

- The absolute error in using Simpson’s Rule to approximate \( \int_{a}^{b} f(x) \, dx \)
  can be no more than \( \frac{M(b - a)^5}{180n^4} \) where \( f(x) \) satisfies \( |f^{(4)}(x)| \leq M \) on \([a, b]\).

Given that \( |(x^2 - 1)e^{-x^2/2}| \leq 1 \) and \( |(x^4 - 6x^2 + 3)e^{-x^2/2}| \leq 3 \) on \([0, 1]\) figure out the maximum possible errors in our estimates in parts (a) and (b). Which is closer to the actual area?