This activity is worth 2% of course credit. See tentative calendar for due dates.

(1) Let’s consider a classic distance problem:

What is the distance \( d \) between a point \( P (x_0, y_0, z_0) \) in \( \mathbb{R}^3 \) and a line \( \ell \) in \( \mathbb{R}^3 \) given by the vector-valued function \( \mathbf{r}(t) = (at + x_1, bt + y_1, ct + z_1) \)?

(a) Draw a line labeled \( \ell \) and a point labeled \( P \) not on the line:

(b) Label a point on the line as \( \mathbf{r}_0 = \mathbf{r}(0) \). Draw the vector \( \mathbf{u} \) from \( \mathbf{r}_0 \) to \( P \).
(c) Calculate the components of \( \mathbf{u} \).

\[
\mathbf{u} =
\]

(d) Determine a direction vector \( \mathbf{v} \) for the line \( \ell \).

\[
\mathbf{v} =
\]

(e) Calculate the projection \( \mathbf{p} = \text{Proj}_\mathbf{v}(\mathbf{u}) \) (of \( \mathbf{u} \) onto \( \mathbf{v} \)).

\[
\mathbf{p} =
\]

(f) Draw the line \( \ell \) again and vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{p} \):
(g) Now finish the problem...

Big Hint:

Express your answer in terms of $k = \frac{a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)}{a^2 + b^2 + c^2}$.

$$d =$$

(h) Use your formula (or follow the steps that derive the formula) to find the distance $d$ from $(1, 2, 3)$ to the line given by $\mathbf{r}(t) = (1 - t, 1 + t, 2t)$.

$$d =$$
(2) We work through an alternate solution to the same problem as in (1):

(a) Define a single-variable function $f(t)$ that is equal to the square of the distance from P $(x_0, y_0, z_0)$ to an arbitrary point on the line $\ell$ given by the vector-valued function $r(t) = \langle at + x_1, bt + y_1, ct + z_1 \rangle$:

$$f(t) =$$

(b) Use the method of optimization (from single-variable calculus) to find and justify the location of the global minimum in $f(t)$:

$$f(t) \text{ has a global minimum at } t_0 =$$

(c) Then the distance is $d = \sqrt{f(t_0)}$. Find $d$.

$$d =$$

(d) Is the answer the same as in (1)?
(3) Come up with yet another way to find the distance using that $|\mathbf{u} \times \mathbf{v}|$ is the area of a certain parallelogram adjacent to the line, with $|\mathbf{v}|$ equal to the base of the parallelogram in the line. Draw a picture to illustrate. Does it produce the same answer as in (1) and (2)?

Hint: The area of a parallelogram is $A = bh$ where $b$ is the base length (any side) and $h$ is the corresponding height (perpendicular to the side chosen as the base).
(4) Now we look at finding the distance from a point to a sphere.

(a) What is the distance \(d\) between a line \(\ell\) given by the vector-valued function 
\[
\mathbf{r}(t) = \langle at + x_1, bt + y_1, ct + z_1 \rangle
\]
and a sphere \(S\) given by the equation 
\[
(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2?
\]

Hint: Find a way to use the result from the previous problems. But also be aware that if the line touches or pierces the sphere then the distance must be taken as zero. One nice way to deal with this is to use the “maximum function” with 0 as the second argument.

\[
d = \]

(b) Find the distance \(d\) from the line \(\mathbf{r}(t) = \langle 1-t, 5+t, t \rangle\) to the sphere \(S\) given by 
\[
(x+2)^2 + (y-1)^2 + (z+5)^2 = 4.
\]

\[
d = \]