(1) (10 pts) Match 4 of 5 of the following first order ordinary differential equations to the (approximate) direction fields (a), (b), (c), (d) shown below. Just write the letter of the direction field next to the o.d.e. to which it corresponds.

\[
\begin{align*}
(d) & \quad y' = \sin(2x) \\
(c) & \quad y' = \frac{y}{x} \\
(a) & \quad y' = x^2 - y \\
(b) & \quad y' = xy - y \\
\end{align*}
\]

(a) $y' = \sin(2x)$

(b) $y' = \frac{2x + 1}{2y}$
(2) (8 pts) Below is the (approximate) slope field of some o.d.e. \( y' = f(t, y) \) on \([-0.5, 4.5]\).

Within this plot, draw the solution to the following initial value problem (IVP):

\[ y' = f(t, y), \quad y(1) = 2 \text{ on the interval } [-0.5, 4.5]. \]
(3) (10 pts) Circle all terms that apply for each first order ordinary differential equation.

(a) \( y' + 2y = e^x \):
- \text{linear}
- \text{homogeneous}
- \text{separable}

(b) \( 2xy' + 2y = x^2 \):
- \text{linear}
- \text{homogeneous}
- \text{separable}

(c) \( y' = 3x^2y \):
- \text{linear}
- \text{homogeneous}
- \text{separable}

(d) \( y' = xy^2 + x \):
- \text{linear}
- \text{homogeneous}
- \text{separable}

(e) \( x \sin(y)y' = \cos(y) \):
- \text{linear}
- \text{homogeneous}
- \text{separable}

(4) (8 pts) Suppose we know that \( y' = 2y \) and \( y(0) = 3 \). Find \( y(\ln(5)) \). Put the final answer in the box provided.

\[ y(\ln(5)) = 75 \]

\[ y' = 2y \quad \implies \quad y = Ce^{2x} \]

\( y(0) = 3 \quad \text{forces} \quad C = 3 \).

So \( y = 3e^{2x} \), and

\[ y(\ln 5) = 3e^{2\ln 5} = 3(5) = 75. \]
(5) \(10\) pts) Solve the non-linear IVP \(2(1+x)y' + y^2 = 1, y(1) = 0\). Justify your answer. An answer without proper support is worth zero credit. For full credit find an explicit solution. Put the final answer in the box provided.

Hint: The o.d.e. is separable.

\[
y = 1 - \frac{y}{2 + 1 + x} = \frac{x - 1}{x + 3}
\]

\[
2(1 + x) y' = 1 - y^2
\]

\[
\frac{2}{1 - y^2} y' = \frac{1}{1 + x}
\]

\[
\int \frac{1 + \frac{1}{1 - y}}{1 + y} dy = \ln |1 - y| + C
\]

\[
-\ln |1 - y| + \ln |1 + y| = \ln |1 + x| + C
\]

\[
\ln \left| \frac{1 + y}{1 - y} \right| = \ln |1 + x| + C
\]

\[
\frac{1 + y}{1 - y} = C(1 + x)
\]

\[
-1 + \frac{2}{1 - y} = C(1 + x)
\]

\[
\frac{2}{1 - y} = 1 + C(1 + x) \Rightarrow 1 - \frac{2}{1 + C(1 + x)} = y
\]

So

\[
y = 1 - \frac{2}{1 + C(1 + x)}
\]

\[
y(1) = 0 \Rightarrow 1 = \frac{2}{1 + 2C} \Rightarrow C = \frac{1}{2}
\]
(6) Consider the following ordinary differential equation:

\[
\left( \frac{\ln(y)}{x} + 2y \right) \, dx + \left( \frac{1}{y} + \frac{1}{y} + x \right) \, dy = 0.
\]

(a) (4 pts) Show that \( \mu(x) = x \) is an integrating factor.

\[
\frac{(\ln(y) + 2xy)}{M} \, dx + \left( \frac{1}{y} + \frac{x}{y} + x^2 \right) \, dy = 0
\]

\[M \land N \land M_y = N_x\]

(b) (4 pts) Find the general solution for this o.d.e. Put the final answer in the box provided.

\[x\ln(y) + x^2y + y = C,\]

\[F = x\ln(y) + x^2y + g(y) \land g'(y) = 1, \text{ so}
\]

set \[g(y) = y,\]

(c) (2 pts) Find the solution where \( y(1) = 1 \). Put the final answer in the box provided.

\[0 + 1 + 1 = C\]

\[x\ln(y) + x^2y + y = 2, \quad C = 2.\]
(7) (10 pts) Find the general solution of \( ty' + 2y = 5t^3 \) for \( y(t) \). Justify your answer. An answer without proper support is worth zero credit. Put the final answer in the box provided.

\[
y = t^3 + \frac{c}{t^2},
\]

\[
y' + \frac{2}{t} y = 5t^2
\]

\[
\int \frac{2}{t} \, dt = 2 \ln t + 1 = t^2.
\]

\[
\mu = e^{2 \ln t + 1} = t^2.
\]

\[
t^2 y' + 2t y = 5t^4
\]

\[
(t^2 y)' = 5t^4.
\]

\[
t^2 y = t^5 + c
\]

\[
y = t^3 + \frac{c}{t^2}.
\]
(8) Consider the following scenario:

A tank contains 20 liters of pure water. Then a process begins where water containing 50 grams per liter of salt is pumped into the tank at a rate of 2 liters per minute. The well-mixed tank is drained at the same rate.

(a) (5 pts) Let \( y(t) \) be the amount of salt in grams at \( t \) minutes (after the process begins). Setup a first order initial value problem (IVP) to model this system. Put the final answer in the box provided.

\[
y(0) = 0, \quad y' = 100 - \frac{1}{10} y, \quad y(0) = 0.
\]

\[
y' = \text{rate in} - \text{rate out}
\]

\[
y' = 100 - \frac{y}{20} \cdot 2
\]

(b) (5 pts) Solve the IVP derived in part (a). Put the final answer in the box provided.

\[
y(t) = 1000 - 1000 e^{-\frac{t}{10}}
\]

If you cannot figure out part (a) then you may solve \( y' + \frac{1}{t+20} y = 100, \ y(0) = 0 \) for up to 3 pts (of the 5 pts).

\[
-10 \int_{0}^{t} \left| 100 - \frac{1}{10} y \right| = t + C
\]

\[
-10 \int_{0}^{t} \left| 100 - \frac{1}{10} y \right| = -\frac{1}{10} t + C
\]

\[
100 - \frac{1}{10} y = C e^{-\frac{t}{10}}
\]

\[
y = 1000 + C e^{-\frac{t}{10}} \quad \& \quad y(0) = 0 \implies C = -1000
\]
(9) (10 pts) What is the largest interval on which the IVP \((x-1)(x-4)y' = 6y, y(5) = 1\) has a unique solution? You are not being asked to solve this o.d.e. Put the final answer in the box provided.

\[
y' = \frac{6y}{(x-1)(x-4)}
\]

\[
y(5) = 1 \quad \Rightarrow \quad x \neq 1, x \neq 4
\]

\[
(4, \infty)
\]

(10) (10 pts) Find the general solution to the second order linear homogeneous constant coefficient o.d.e. \(y'' + y' = 2y\). Put the final answer in the box provided.

\[
y'' + y' - 2y = 0
\]

\[
\lambda^2 + \lambda - 2 = 0
\]

\[
(\lambda - 1)(\lambda + 2) = 0
\]

\[
\lambda = -2, 1
\]

\[
y = C_1 e^{-2x} + C_2 e^x
\]
(11) (4 pts) Did you write your name on the first page? If not, please do so now.

NOTE: YOU WILL GET THESE POINTS IF AND ONLY IF YOU WRITE YOUR NAME ON THE FIRST PAGE.