**Math 256 - Applied Differential Equations** **Notes**

**Basic Definitions and Concepts**

A differential equation is an equation that involves one or more of the derivatives (first derivative, second derivative, etc.) of an unknown function.

For example, consider the equation $xy''' + y' = e^x y$.

The order of a differential equation is the order of the of the highest derivative that it contains. For example, the differential equation above is a third order differential equation.

An ordinary differential equation only involves an unknown function of a single variable, while a partial differential equation involves partial derivatives of a function of two or more variables.

This class is just about ordinary differential equations.

A standard abbreviation for an ordinary differential equation is "o.d.e." or "ode."

A solution of an o.d.e. is a function that satisfies the o.d.e. on some open interval.

For example: $y = \sin(2x)$ satisfies $y'' = -4y$ since $y'' = -4\sin(2x) = -4y$.

The graph of a solution to an o.d.e. is a solution curve.

An integral curve of a differential equation is the graph of a relation (not necessarily a function) such that any segment of the curve that is the graph of a function on some open interval, is a solution to the o.d.e.

While all solution curves are integral curves, not all integral curves are solution curves. For example, $x^2 + y^2 = 1$ can be an integral curve but not a solution curve, because it's not a function.

Consider the equation $y^3 y' = -x$. Let's verify that $y = \pm\sqrt[4]{1-2x^2}$ are solutions to this o.d.e. by taking derivatives:

$$y^3 y' = \left(\pm\sqrt[4]{1-2x^2}\right)^3 \left(\frac{-4x}{\pm 4\sqrt[4]{(1-2x^2)^3}}\right) = y^3 \frac{-x}{y^3} = -x.$$

Consequently the graph of $2x^2 + y^4 = 1$ is an integral curve, but not a solution curve.

Sometimes *integrating both sides* of an o.d.e. yields all its solutions. Consider the o.d.e. $y^{(4)} = e^{x/2}$ where $y^{(4)}$ denotes the 4th derivative of $y$ (not the 4th power).

By integrating both sides we get $y^{(3)} = 2e^{x/2} + c_1$ where $c_1$ is an arbitrary constant. By doing it again, $y'' = 4e^{x/2} + c_1 x + c_2$ where $c_1, c_2$ are arbitrary constants. After doing it twice more,

$$y = 16e^{x/2} + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3 x + c_4$$

where $c_1, c_2, c_3, c_4$ are arbitrary constants.

Since $c_1$ and $c_2$ are arbitrary so are $c_1/6$ and $c_2/2$. So we may express all solutions as

$$y = 16e^{x/2} + c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

where $c_1, c_2, c_3, c_4$ are arbitrary constants.

We can call such an expression the general solution of an o.d.e.

We will often replace an expression involving an arbitrary constant with an arbitrary constant (generally without picking a new symbol – instead just relabeling expressions like $c_1/6$ by $c_1$ when $c_1/6$ is arbitrary).

Let's find a solution to the o.d.e. $2yy' = (1 + x^2)^{-1}$ such that $y(0) = 1$. Such a problem is called an initial value problem which is abbreviated IVP.

By integrating both sides we get $y^2 = \tan^{-1}(x) + c$. Then $y = \pm\sqrt{\tan^{-1}(x) + c}$. Since $y(0) = 1$ we have $y = \sqrt{\tan^{-1}(x) + 1}$.

An initial value problem can also be posed for a higher order o.d.e. Consider the following one:

$$y'' + 2y' - 3y = x, \;\; y(1) = 5, \;\; y'(1) = 2.$$

The initial conditions (in this example $y(1) = 5$ and $y'(1) = 2$) of an $n$th order IVP are values for $y, y', ..., y^{(n-1)}$ at some point $x_0$.

Near the Earth's surface the height $h(t)$ of an object falling under the influence of gravity satisfies $h'' = -g$ where $g$ is the constant acceleration due to gravity. Then the solution to the IVP

$$h'' = -g, \;\; h(0) = h_0, \;\; h'(0) = v_0$$

corresponds to the motion of an object that is at an initial height of $h_0$ with an initial vertical velocity of $v_0$.

(ELFY) "Exercise left for you:" Show that $h(t) = -\frac{g}{2}t^2 + v_0 t + h_0$.

**Direction/Slope Fields**

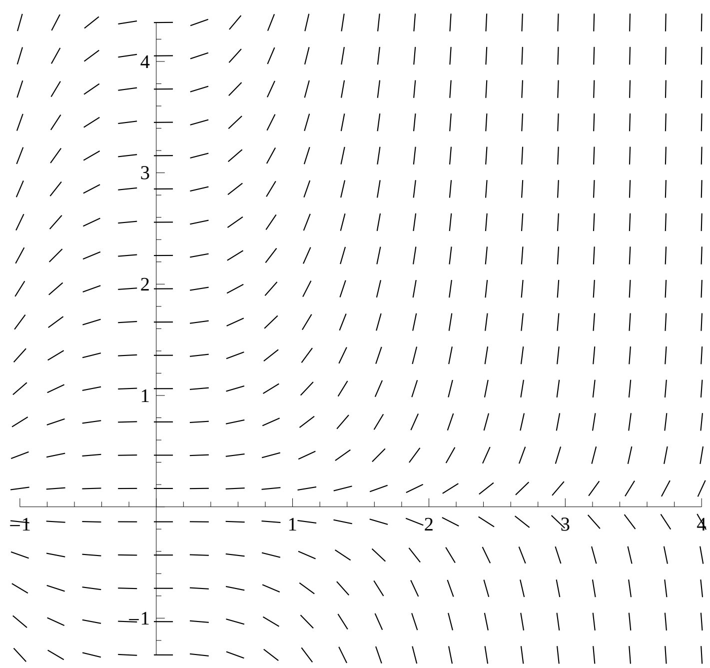
Consider a first order o.d.e. of the form $y' = f(x, y)$ where $f$ is real-valued function of two variables.

If $f$ is defined on a region $R$ in the $xy$-plane then a direction field (aka slope field) for the o.d.e. $y' = f(x, y)$ is a graph consisting of short line segments with slope determined by $f(x, y)$ at each $(x, y)$ in $R$.

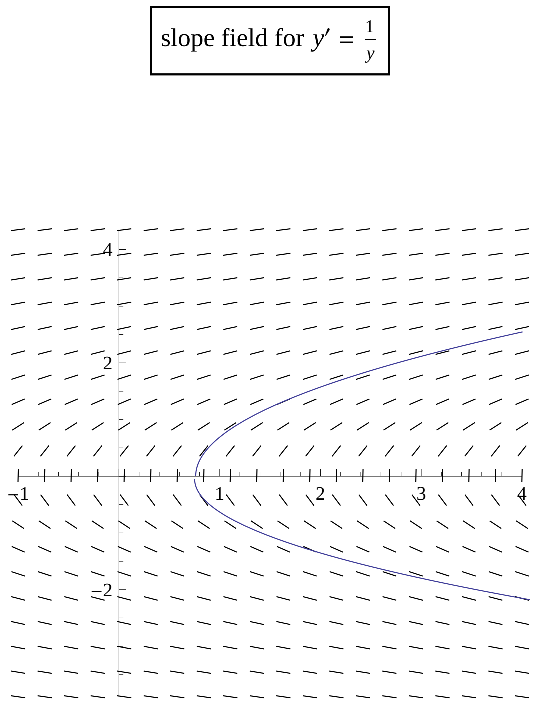
As a practical matter, we draw these line segments at ENOUGH points in $R$ to get an a representation of the direction field (an approximation).

Here is some of the (approximate) slope field determined by $y' = x^2 y$ using 400 sample points:

slope field for $y' = x^2\, y$



Here is another (approximate) slope field with an integral curve going through it.



**Linear First Order ODEs**

A first order o.d.e. is <u>linear</u> if it can be written in the form $y' + p(x)y = f(x)$; otherwise it's called <u>non-linear</u>.

If $f(x) = 0$ then the linear equation (above) is called <u>homogeneous</u>; otherwise it's called <u>non-homogeneous</u>.

Obviously $y = 0$ is a solution of the homogeneous equation. It's called the <u>trivial</u> solution. Any non-zero solution is called <u>non-trivial</u>.

Consider the o.d.e. $xy' + \ln(x)y = x\sin(x)$. Since we can rewrite this as $y' + \frac{\ln(x)}{x}y = \sin(x)$, it's linear. Clearly it's non-homogeneous.

Consider the o.d.e. $xy' + \ln(y) = 5$. Due to the term $\ln(y)$ we cannot rewrite this equation in the form $y' + p(x)y = f(x)$, so it's non-linear.

Consider the o.d.e. $yy' = e^x$. Due to the product $yy'$ we cannot rewrite this equation in the form $y' + p(x)y = f(x)$, so it's non-linear.

Consider the o.d.e. $\sec(x)y' = xy$. Since we can rewrite this as $y' + (-x\cos(x))y = 0$, it's linear. Clearly it's also homogeneous.

Consider the o.d.e. $y' + x^2y = xy^2$. Due to the term $xy^2$ we cannot rewrite this equation in the form $y' + p(x)y = f(x)$, so it's non-linear.

Suppose we are working with a linear o.d.e. $y' + p(x)y = f(x)$. Assume $p(x), f(x)$ are continuous on $(a, b)$. We will show that then there is a unique function $y = y(x, c)$ such that the following two results hold:

(i) For any real value of $c$, the function of $x$ given by $h_c(x) = y(x, c)$ is a solution to the o.d.e. on $(a, b)$.

(ii) Every solution $g(x)$ of the o.d.e. on $(a, b)$ satisfies $g(x) = h_c(x)$ for some real value of $c$.

The function described by $y(x, c)$ is the general solution (for $y' + p(x)y = f(x)$ it exists and is unique with the assumption of continuity on $p(x), f(x)$ on $(a, b)$).

We begin with considering the homogeneous first order o.d.e. $y' + p(x)y = 0$.

**Proposition:** If $p(x)$ is continuous on $(a, b)$ then the general solution to the homogeneous o.d.e. $y' + p(x)y = 0$ is $y = ce^{-P(x)}$ where $P'(x) = p(x)$ on $(a, b)$.

*Proof:* Suppose $y = ce^{-P(x)}$ where $P'(x) = p(x)$ on $(a, b)$. Then

$$y' = ce^{-P(x)}(-P'(x)) = -p(x)ce^{-P(x)} = -p(x)y$$

on $(a, b)$. Hence $y' + p(x)y = 0$ on $(a, b)$. Hence $y = ce^{-P(x)}$ where $P'(x) = p(x)$ on $(a, b)$ is a solution to the o.d.e.

So it only remains to show that every solution is of the form $y = ce^{-P(x)}$ for some constant $c$.

Clearly $c = 0$ shows that the trivial solution is of the form $y = ce^{-P(x)}$. Suppose $y$ is a non-trivial solution on $(a, b)$ (on which $y$ is not zero at any point). Then $y' + p(x)y = 0$ implies that

$$\frac{y'}{y} = -p(x).$$

By integrating both sides, we get $\ln|y| = -P(x) + C$ for some constant $C$. This implies that

$$y = ke^{-P(x)} \text{ where } k = \pm e^C.$$

So we can rewrite this as $y = ce^{-P(x)}$ on $(a, b)$ for some constant $c \neq 0$. This completes the proof $\square$

Examples:

(1) The general solution to the o.d.e. $y' + \cos(x)y = 0$ is $y = ce^{-\sin(x)}$ (on any interval $(a, b)$).

(2) Let's find the general solution to $(x^2 + x)y' = y$ on any open interval not including 0 nor $-1$. This can be rewritten as

$$y' - \frac{1}{x(x+1)}y = 0.$$

With $P(x) = \int -\frac{1}{x(x+1)}dx = \int \left(\frac{1}{x+1} - \frac{1}{x}\right) dx = \ln\left|\frac{x+1}{x}\right|$, the general solution is $y = ce^{-P(x)} = c\left|\frac{x}{x+1}\right|$ where $c$ is a constant. We can drop the absolute value and relabel $c$, so $y = \frac{cx}{x+1}$.

Now let's consider the non-homogeneous first order o.d.e. $y' + p(x)y = f(x)$.

We will call the corresponding homogeneous first-order o.d.e. $y' + p(x)y = 0$ the <u>complementary equation</u>.

Consider a first-order linear differential equation

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

By dividing through by $a_1(x)$ we get the **standard form**:

$$y' + p(x)y = f(x) \qquad (*)$$

The left hand side *almost* looks like the product rule applied to $y(x)$ and some function $\mu(x)$:

$$\frac{d}{dx}[\mu y] = \mu y' + \mu' y.$$

Multiplying both sides of $(*)$ by $\mu$ yields

$$\mu y' + \mu p y = \mu f.$$

Matching the left hand side to the form of the product rule for $\mu y$, we want a function $\mu$ such that

$$\mu' = \mu p.$$

From here we get $\dfrac{\mu'}{\mu} = p$, and by integrating both sides

$$\ln|\mu| = \int p\,dx.$$

So the possibilities for $\mu$ are

$$\mu = \pm e^{\int p\,dx}.$$

These are called **integrating factors**.

Note: The textbook uses an alternate technique (variation of parameters) to start, but eventually gets integrating factors.

Since we only need one we will always choose $\mu = e^{\int p\,dx}$ and set the constant of integration to 0.

With this choice $\mu y' + \mu p y = \mu f$ becomes $[\mu y]' = \mu f$. From here we can integrate both sides and solve for $y$.

If you like memorizing things,

$$y = \frac{1}{\mu}\int \mu f\ dx.$$

Examples:

(1) $y' + \frac{3}{x}y = \frac{4}{x^5}$.

We can use an integrating factor of $\mu(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|} = \pm x^3$.

By multiplying both sides of $y' + \frac{3}{x}y = \frac{4}{x^5}$ by $\mu = x^3$ we get

$$x^3y' + 3x^2y = \frac{4}{x^2}.$$

This equation can be rewritten as

$$(x^3y)' = \frac{4}{x^2}.$$

By integrating both sides,

$$x^3y = -\frac{4}{x} + C.$$

So $y = \frac{C}{x^3} - \frac{4}{x^4}$ is the general solution.

(2) $xy' - 3y = x^4e^x$.

First we rewrite this in the standard form to get $y' - \frac{3}{x}y = x^3e^x$.

So an integrating factor is $\mu(x) = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln|x|} = \pm x^{-3}$. Notice the integrand's sign here.

By multiplying both sides of $y' - \frac{3}{x}y = x^3e^x$ by $\mu = x^{-3}$ we get

$$x^{-3}y' - 3x^{-4}y = e^x.$$

This equation can be rewritten as

$$(x^{-3}y)' = e^x.$$

By integrating both sides,

$$x^{-3}y = e^x + C.$$

So $y = x^3e^x + Cx^3$ is the general solution.

(3) $y' + 2xy = x^3$.

An integrating factor is $\mu = e^{\int 2x\,dx} = e^{x^2}$.

By multiplying both sides of $y' + 2xy = x^3$ by $\mu$ we get

$$e^{x^2}y' + 2xe^{x^2}y = x^3e^{x^2}.$$

This equation can be rewritten as

$$(e^{x^2}y)' = x^3e^{x^2}.$$

By integration by parts,

$$e^{x^2}y = \frac{1}{2}e^{x^2}(x^2 - 1) + C.$$

Hence $y = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}$ is the general solution.

(4) $y' \cos(t) + y \sin(t) = 1$, $y(0) = 2$.

First we rewrite this in the standard form to get $y' + y \tan(t) = \sec(t)$.

So an integrating factor is $e^{\int \tan(t)\,dt} = e^{\ln|\sec(t)|} = \pm \sec(t)$.

By multiplying both sides of $y' + y \tan(t) = \sec(t)$ by $\mu = \sec(t)$ we get

$$\sec(t)y' + \sec(t)\tan(t)y = \sec^2(t).$$

This equation can be rewritten as

$$(\sec(t)y)' = \sec^2(t).$$

By integrating both sides,

$$\sec(t)y = \tan(t) + C.$$

Then $y = \sin(t) + C\cos(t)$ is the general solution. Imposing the initial condition of $y(0) = 2$ yields $C = 2$ and a solution of

$$y = \sin(t) + 2\cos(t).$$

Sometimes you have to leave your answer in an integral form and use numerical integration.

(5) $y' - 2xy = 1$, $y(1) = e$.

An integrating factor is $\mu = e^{\int -2x\,dx} = e^{-x^2}$.

By multiplying both sides by $\mu$ we get

$$e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2}$$

This equation can be rewritten as

$$(e^{-x^2}y)' = e^{-x^2}.$$

But the integral of $e^{-x^2}$ does not have an expression in terms of elementary functions. Looking at the initial condition of $y(1) = 2$ we will use the integral function $F(x) = \displaystyle\int_1^x e^{-t^2}\,dt$ as an antiderivative of $f(x) = e^{-x^2}$.

So we may write that $e^{-x^2}y = \displaystyle\int_1^x e^{-t^2}\,dt + C$. Then the general solution is

$$y = e^{x^2}\int_1^x e^{-t^2}\,dt + Ce^{x^2}.$$

Imposing the initial condition of $y(1) = e$ yields $C = 1$. So the solution is

$$y = e^{x^2}\int_1^x e^{-t^2}\,dt + e^{x^2}.$$

*Mixing problem*:

Jack the immortal vampire retires at 666. He expects his retirement savings to grow at an annualized rate equal to 4 percent of total savings (i.e. money is earned at a rate equal to 0.04 times the money Jack has), while spending $40 + t$ thousands of dollars of his savings per year at $t$ years after retirement.

Let $y(t)$ be the amount of savings Jack has in thousands of dollars at $t$ years after his retirement. Let $N$ represent the amount of savings Jack had accumulated entering retirement. Let's find a first order IVP that $y(t)$ will satisfy.

Here is how we setup such a problem: $y' =$ (rate of money coming in) - (rate of money going out).

So $y' = 0.04y - (40 + t)$, $y(0) = N$.

Let's solve the IVP to find the minimum value of $N$ such that Jack never runs out of money:

We can rewrite this equation as $y' - 0.04y = -(40 + t)$. An integrating factor is $e^{-0.04t}$. So $(e^{-0.04t}y)' = -(40 + t)e^{-0.04t}$.

Integrating both sides, using Integration by Parts on the right-hand side, yields

$$e^{-0.04t}y = 25te^{-0.04t} + 1,625e^{-0.04t} + C.$$

So $y = 25t + 1,625 + Ce^{0.04t}$.

We need $C \geq 0$ for Jack to not out-survive his money! Let's look at the initial condition: $y(0) = N$ so $1,625 + C = N$ or $C = N - 1,625$. So as long as $N \geq 1,625$, Jack is fine. This means Jack needs $1,625$ thousands of dollars (so $\$1,625,000$).

**Proposition** Suppose $p(x)$ and $f(x)$ are continuous on $(a,b)$ and let $y_1$ be a non-trivial solution to the homogeneous o.d.e. $y' + p(x)y = 0$. Then the general solution to $y' + p(x)y = f(x)$ on $(a,b)$ is

$$y = y_1 \left( \int \frac{f(x)}{y_1}\, dx + c \right).$$

Furthermore, if $x_0$ is in $(a,b)$ then on $(a,b)$ the IVP , $y' + p(x)y = f(x)$, $y(x_0) = y_0$ has the unique solution

$$y = y_1 \left( \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\, dt + \frac{y_0}{y_1(x_0)} \right).$$

*Proof*: If $y = y_1 \left( \int \frac{f(x)}{y_1}\, dx + c \right)$ then

$$y' = y_1' \left( \int \frac{f(x)}{y_1}\, dx + c \right) + y_1 \frac{f(x)}{y_1} = -p(x) y_1 \left( \int \frac{f(x)}{y_1}\, dx + c \right) + f(x) = -p(x)y + f(x).$$

So if $y = y_1 \left( \int \frac{f(x)}{y_1}\, dx + c \right)$ then $y' + p(x)y = f(x)$.

Now suppose that $y$ is a solution to $y' + p(x)y = f(x)$. Then $y' = -p(x)y + f(x)$ and $y_1' = -p(x)y_1$. As we already know $y_1$ is never zero on $(a,b)$. Let $u = y/y_1$. Then

$$u' = \frac{y' y_1 - y y_1'}{y_1^2}.$$

By substitution,

$$u' = \frac{(-p(x)y + f(x))y_1 + p(x) y y_1}{y_1^2} = \frac{-p(x)y + f(x) + p(x)y}{y_1} = \frac{f(x)}{y_1}.$$

Hence $u = \left( \int \frac{f(x)}{y_1}\, dx + c \right)$ and therefore $y = u y_1 = y_1 \left( \int \frac{f(x)}{y_1}\, dx + c \right)$.

Now consider the IVP. For convenience, choose an antiderivative of $f(x)/y_1$ in the general solution such that the antiderivative is zero at $x_0$ to get

$$y = y_1 \left( \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\, dt + c \right).$$

Then $y_0 = y(x_0) = y_1(x_0)c$. So $y$ above is a general solution the IVP if and only if $c = \frac{y_0}{y_1(x_0)}$.

Finally, suppose that $z$ is another solution to the IVP. Then $z' + p(x)z = f(x)$. Then $(z' - y') + p(x)(z - y) = 0$. So $z - y$ is a solution to the complementary equation. So $z - y = k y_1$. Thus

$$z = y + k y_1 = y_1 \left( \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\, dt + c \right) + k y_1 = y_1 \left( \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\, dt + C \right),$$

where $C = c + k$. $z(x_0) = y_0$ if and only if $C = \frac{y_0}{y_1(x_0)}$, which implies that $z = y$ $\square$

**Separable Equations**

A first-order differential equation is separable if it can be written in the following form:

$$h(y)y' = g(x).$$

From this form, integrate both sides with respect to the independent variable to solve for the general solution.

Example: Consider the non-linear first order separable o.d.e. $(x^2+1)y' - \sqrt{y} = 0$. Find the general solution, and then find a solution for the IVP with the initial condition $y(0) = 1$.

First note that $y = 0$ works. Now let's look for nontrivial solutions.

$$(x^2+1)\frac{dy}{dx} = \sqrt{y} \rightarrow$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{1}{1+x^2} \rightarrow$$

$$\int \frac{1}{\sqrt{y}}\frac{dy}{dx}dx = \int \frac{1}{1+x^2}dx \rightarrow$$

$$\int \frac{dy}{\sqrt{y}} = \tan^{-1}(x) + C \rightarrow$$

$$2\sqrt{y} = \tan^{-1}(x) + C \rightarrow$$

$$y = \left(0.5\tan^{-1}(x) + c\right)^2$$

Let's find a solution for the IVP with $y(0) = 1$. Set $1 = (0+c)^2$ to get $c = 1$ so

$$y = \left(0.5\tan^{-1}(x) + 1\right)^2.$$

Note: $c = -1$ is extraneus as in that case $\sqrt{y} = |0.5\tan^{-1}(x) - 1| = -(0.5\tan^{-1}(x) - 1)$ won't match the left hand side of the equation, which simplifies to $(0.5\tan^{-1}(x) - 1)$.

*Mixing problem*:

A tank contains 10 kg of salt dissolved in 3200 L of water. Brine that contains 0.04 kg of salt per liter enters the tank at a rate of 20 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 10 minutes?

Let $y(t)$ be the amount of salt in kg, at time $t$ minutes, with $y(0) = 10$.

$y'(t) =$ rate in $-$ rate out $= \left(0.04\frac{kg}{L}\right)\left(20\frac{L}{min}\right) - \left(\frac{y(t)}{3200}\frac{kg}{L}\right)\left(20\frac{L}{min}\right).$

So $y'(t) = 0.8 - 0.00625y(t) \rightarrow$

$$\int \frac{dy}{0.8 - 0.00625y} = \int dt = t + C \rightarrow$$

$$-160 \ln |0.8 - 0.00625y| = t + C \rightarrow$$

$$0.8 - 0.00625y = ce^{-\frac{t}{160}} \rightarrow$$

$$y = 128 + ce^{-\frac{t}{160}} \rightarrow$$

$c = -118$ so that $y(0) = 10$.

So $y = 128 - 118e^{-\frac{t}{160}}$ is the solution to the IVP.

After 10 minutes, the amount of salt in the tank is $y(10) \approx 17$ kg. The equilibrium solution (as $t \rightarrow \infty$) is 128 kg.

Let's solve the first order non-linear separable o.d.e. $y' = ry(M - y)$ for $y, t > 0$ where $M, r$ are positive constants. Clearly $y = M$ is a constant solution. Let's look for nonconstant solutions.

$$y' = ry(M - y) \to$$

$$\frac{y'}{y(M - y)} = r \to$$

$$\int \frac{1}{y(M - y)}\, y'\, dt = \int r\, dt \to$$

$$\int \left( \frac{1/M}{M - y} + \frac{1/M}{y} \right) dy = rt \to$$

$$\int \left( \frac{1}{M - y} + \frac{1}{y} \right) dy = Mrt \to$$

$$\ln|y| - \ln|M - y| = Mrt + C \to$$

$$\ln\left| \frac{y}{M - y} \right| = Mrt + C \to$$

$$\frac{y}{M - y} = ke^{Mrt} \text{ (where } k = \pm e^C \neq 0) \ \to$$

$$\frac{1}{M/y - 1} = ke^{Mrt} \to$$

$$M/y - 1 = ce^{-Mrt} \text{ (where } c = 1/k) \ \to$$

$$M/y = 1 + ce^{-Mrt} \to$$

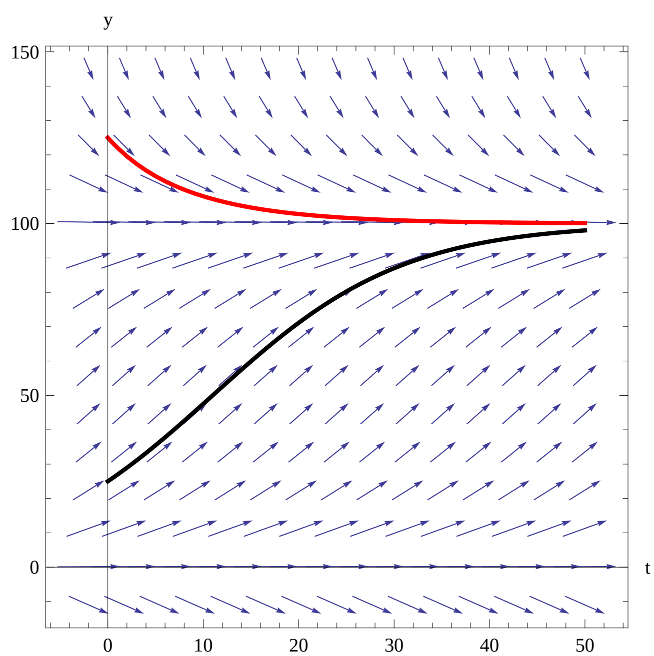
$$y = \frac{M}{1 + ce^{-Mrt}}.$$

Let's compare solutions when $M = 100$ and $r = 0.001$ for initial values $y(0) = 25$ and $y(0) = 125$.

For $y(0) = 25$: $25 = \dfrac{100}{1+c}$ implies that $c = 3$ and $y = \dfrac{100}{1 + 3e^{-0.1t}}$.

For $y(0) = 125$: $125 = \dfrac{100}{1+c}$ implies that $c = -0.2$ and $y = \dfrac{100}{1 - 0.2e^{-0.1t}}$.

Here are the graphs (shown within the slope field):



**Proposition**: Suppose $g(x)$ is continuous on $(a, b)$ and $h(y)$ is continuous on $(c, d)$. Let $G$ and $H$ be antiderivatives of $g$ and $h$ on $(a, b)$ and $(c, d)$ respectively. Let $x_0$ be an arbitrary point in $(a, b)$, let $y_0$ be a point on $(c, d)$ such that $h(y_0) \neq 0$, and define

$$c = H(y_0) - G(x_0).$$

Then there is a function $y(x)$ defined on the open interval $(a_1, b_1)$ that is contained in $(a, b)$ and contains $x_0$ such that

$$H(y) = G(x) + c.$$

In particular, $y$ is a solution to the IVP $h(y)y' = g(x)$, $y(x_0) = y_0$ on $(a_1, b_1)$ (called an implicit solution to the IVP).

To save time we skip the proof, which requires advanced ideas (such as the implicit function theorem).

For convenience, we call a function $y$ that satisfies $H(y) = G(x) + c$ with an arbitrary $c$ an implicit solution. However, for some values of $c$ there may not be a differentiable $y$ that satisfies $H(y) = G(x) + c$ (hence not yielding a solution to the o.d.e. as y' does not exist).

Find implicit solutions for the first order non-linear non-homogeneous separable o.d.e.

$$y' = \frac{2x+1}{6y^5+1}.$$

Then find an implicit solution to the IVP with this o.d.e. and the initial condition $y(2) = 1$.

$$y' = \frac{2x+1}{6y^5+1} \rightarrow$$

$$(6y^5+1)y' = 2x+1 \rightarrow$$

$$\int (6y^5+1)y'\,dx = \int (2x+1)\,dx \rightarrow$$

$$\int (6y^5+1)\,dy = x^2 + x \rightarrow$$

$$y^6 + y = x^2 + x + c.$$

This is an implicit solution to the o.d.e.

$$y' = \frac{2x+1}{6y^5+1}.$$

Imposing the initial condition yields $2 = 6 + c$ so $c = -4$. Hence $y^6 + y = x^2 + x - 4$ is an implicit solution to the IVP.

Now consider an o.d.e. of the form $y' = p(y)g(x)$. It is separable, but rewriting it as

$\frac{1}{p(y)}y' = g(x)$ is not valid if $p(y) = 0$ for some values of $y$.

Let's see how to deal with this issue.

Find solutions to $y' = (xy)^2$. Suppose $y$ is a solution that isn't identically 0. Since $y$ is continuous (as it's differentiable) there must be an interval $(a, b)$ on which $y$ is never zero. On this interval it makes sense to separate the variables and solve:

$$\frac{1}{y^2}y' = x^2 \rightarrow$$

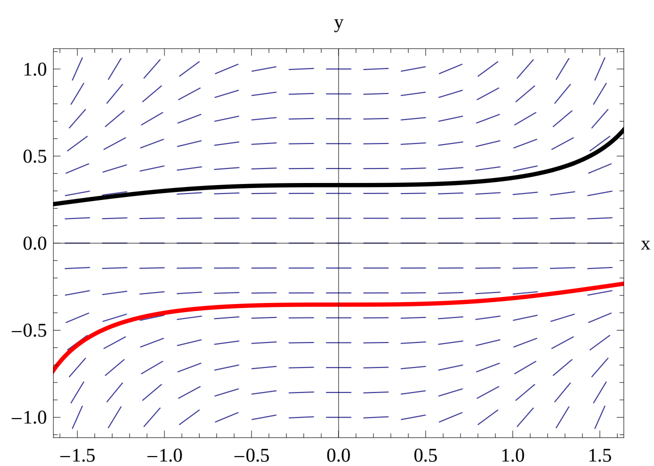
$$\int \frac{1}{y^2}y'\,dx = \int x^2\,dx \rightarrow$$

$$\int \frac{1}{y^2}dy = \frac{1}{3}x^3 \rightarrow$$

$$-\frac{1}{y} = \frac{1}{3}x^3 + k \rightarrow$$

$$y = \frac{3}{c - x^3} \text{ (where } c = -3k\text{).}$$

Here is the slope field with a pair of solutions (on intervals where they are continuous):



Find a solution to $y' = (xy)^2$ where $y(0) = 0.375$ and determine the largest interval on which it is valid.

From the previous page we have that $y = \dfrac{3}{c - x^3}$. Imposing the initial condition yields $0.375 = \dfrac{3}{c}$, so $c = 8$.

So $y = \dfrac{3}{8 - x^3}$. The largest interval containing $x_0 = 0$ on which this function is positive and continuous is $(-\infty, 2)$.