Existence and Uniqueness

The following theorem gives sufficient conditions for the existence and uniqueness of a solution to the IVP for first order nonlinear o.d.e. We omit the proof as it is beyond the scope of this class.

Proposition: If $f(x, y)$ is cont. on the *open rectangle* $R = \{(x, y)|a < x < b, c < y < d\}$ that contains (x_0, y_0) then the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has a solution on some open interval contained in (a, b) that contains x_0. If the partial f_y is also continuous on R then the IVP has a unique solution.

Consider the IVP $y' = 2x\sqrt[3]{y}$, $y(x_0) = y_0$. For what points (x_0, y_0) does this IVP have a solution? For what points (x_0, y_0) does this IVP have a unique solution (on some open interval containing x_0)?

Since $f(x, y) = 2x\sqrt[3]{y}$ is continuous for all points in the xy-plane, there exists a solution to the IVP for any point (x_0, y_0).

$f_y(x, y) = \dfrac{2x}{3\sqrt[3]{y^2}}$ is continuous everywhere except where $y = 0$. So for any (x_0, y_0) such that $y_0 \neq 0$ there is an open rectangle R on which both f and f_y are continuous, and hence there is a unique solution to the IVP on some open interval containing x_0.

$y' = 2x\sqrt[3]{y}$, $y(0) = 0$ has at least two solutions on any interval containing 0:

$$y = 0 \text{ and } y = \sqrt{\frac{8}{27}}x^3.$$

Why didn't we get a unique solution on some interval containing 0?

Because for $f(x, y) = 2x\sqrt[3]{y}$, the partial $f_y = \dfrac{2x}{3\sqrt[3]{y^2}}$ isn't continuous at $y = 0$.

Transformations of Nonlinear o.d.e.

Sometimes a nonlinear first order o.d.e. isn't separable, but can be "transformed" into a separable equation by having $y = uy_1$ where y_1 is a suitably chosen known function, and u satisfies a separable equation.

A special kind of non-linear first order o.d.e. is an equation of the form $y' + p(x)y = f(x)y^r$ where r is any real other than 0 or 1 (the only values of the r that actually make the o.d.e. linear). Such an o.d.e. is called a Bernoulli equation. If $r > 0$ then $y = 0$ is a solution to a Bernoulli equation.

Suppose y_1 is a non-trivial solution to the complementary equation $y' + p(x)y = 0$. Let's substitute $y = uy_1$ where u is a function of x to see how that transforms the equation:

$$y' + p(x)y = f(x)y^r \text{ becomes } u'y_1 + uy_1' + p(x)uy_1 = f(x)(uy_1)^r.$$

Since this can be rewritten as $u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r$ and $y_1' + p(x)y_1 = 0$ we get

$$u'y_1 = f(x)u^r y_1^r.$$

This transformed equation is separable!

Example: Let's solve $y' - 2xy = (xy)^3$. First we solve $y' - 2xy = 0$.

This equation has a non-trivial solution of $y_1 = e^{x^2}$. Then with $y = uy_1$ we have

$$u'e^{x^2} = x^3u^3e^{3x^2}.$$

This becomes

$$\frac{u'}{u^3} = x^3e^{2x^2}.$$

Integrating both sides (using integration by parts) yields

$$-\frac{1}{2u^2} = \frac{x^2}{4}e^{2x^2} - \frac{1}{8}e^{2x^2} + c.$$

By multiplying both sides by -8, factoring and relabeling the arbitrary constant we get

$$\frac{4}{u^2} = (1 - 2x^2)e^{2x^2} + c.$$

Then

$$u = \pm\frac{2}{\sqrt{(1 - 2x^2)e^{2x^2} + c}}.$$

So the general solution is

$$y = uy_1 = \pm\frac{2e^{x^2}}{\sqrt{(1 - 2x^2)e^{2x^2} + c}},$$

or $y = 0$.

Consider a non-linear first order o.d.e. $y' = f(x, y)$. Suppose x, y occur in $f(x, y)$ in such a way that $y' = q(y/x)$ where $q(u)$ is a function of a single variable. In this case the o.d.e. is said to be nonlinear homogeneous.

Consider the non-linear first order o.d.e. $y' = \dfrac{y - x\sec\left(\frac{y}{x}\right)}{x}$. It's (nonlinear) homogeneous since it can be rewritten as

$$y' = \frac{y}{x} - \sec\left(\frac{y}{x}\right),$$

which is of the form $y' = q(y/x)$ for $q(u) = u - \sec(u)$.

Here is the transformation: Substitute $y = ux$ into the equation $y' = q(u)$ to get

$$u'x + u = q(u).$$

This is separable (ignoring the constant solution $u = u_0$ where $q(u_0) = u_0$) as it can be rewritten as

$$\frac{u'}{q(u) - u} = \frac{1}{x} \text{ on an interval not containing } x = 0.$$

Let's apply this to $y' = q(u) = u - \sec(u)$. This yields

$$-\cos(u)u' = \frac{1}{x}.$$

Then

$$-\int \cos(u)\, du = \ln|x|.$$

This becomes

$$-\sin(u) = \ln|x| + c.$$

Hence (relabeling the constant) $u = \sin^{-1}(c - \ln|x|)$ and therefore

$$y = ux = x\sin^{-1}(c - \ln|x|).$$

Now consider the IVP with this o.d.e. and the initial condition $y(e) = 0$. By imposing the initial condition, we get $c = 1$, so the IVP has the solution

$$y = x\sin^{-1}(1 - \ln(x)).$$

Exact Equations

A convenient way to write a first order o.d.e. is with differentials:

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

This is equivalent to $M(x,y) + N(x,y)y' = 0$ for y is a function of x.

We will say that $F(x,y) = c$ is an implicit solution of $M(x,y)\,dx + N(x,y)\,dy = 0$ if every differentiable function $y(x)$ such that $F(x, y(x)) = c$ is a solution of $M(x,y) + N(x,y)y' = 0$.

Let's show that $x^2y + 2xy^3 = c$ is an implicit solution of $2y(x+y^2)\,dx + x(x+6y^2)\,dy = 0$. By implicit differentiation, regarding y as a function of x,

$$2xy + x^2y' + 2y^3 + 6xy^2y' = 0 \rightarrow$$

$$2y(x+y^2) + x(x+6y^2)y' = 0.$$

More generally, $F(x,y) = c$ is an implicit solution of $F_x\,dx + F_y\,dy = 0$ where F_x, F_y denote the x, y partials respectively.

This motivates the following important definition:

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact on an open rectangle R if there is a function $F(x,y)$ such that F_x, F_y are continuous on R, and

$$F_x(x,y) = M(x,y) \text{ and } F_y(x,y) = N(x,y),$$

for all (x,y) in R.

It turns out (we skip the proof) that $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact on an open rectangle R if and only if $M_y = N_x$ (assuming M_y, N_x are continuous on R).

An implicit solution for an exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is straightforward:

Integrate M with respect to x, producing a constant of integration that can be a function of y. Then take the y-partial of the result to determine the constant of integration.

Note: One can do the above with the roles of M and N switched and the roles of x and y switched.

This gives $F(x,y)$ and an implicit solution of $F(x,y) = c$. If possible, find an *explicit solution* $y(x)$ from $F(x,y) = c$.

Determine if the equation is exact, and if so, solve it:

(1) $(2xy + 3x^2)\,dx + (x^2 + \cos(y))\,dy = 0$.

The y-partial of $M(x, y) = 2xy + 3x^2$ is $M_y = 2x$.
The x-partial of $N(x, y) = x^2 + \cos(y)$ is $N_x = 2x$.

Since $M_y = N_x$ everywhere in the xy-plane, this equation is exact (on any open rectangle).

By integrating M with respect to x we get $F(x, y) = x^2y + x^3 + g(y)$. By taking the y-partial we get $F_y = x^2 + g'(y)$. So $g'(y) = \cos(y)$ and hence $g(y) = \sin(y)$ works (we may set the constant of integration to 0).

Therefore, $x^2y + x^3 + \sin(y) = c$ is an implicit solution.

(2) $(y^3 - 1)e^x\,dx + 3y^2(e^x + 1)\,dy = 0$.

The y-partial of $M(x, y) = (y^3 - 1)e^x$ is $M_y = 3y^2e^x$.
The x-partial of $N(x, y) = 3y^2(e^x + 1)$ is $N_x = 3y^2e^x$.

Since $M_y = N_x$ everywhere in the xy-plane, this equation is exact (on any open rectangle).

Integrating M with respect to x yields $F(x, y) = (y^3 - 1)e^x + g(y)$. Then $F_y = 3y^2e^x + g'(y)$. We want $g'(y) = 3y^2$. So set $g(y) = y^3$.

Thus, $(y^3 - 1)e^x + y^3 = c$ is an implicit solution. We can take this one further!

This implicit solution is equivalent to $y^3(e^x+1) = e^x+c$ and therefore, $y = \sqrt[3]{\dfrac{e^x + c}{e^x + 1}}$ is the explicit general solution.

(3) $3x^2y^2\,dx + 4x^3y\,dy = 0$.

The y-partial of $M(x, y) = 3x^2y^2$ is $M_y = 6x^2y$ and the x-partial of $N(x, y) = 4x^3y$ is $N_x = 12x^2y$.

Except for on the x- or y-axis, $M_y \neq N_x$. Hence there is no open rectangle on which this equation is exact.

Now suppose $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, but can be made exact by multiplying both sides by some function $\mu(x,y)$. Such a function is called an integrating factor.

Of course we have seen this terminology already:

The equation $y' + p(x)y = f(x)$ can be rewritten as $dy + (p(x)y - f(x))\,dx = 0$. Then $\mu = e^{\int p(x)\,dx}$ is an integrating factor since $\mu\,dy + (\mu p(x)y - \mu f(x))\,dx = 0$ is exact as

$$\frac{\partial}{\partial x}(\mu) = p(x) = \frac{\partial}{\partial y}(\mu p(x) - \mu f(x)).$$

Consider the o.d.e. $\left(3x + 2 + \frac{2}{y}\right)\,dx + \left(\frac{x^2+x}{y}\right)\,dy = 0.$

With $M = 3x + 2 + \frac{2}{y}$ and $N = \frac{x^2+x}{y}$ we get $M_y = -\frac{2}{y^2}$ and $N_x = \frac{2x+1}{y}$, so the equation isn't exact.

Multiplying both sides by $\mu(x,y) = xy$ we get $(3x^2y + 2xy + 2x)\,dx + (x^3 + x^2)\,dy = 0$.

This equation is exact since for $M = 3x^2y + 2xy + 2x$ and $N = x^3 + x^2$ we get

$$M_y = 3x^2 + 2x = N_x.$$

Now we can solve as before:

Taking the integral of $N = x^3 + x^2$ with respect to y yields $x^3y + x^2y + g(x)$. Taking the x-partial and equating with $M = 3x^2y + 2xy + 2x$ implies that we may set $g(x) = x^2$.

So an implicit solution is $x^3y + x^2y + x^2 = c$ which we can solve for y to get the explicit solution:

$$y = \frac{c - x^2}{x^3 + x^2}.$$

Let's consider how to find an integrating factor in a special case:

Proposition: Let M, N, M_y, N_x be continuous functions on an open rectangle R in the xy-plane. Assume that $p = (M_y - N_x)/N$ is independent of y. Then

$$\mu(x) = \pm e^{\int p(x)\,dx}$$

is an integrating factor of $M\,dx + N\,dy = 0$ on R.

Proof: $(\mu M)_y = \mu_y M + \mu M_y = \mu M_y = \mu(pN + N_x)$.

$$(\mu N)_x = \mu_x N + \mu N_x = \mu p N + \mu N_x = (\mu M)_y.$$

Hence $\mu M\,dx + \mu N\,dy = 0$ is exact $\square$

Example: Let's solve $(2xy + x + 3y)\,dx + (2x + 1)\,dy = 0$. This equation isn't exact as for $M = 2xy + x + 3y$ and $N = 2x + 1$ we have $M_y = 2x + 3$ and $N_x = 2$ which aren't equal.

$p = (M_y - N_x)/N = (2x + 1)/(2x + 1) = 1$ is independent of y. So we get an integrating factor:

$$\mu = e^{\int p(x)\,dx} = e^x.$$

Now consider the equation $e^x(2xy+x+3y)\,dx+e^x(2x+1)\,dy = 0$. Integrating $N = e^x(2x+1)$ with respect to y yields

$$ye^x(2x + 1) + g(x).$$

Differentiating this with respect to x gives

$$y(e^x(2x + 1) + 2e^x) + g'(x) = ye^x(2x + 3) + g'(x).$$

Setting this equal to $e^x(2xy + x + 3y)$ tells us that $g'(x) = xe^x$.

By integrating by parts we can set $g(x) = xe^x - e^x$. So $(2x + 1)e^x y + xe^x - e^x = c$ is an implicit solution of $(2xy + x + 3y)\,dx + (2x + 1)\,dy = 0$.

We can solve for the explicit solution:

$$y = \frac{ce^{-x} - x + 1}{2x + 1}.$$

Consider the IVP $\left(\dfrac{\tan^{-1}(y)}{x} + 2\right) dx + \dfrac{1}{1 + y^2}\,dy = 0$, $y(1) = 0$.

For $M = \dfrac{\tan^{-1}(y)}{x} + 2$ and $N = \dfrac{1}{1 + y^2}$ we have $M_y = \dfrac{1}{x(1 + y^2)}$ and $N_x = 0$.

So the equation isn't exact.

But $(M_y - N_x)/N = \dfrac{1}{x}$ is independent of y. This yields an integrating factor of

$$\mu = \pm e^{\int \frac{1}{x}\,dx} = \pm e^{\ln|x|} = \pm x.$$

So let's solve $\left(\tan^{-1}(y) + 2x\right) dx + \dfrac{x}{1 + y^2}\,dy = 0$.

By integrating $\dfrac{x}{1 + y^2}$ with respect to y we get $x\tan^{-1}(y) + g(x)$. The x-partial of this is $\tan^{-1}(y) + g'(x)$. Setting this equal to $\tan^{-1}(y) + 2x$ tells us we can set $g(x) = x^2$.

Thus $x\tan^{-1}(y) + x^2 = c$ is an implicit solution, which can be solved for the explicit solution:

$$y = \tan\left(\frac{c - x^2}{x}\right).$$

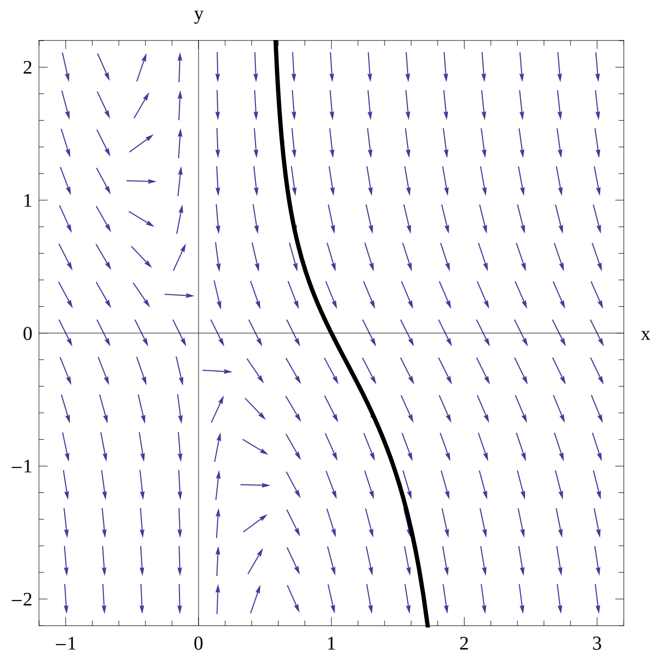
Finally, let's impose the initial condition $y(1) = 0$:

$$0 = \tan(c-1) \to c-1 = 0 \to c = 1.$$

So the IVP has the solution

$$y = \tan\left(\frac{1-x^2}{x}\right).$$

Here is a graph of the solution curve (through the direction field):



Autonomous First Order Equations

Note: This material is not covered in the textbook.

A first order differential equation of the form $\frac{dy}{dt} = f(y)$ is said to be **autonomous**.

The rate function f does not depend on the independent variable t.

Here are a few useful properties of first order autonomous differential equations

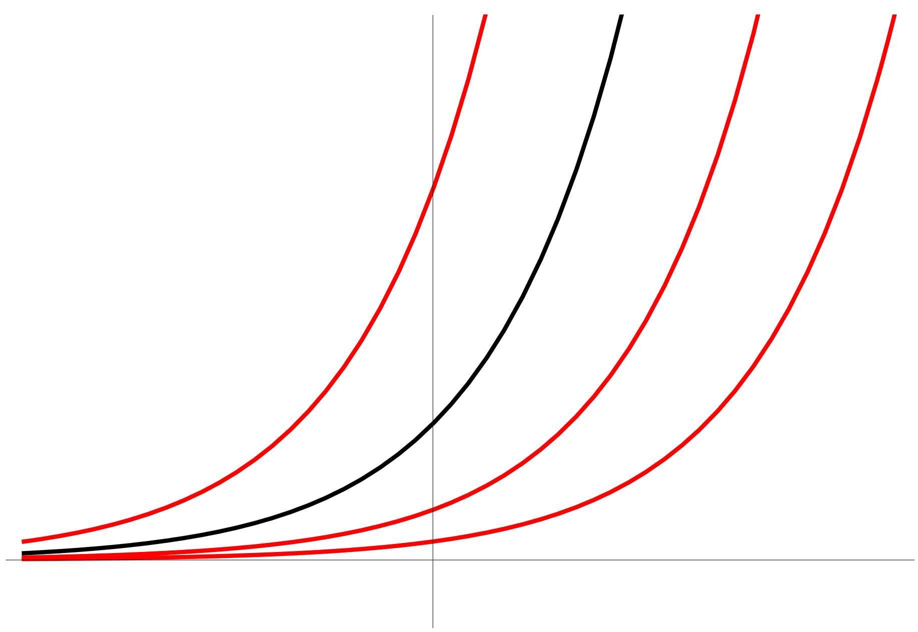
(1) They are separable.
(2) The slopes in the direction field only depend on y.
(3) The general solution is invariant under horizontal translations.

Example: Exponential growth or decay.

Let $q(t)$ be the quantity of something. This common model assumes that the rate of growth in q is proportional to itself:

$$\frac{dq}{dt} = rq.$$

Examples include population growth ($r > 0$) and radioactive decay ($r < 0$). As we have already seen, with an initial condition $q(t_0) = q_0$, we have $q(t) = q_0 e^{r(t-t_0)}$.



Example: Population dynamics.

Let $p(t)$ represent the size of a population at time t. When resources are restricted we can employ a competition factor, $-bp^2$, so that

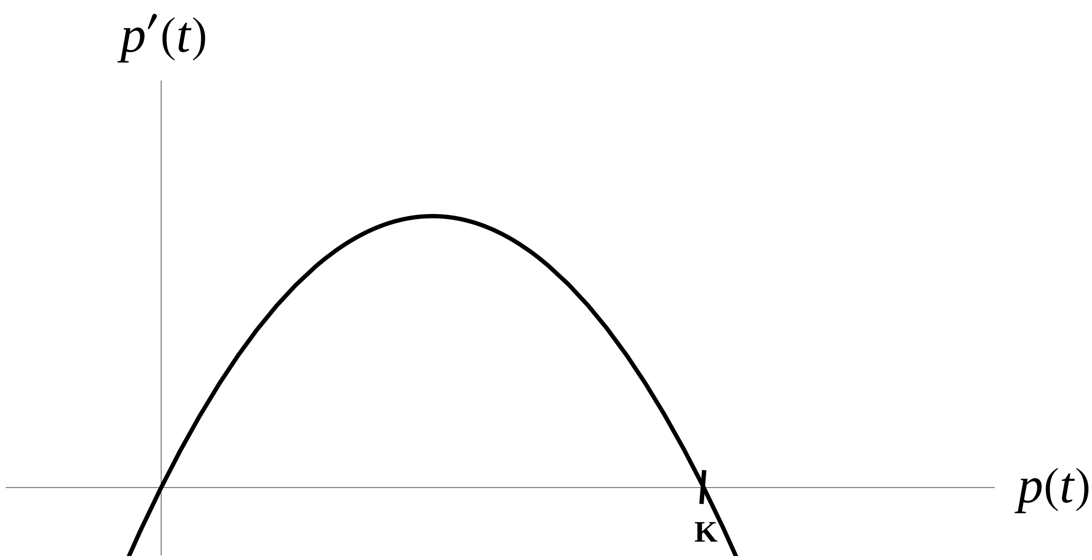
$$\frac{dp}{dt} = rp - bp^2 = rp\left(1 - \frac{p}{K}\right),$$

where $K = \dfrac{r}{b}$.

This equation is called the **logistic equation**. This differential equation has equilibrium solutions at $y = 0$ and $y = K$. We call K the **carrying capacity** and r the **intrinsic growth rate**.

Let's determine the "stability" of these equilibrium values. Assume $r > 0$ and $K > 0$ (negative values would make no sense for this model).

To do this we plot $\dfrac{dp}{dt}$ versus p for $\dfrac{dp}{dt} = rp\left(1 - \dfrac{p}{K}\right)$:

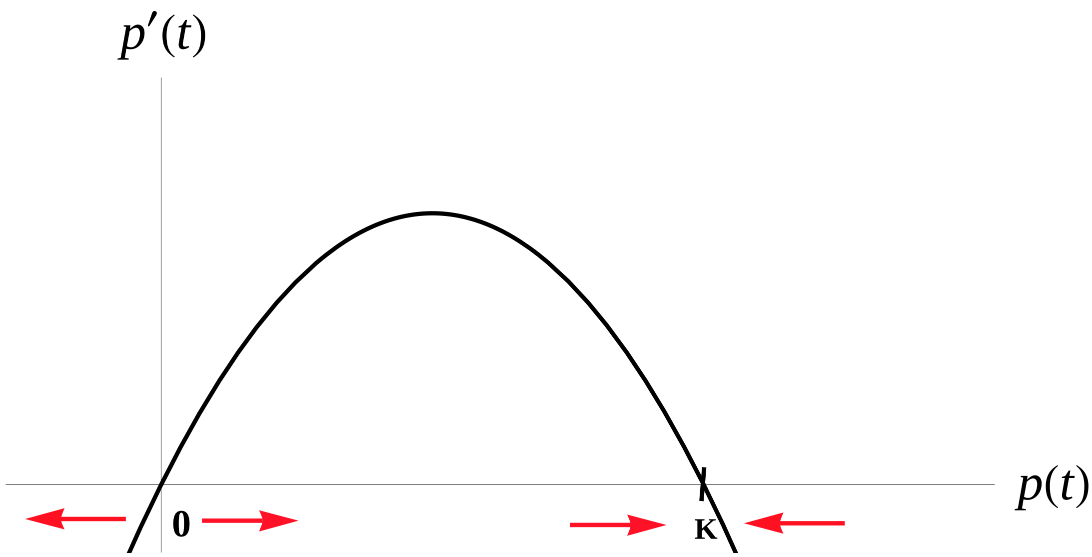


The graph is a parabola with x-intercepts at $x = 0$ and $x = K$.

Note:
If $0 < p < K$ then $\dfrac{dp}{dt} > 0$ and p is increasing. If $p > K$ then $\dfrac{dp}{dt} < 0$ and p is decreasing. So if p is close to K then p is "pushed" towards K. A similar analysis shows that if p is close to 0 it will be "pushed" away from 0.

(While we get $\dfrac{dp}{dt} < 0$ when $p < 0$, it's not relevant to the model as p cannot be negative.)

We can use arrows to indicate how p is "pushed." Here it is with the logistic equation:

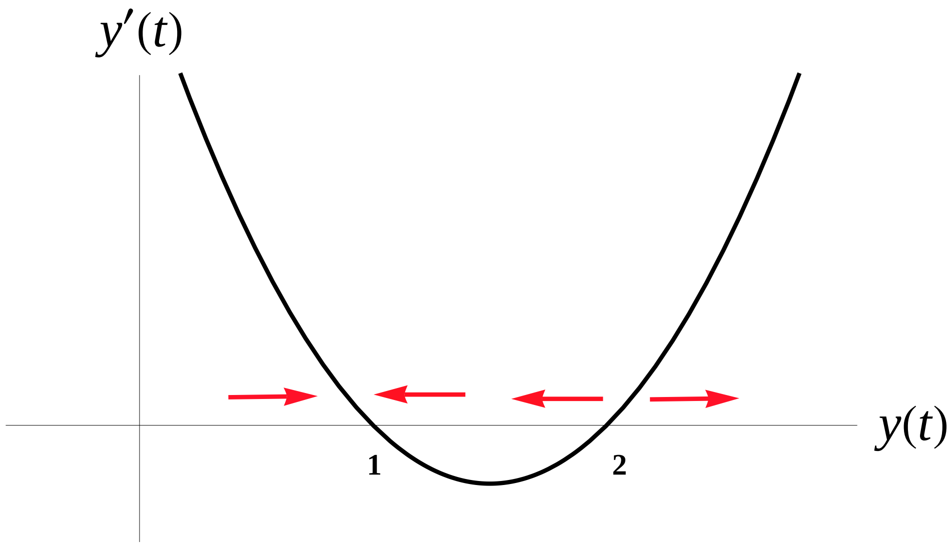


The solution $p = K$ is an **asymptotically stable equilibrium solution**, since small perturbations (changes) are "pushed" back towards K. The solution $p = 0$ is an **unstable solution**, since any perturbations will "push" p away from 0.

With the arrows, the p-axis is called the **phase line**.

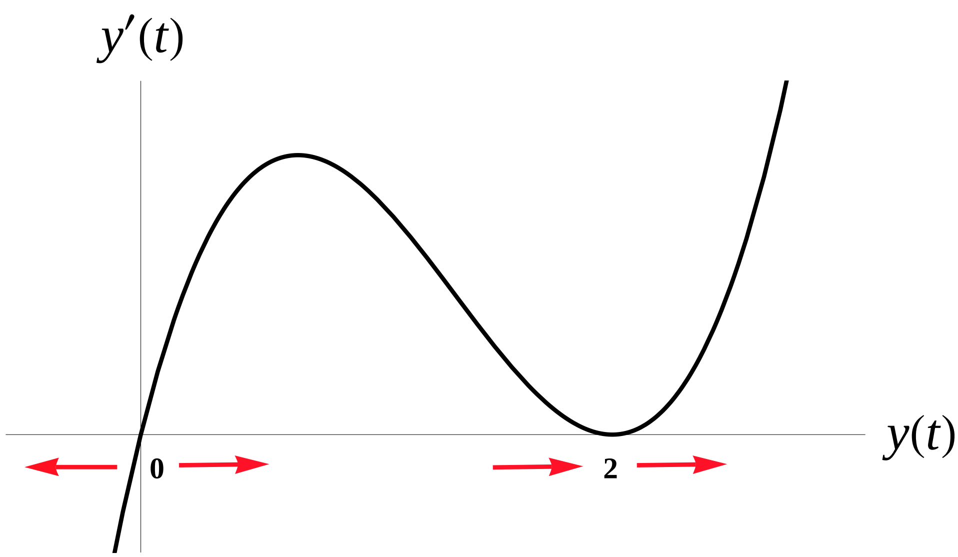
More Examples:

Find and classify the equilibrium solutions of $\frac{dy}{dt} = (1-y)(2-y)$.



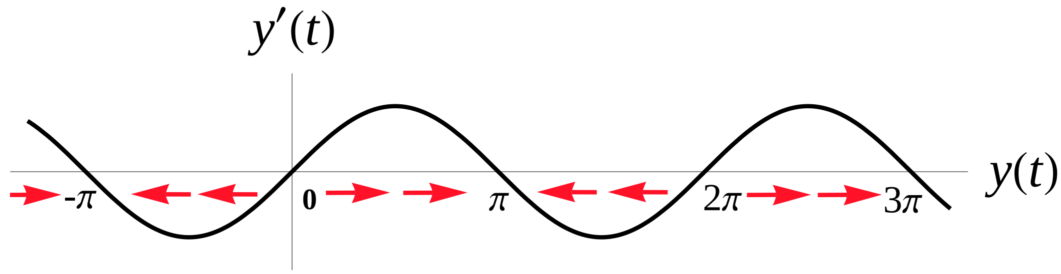
The equilibrium solutions are $y = 1$ and $y = 2$. Since y' is positive outside of the interval $[1, 2]$ and negative inside $(1, 2)$, $y = 1$ is an asymptotically stable solution and $y = 2$ is an unstable solution

Find and classify the equilibrium solutions of $\frac{dy}{dt} = y(2-y)^2$.



The equilibrium solutions are $y = 0$ and $y = 2$. $y = 0$ is an unstable solution and $y = 2$ is a **semistable equilibrium solution** since it is stable from one side (in this case from below) while unstable from the other side (in this case from above).

Find and classify the equilibrium solutions of $\dfrac{dy}{dt} = sin(y)$.



Unstable equilibria exist at $t = 0, \pm 2\pi, \pm 4\pi, ...$

Stable equilibria exist at $t = \pm\pi, \pm 3\pi, ...$

Applications of First Order o.d.e.

We start with the classic *exponential model*: $y'(t) = ky$, $y(t_0) = y_0$.

The general solution (without an initial value) is $y(t) = ce^{kt}$. If $k > 0$ the model is called exponential growth and if $k < 0$ the model is called exponential decay. The constant k is called a growth constant and decay constant respectively.

Applying the initial value, $y_0 = ce^{kt_0}$, so $c = y_0 e^{-kt_0}$. The initial value problem has the solution

$$y = y_0 e^{k(t-t_0)}.$$

We assume that you have seen examples (in algebra and calculus) of using the exponential model to model at least population growth, compound interest, and radioactive decay (using the half-life).

Let's consider a variation on the exponential model:

Suppose that a radioactive substance has decay constant $k > 0$ (reported as a positive number, so use $y' = -ky$). At the same time the substance is being produced at a constant rate of a units of mass per unit time.

Let $y(t)$ be the amount of the substance at time t and $y(0) = y_0$ be the initial mass. Let's derive an IVP and solve it. Then we can take the limit of $y(t)$ as $t \to \infty$ to find the equilibrium solution.

First we derive a differential equation: $y'(t) =$ the rate of increase in $y(t)$ $-$ the rate of decrease in y.

So $y'(t) = a - ky$. We can solve this in various ways. Since it's separable let's solve it by integrating.

$$\frac{1}{a-ky}y' = 1 \to -\frac{1}{k}\ln|a-ky| = t + c \to a - ky = ce^{-kt} \to y = \frac{a}{k} + ce^{-kt}.$$

Imposing the initial condition we get:

$$y_0 = \frac{a}{k} + c \to c = y_0 - \frac{a}{k}.$$

So the solution to the IVP is

$$y = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right)e^{-kt}.$$

The equilibrium solution is $y = \dfrac{a}{k}$. In particular, if $y_0 > a/k$ then y decreases toward this solution, and if $y_0 < a/k$ then y increases towards this solution.

A *mixing problem*:

An immortal alchemist creates bitcoin at the rate of 1 bitcoin per day. Going unnoticed, a hacker has written code that is continuously stealing it at a rate that is equivalent, on a daily basis, to 2.5 percent of whatever is there. What is the number of bitcoins $b(t)$ the alchemist has at time t? Assume $b(0) = 1$. How many bitcoins will the alchemist have in the long run (as $t \to \infty$)?

$b'(t) =$ rate of increase in bitcoin $-$ rate of decrease in bitcoin.

So $b' = 1 - 0.025b$. Using $a = 1$, $k = 0.025$, and $b_0 = 1$ we get $b(t) = 40 - 39e^{-0.03t}$.

In the long run the alchemist will have 40 bitcoins.

Newton's Law of Cooling:

If an object of temperature $T(t)$ at time t is in a medium of temperature $T_m(t)$ at time t then the rate of change in $T(t)$ is proportional to the $\Delta T = T(t) - T_m(t)$. When $T(t) > T_m(t)$ we have $T'(t) < 0$ and when $T(t) < T_m(t)$ we have $T'(t) > 0$.

So $T(t)$ satisfies $T' = -k(T - T_m)$ where $k > 0$.

Suppose a cup of coffee brewed at 150 degrees Fahrenheit is cooling in a room of constant temperature 70 degrees Fahrenheit. After 10 minutes the temperature of the coffee is 100 degrees Fahrenheit. How hot is the temperature after 20 minutes?

First we setup the IVP: $T' = -k(T - 70)$, $T(0) = 150$.

Then $\ln|T - 70| = -kt + c$ implies $T - 70 = ce^{-kt}$ and hence $T = 70 + ce^{-kt}$. By imposing the initial condition, $T(t) = 70 + 80e^{-kt}$.

Since we are given $T(10) = 100$ we can solve for k:

$100 = 70 + 80e^{-10k} \to k = -0.1 \ln(3/8) = \ln(8/3)/10$

Then after 20 minutes the cup of coffee is at $T(20) = 70 + 80e^{-2\ln(8/3)} = 70 + 80(9/64) = 81.25$ degrees Fahrenheit.

Another *mixing problem*:

A tank initially contains 10 kg of salt dissolved in 200 L of water. Brine that contains 0.5 kg of salt per liter is pumped into the tank at 5 liters per minute and at the same time water is drained from the tank at 5 liters per minute. Assume the tank is always kept uniformly mixed. Calculate the amount of salt in the tank in the long run (the equilibrium solution as $t \to \infty$).

Let's pose this as an IVP: Let $y(t)$ be the amount of salt in kg in the tank (which always contains 200 L of water).

Then $y' =$ rate of salt in $-$ rate of salt out.

So

$$y' = \left(\frac{0.5 \text{ kg}}{\text{L}}\right)\left(\frac{5 \text{ L}}{\text{min}}\right) - \left(\frac{y(t) \text{ kg}}{200 \text{ L}}\right)\left(\frac{5 \text{ L}}{\text{min}}\right), \; y(0) = 10.$$

Again, we can solve $y' = 2.5 - y/40$ in more than one way, but it is separable:

$$\frac{1}{2.5 - y/40} y' = 1 \to -40 \ln |2.5 - y/40| = t + c \to 2.5 - y/40 = ce^{-t/40} \to y = 100 + ce^{-t/40}.$$

Imposing the initial condition yields $c = -90$, so $y = 100 - 90e^{-t/40}$. Thus as $t \to \infty$, $y \to 100$. So in the long run the amount of salt in the tank is 100 kg.

Here is a variation on the problem above:

A tank of maximum capacity $1,000$ gallons initially contains 10 lb of salt dissolved in 500 gallons of water. Brine that contains 0.5 lb of salt per gallon is pumped into the tank at 10 gallons per minute and at the same time water is drained from the tank at 5 gallons per minute. Assume the tank is always kept uniformly mixed. Calculate the amount of salt in the tank the moment it begins to overflow.

The change here is the amount of water in the tank at time t isn't constant: $V(t) = 500 + 5t$ is the volume of the tank in gallons after t minutes where $V(0) = 500$.

We still set it up like before: Let $y(t)$ be the amount of salt in the tank in lb at time t, where $y(0) = 10$.

Then $y' =$ rate of salt in $-$ rate of salt out.

So

$$y' = \left(\frac{0.5 \text{ lb}}{\text{gal}}\right)\left(\frac{10 \text{ gal}}{\text{min}}\right) - \left(\frac{y(t) \text{ lb}}{500 + 5t \text{ gal}}\right)\left(\frac{5 \text{ gal}}{\text{min}}\right), \; y(0) = 10.$$

So we have to solve the following IVP: $y' = 5 - \dfrac{1}{100 + t} y$, $y(0) = 10$.

Let's rewrite the o.d.e. as $y' + \dfrac{1}{100+t}y = 5$. An integrating factor is found by computing

$$\mu = e^{\int \frac{1}{100+t}\,dt} = e^{\ln|100+t|} = \pm(100+t).$$

Let's multiply both sides of the o.d.e. by $100+t$:

$$(100+t)y' + y = 5(100+t).$$

This becomes

$$((100+t)y)' = 5(100+t).$$

Integrating both sides yields

$$(100+t)y = 5(100t + 0.5t^2) + c \to y = \frac{500t + 2.5t^2 + c}{100+t}.$$

By imposing the initial condition $y(0) = 10$ we get $10 = \dfrac{c}{100}$, so $c = 1,000$. So

$$y = \frac{1,000 + 500t + 2.5t^2}{100+t}.$$

Now to find the amount of salt at the moment the tank overflows: Set $V(t) = 500+5t = 1,000$ to get $t = 100$ minutes.

Then $y(100) = 380$ lbs of salt when the tank overflows.

Motion with resistance: We assume an object is traveling vertically through a medium (air or water for example) and that the only forces acting on the object are a constant gravitational force pulling the object down and the resistance of the medium (directly proportional to the speed of the object).

Suppose an object of mass $m > 0$ moves through vertically a medium under a constant downward gravitational force $F_g = mg$ (where g is the constant acceleration due to gravity) and the medium exerts a force of $F_{med} = k|v|$, $k > 0$, in the *opposite direction of the motion*, where v is the velocity of the object.

Let's derive a model for this scenario:

Let v be the vertical velocity of an object . Then Newton's second law of motion asserts that $F_{net} = ma$ where $a = v'$.

Suppose the object is moving up. Then the resistance acts downward: $F_{med} = -k|v| = -kv$. Now suppose the object is moving down. Then the resistance acts upward: $F_{med} = k|v| = k(-v) = -kv$.

Either way, the net force on the object is and $F_{net} = -mg - kv$.

So we get the following first order linear o.d.e. $mv' = -mg - kv$.

This can be expressed as $v' + \dfrac{k}{m}v = -g$.

While this is separable, we can solve quickly it by finding an integrating factor: $\mu = e^{(k/m)t}$.

$$e^{(k/m)t}v' + \frac{k}{m} = -ge^{(k/m)t} \to (e^{(k/m)t}v)' = -ge^{(k/m)t} \to e^{(k/m)t}v = -\frac{mg}{k}e^{(k/m)t} + c.$$

(ELFY) Solve it as a separable ode by separating the variables and integrating both sides.

Hence

$$v(t) = -\frac{mg}{k} + ce^{-(k/m)t}.$$

Finally, we see that $\lim_{t\to\infty} v(t) = -\dfrac{mg}{k}$ is the equilibrium solution, which is called the terminal velocity in this context.

Given an initial value v_0 for the velocity, we get $c = v_0 + \frac{mg}{k}$.

A 4 kg object is launched vertically into the air at 40 meters per second near the surface of the Earth (we assume far enough up that the object will attain terminal velocity before reaching the surface of the Earth).

Suppose the air resists motion with a force of 4.9 Newtons for each m/s of speed. Using $g = 9.8$ meters per square second, find a model for the objects vertical velocity and find the terminal velocity.

We can just plug-in to the general model we just derived: $v(t) = -\frac{4(9.8)}{4.9} + (40 + \frac{4(9.8)}{4.9})e^{-(4.9/4)t}$.

So $v(t) = -8 + 48e^{-(49/40)t}$.

The terminal velocity is -8. Meaning that in the end the object falls at a constant rate of 8 meters per second.

[IF TIME PERMITS COVER 2ND ORDER AUTONOMOUS AND ESCAPE VELOCITY]

A Brief Treatment of Second Order Autonomous

A second order equation is autonomous if it can expressed in the form $y'' = F(y, y')$. The independent variable (x or t) does not appear in the differential equation. We will use t for the independent varible.

Let $v = y'$. By the Chain Rule we can write that $y'' = v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}y' = \frac{dv}{dt}v$.

So the second order autonomous equation $y'' = F(y, y')$ reduces to the first order equation $v\frac{dv}{dy} = F(y, v)$, where we treat y as the independent variable in v. We will only look at some very basic examples, ones where $y'' = p(y)$.

Consider the equation $y'' + y = 0$. By letting $v = y'$, this equation reduces to $v\frac{dv}{dy} + y = 0$, which is separable. The reader is left to check that we do indeed get

$$v = \pm\sqrt{C - y^2}, \text{ where } C \geq 0.$$

If we plot these curves (for various values of C) in the (y, v)-plane (a so-called "Poincaré phase plane) we get a bunch of circles. On one circle we see that if $|y|$ is larger, $|v|$ is smaller and vice versa (going around clockwise). This represents oscillations!

(Exercise left for you!) Find $y(t)$ given that it satisfies $y' = v = \pm\sqrt{R^2 - y^2}$. Confirm the oscillations?

In general, finding $y(t)$ from $v(y)$ just requires solving a separable equation, which is something we know how to do, but often it is really quite hard because of the integration required.

Let's return to $y'' + y = 0$. Notice that if $y = 0$ then $y' = 0$ and so $y = 0$ is a constant solution (equilibrium of the second order autonomous equation)!

The corresponding point $(0, 0)$ in the Poincaré phase plane (horizontal y-axis, vertical v axis) is called a critical point. Such a equilibrium is stable if small perturbations in either y tend to keep the values of y and y' relatively "near" to the critical point (y equal to the constant, $v = y'$ equal to 0). Otherwise it's unstable.

In the case of $y = 0$ in $y'' + y = 0$, it's stable since small perturbations move to a circular trajectory near to and about the origin.

Consider $y'' + \sin(y) = 0$. It has equilibria at $y = n\pi$ where n is an arbitrary integer.

The equation reduces to $v\dfrac{dv}{dy} + \sin(y) = 0$, which is separable. The reader is left to check that we do indeed get

$$v^2 = 2\cos(y) + C.$$

Consider the equilibrium $y = n\pi$ where n is an even integer (i.e. $y = 0$).

$v\dfrac{dv}{dy} = -\sin(y)$. If y is increased a little (so $v > 0$) then $v\dfrac{dv}{dy} < 0$ meaning v and $\dfrac{dv}{dy}$ have opposite signs. If y is decreased a little (so $v < 0$) then $v\dfrac{dv}{dy} > 0$ meaning v and $\dfrac{dv}{dy}$ have the same sign. Either way, it follows that $\dfrac{dv}{dy} < 0$ and $\dfrac{dy}{dv} < 0$ (Inverse Derivative Rule).
This means that y and v move in opposition. As y increases, v decreases, causing y to slow and then (after v becomes negative) decrease. As y decreases, v increases, causing the decrease in y to slow and then (after v becomes positive) increase. This makes these equilibria stable.

Consider the equilibrium $y = n\pi$ where n is an odd integer (i.e. $y = \pi$).

The exact opposite thing happens at these equilibria. y and v move in concert. As y increases, so does $v > 0$ only causing y to increase more. That makes these equilibria unstable.

Over large vertical distances, gravitational models assume that gravity is inversely proportional to the square of the distance from the center of the massive object (assumed to be a sphere). This is an example of an inverse square law.

Assume a spacecraft launches from the surface of the Earth and exhausts its fuel as it reaches a height h that sufficiently large that atmospheric resistance is negligible and can be assumed to be effectively 0. Let $t = 0$ be this time (known as "burnout") and consider that the force of gravity on the spacecraft at altitude $y \geq h$ is

$$F_g = -\frac{c}{(R+y)^2} \text{ where } c \text{ is a constant } R \text{ is the radius of the Earth.}$$

Using $F_g = -mg$ when $y = 0$ where g is the acceleration due to gravity on the surface of the Earth, we get $-mg = -\frac{c}{R^2}$, so $c = mgR^2$.

Thus

$$F_g = -\frac{mgR^2}{(R+y)^2}.$$

Since F_g is assumed to be the only force acting on the spacecraft for $t \geq 0$,

$$F_g = ma = my'' = -\frac{mgR^2}{(R+y)^2} \rightarrow$$

$$y'' = -\frac{gR^2}{(R+y)^2}.$$

This equation is autonomous, so with $v = y'$ it reduces to $v\dfrac{dv}{dy} = -\dfrac{gR^2}{(R+y)^2}$.

Integrating both sides yields

$$0.5v^2 = \frac{gR^2}{R+y} + c \rightarrow v = \sqrt{\frac{2gR^2}{R+y} + c}.$$

Now suppose we have an initial velocity of $v(h) = v_0$.

Then $v_0^2 = \dfrac{2gR^2}{R+h} + c$.

So $c = v_0^2 - \dfrac{2gR^2}{R+h}$ and hence

$$v = \sqrt{\frac{2gR^2}{R+y} + v_0^2 - \frac{2gR^2}{R+h}}.$$

If $v_0 \geq \sqrt{\dfrac{2gR^2}{R+h}}$ then $v \geq 0$ for all $y \geq h$ and the spacecraft continues on up into space!

The expression $v_e = \sqrt{\dfrac{2gR^2}{R+h}}$ is called escape velocity.

We will show that if $v_0 < v_e$ the spacecraft falls back to Earth:

If $v_0 < v_e$ then $v = \sqrt{\dfrac{2gR^2}{R+y} + v_0^2 - \dfrac{2gR^2}{R+h}} = 0$ for y such that

$$\frac{2gR^2}{R+y} + v_0^2 - \frac{2gR^2}{R+h} = 0,$$

which occurs at $y = \dfrac{2gR^2 h + R(R+h)v_0^2}{2gR^2 - (R+h)v_0^2} \geq h$ since

$$\frac{2gR^2}{R+y} = \frac{2gR^2}{R+h} - v_0^2 \leq \frac{2gR^2}{R+h}.$$