Linear Second Order o.d.e.

A linear second order o.d.e. can be written in the form $P_0(x)y'' + P_1(x)y' + P_2(x)y = f(x)$ where $P_0(x) \neq 0$ (as a function). $f(x)$ is called the forcing function.

As before, if $f(x) = 0$ then the o.d.e. is homogeneous. Otherwise it’s non-homogeneous. If $P_0, P_1, P_2$ are constants, then the equation is called a constant coefficient equation.

We begin by considering the homogeneous case. In the homogeneous case we may write the equation in the form $y'' + p(x)y' + q(x)y = 0$.

This theorem gives sufficient conditions for existence and uniqueness of a solution to a linear homogeneous second order IVP. We omit the proof.

Proposition: Suppose $p, q$ are continuous functions on $(a, b)$, $x_0$ is in $(a, b)$ and $k_0, k_1$ are arbitrary real numbers. Then the IVP

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1,$$

has a unique solution on $(a, b)$.

As before $y = 0$ will be called the trivial solution.

Consider the linear homogeneous second order o.d.e. $y'' + \frac{4x}{x^2 - 1}y' + \frac{2}{x^2 - 1}y = 0$.

Let’s verify that $y = \frac{1}{x + 1}$ is a solution:

Starting with $y = \frac{1}{x + 1}$. $y' = -\frac{1}{(x + 1)^2}$ and $y'' = \frac{2}{(x + 1)^3}$. Then

$$y'' + \frac{4x}{x^2 - 1}y' + \frac{2}{x^2 - 1}y = \frac{2}{(x + 1)^3} + \frac{4x}{x^2 - 1} \left( -\frac{1}{(x + 1)^2} \right) + \frac{2}{x^2 - 1} \left( \frac{1}{x + 1} \right)$$

$$= \frac{2}{(x + 1)^3} - \frac{4x}{(x - 1)(x + 1)^3} + \frac{2}{(x - 1)(x + 1)^2}$$

$$= \frac{2(x - 1) - 4x + 2(x + 1)}{(x - 1)(x + 1)^3}$$

$$= 0.$$

Exercise Left For You (ELFY): Show that $y = \frac{1}{x - 1}$ is also a solution.
**Proposition:** If $y_1$ and $y_2$ are solutions to $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$ then any linear combination $y = c_1y_1 + c_2y_2$ is also a solution on $(a, b)$.

**Proof:** Assume $y_1$ and $y_2$ are solutions to $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$. Let $c_1, c_2$ be constants and $y = c_1y_1 + c_2y_2$.

Then

$$y'' + p(x)y' + q(x) = (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2)$$

$$= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2)$$

$$= 0.$$ 

That completes the proof □

So $y = \frac{c_1}{x + 1} + \frac{c_1}{x - 1}$ are all solutions of $y'' + \frac{4x}{x^2 - 1}y' + \frac{2}{x^2 - 1}y = 0$.

Assume $y_1$ and $y_2$ are solutions to $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$. \{y_1, y_2\} is called a fundamental set of solutions on $(a, b)$ if every solution can be expressed as $c_1y_1 + c_2y_2$ for an appropriate choice of constants.

When \{y_1, y_2\} is a fundamental set of solutions on $(a, b)$, $c_1y_1 + c_2y_2$ is said to be the general solution on $(a, b)$.

Two functions $y_1, y_2$ defined on $(a, b)$ are linearly independent on $(a, b)$ if neither is a constant multiple of the other on $(a, b)$. Otherwise they are said to be linearly dependent.

**Proposition:** Suppose $p, q$ are continuous on $(a, b)$. Then a set \{y_1, y_2\} of solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on $(a, b)$ is a fundamental set of solutions on $(a, b)$ if and only if $y_1, y_2$ are linearly independent on $(a, b)$.

For now we postpone the proof. We begin by understanding what is must be shown in order to prove \{y_1, y_2\} is a fundamental set...

\{y_1, y_2\} is a fundamental set if and only if every IVP has a solution of the form $c_1y_1 + c_2y_2$. 
That equivalent to always being able to solve the system of equations for $c_1, c_2$

\[
c_1 y_1(x_0) + c_2 y_2(x_0) = k_0
\]

\[
c_1 y'_1(x_0) + c_2 y'_2(x_0) = k_1
\]

for any choice of $k_0, k_1$.

This system has a solution for all $k_0, k_1$ if and only if $y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0) \neq 0$ (forces the lines to not be parallel in the $c_1c_2$-plane).

In the case where $y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0) \neq 0$, Cramer’s Rule gives us

\[
c_1 = \frac{y'_2(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0)} \quad \text{and} \quad c_2 = \frac{y_1(x_0)k_1 - y'_1(x_0)k_0}{y_1(x_0)y'_2(x_0) - y'_1(x_0)y_2(x_0)}.
\]

We define the Wronskian of set of two functions $\{y_1, y_2\}$ as the function $W = y_1y'_2 - y'_1y_2$, which can be written as the determinant:

\[
W = \det\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}.
\]

**Proposition:** Suppose $p, q$ are continuous on $(a, b)$ and $\{y_1, y_2\}$ are solutions to

\[
y'' + p(x)y' + q(x)y = 0
\]
on $(a, b)$. Let $x_0$ be any point in $(a, b)$. Then the Wronskian of $\{y_1, y_2\}$ satisfies

\[
W(x) = W(x_0)e^{-\int_{x_0}^x p(t)dt} \quad \text{(called Abel’s formula)}.
\]

As a consequence, either $W = 0$ on $(a, b)$ or $W$ has no zeros on $(a, b)$.

**Proof:** $W = y_1y'_2 - y'_1y_2$ so $W' = y'_1y'_2 + y_1y''_2 - y'_1y'_2 - y'_1y'_2 = y_1y''_2 - y'_1y'_2$.

We can replace $y''_i = -p(x)y'_i - q(x)y_i$ for $i = 1, 2$ to get

\[
W' = y_1(-p(x)y'_2 - q(x)y_2) - (-p(x)y'_1 - q(x)y_1)y_2 = -p(x)y_1y'_2 + p(x)y'_1y_2 = -p(x)W.
\]

So $W$ is a solution to the IVP $W' + p(x)W = 0$. We know that for this homogeneous first order o.d.e. either $W = 0$ on $(a, b)$ or $W = ce^{-\int p(x)dx}$ with $c \neq 0$ on $(a, b)$. In the latter case $W \neq 0$ for all points on $(a, b)$.

So if $W(x_0) = 0$ then $W = 0$. If $W(x_0) \neq 0$ then

\[
W(x) = W(x_0)e^{-\int_{x_0}^x p(t)dt} \quad \text{which actually works in both cases} \quad \square
Let’s verify Abel’s formula works for solutions $y_1 = x^2$ and $y_2 = x^{-2}$ of $x^2y'' + xy' - 4y = 0$.

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} = \det \begin{pmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{pmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1} = -\frac{4}{x}.$$ 

By rewriting the equation in the form $y'' + \frac{1}{x}y' - 4\frac{x^2}{x^2}y = 0$ we see that $p(x) = \frac{1}{x}$. Then on an interval avoiding $x = 0$,

$$e^{-\int_{x_0}^x p(t) \, dt} = e^{-\int_{x_0}^x \frac{1}{t} \, dt} = e^{-\ln(x/x_0)} = \frac{x_0}{x}.$$ 

Thus $W(x_0)e^{-\int_{x_0}^x p(t) \, dt} = -\frac{4}{x_0} \cdot \frac{x_0}{x} = -\frac{4}{x} = W(x)$.

**Proposition:** Suppose $p, q$ are continuous on $(a, b)$ and $\{y_1, y_2\}$ are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on $(a, b)$. Then $y_1, y_2$ are linearly independent on $(a, b)$ if and only if the Wronskian $W = y_1y'_2 - y'_1y_2$ has no zeros on $(a, b)$.

**Proof:** First we assume $W(x_0) = 0$ for some $x_0$ on $(a, b)$ and show that it must be the case that $y_1, y_2$ are linearly dependent on $(a, b)$. Clearly $y_1, y_2$ are linearly dependent on $(a, b)$ if $y_1 = 0$ on $(a, b)$. Assume otherwise. Then $y_1$ is non-zero on some subinterval $I$ of $(a, b)$ as it’s continuous (because it’s differentiable).

Then on $I$, $(y_2/y_1)$ is defined and

$$\left( \frac{y_2}{y_1} \right)' = \frac{y_1y'_2 - y'_1y_2}{y_1^2} = \frac{W}{y_1^2}.$$ 

Since $W(x_0) = 0$ a previous proposition implies that $W = 0$ on $I$. Hence $y_2/y_1 = c$ on $I$. So $y_2 = cy_1$ on $I$. We’d like to show this is the case on $(a, b)$. Let $Y = cy_1 - y_2$. Since $Y$ is a linear combination of $y_1$ and $y_2$, it is a solution of the IVP $y'' + p(x)y' + q(x)y = 0$, with $y(x_0) = y'(x_0) = 0$ for some $x_0$ in $I$.

Since the trivial solution $Y = 0$ works and a proposition guarantees the IVP has a unique solution, it must be the case that $Y = 0$ and hence $y_2 = cy_1$ on $(a, b)$.

Now assume $W$ has no zeros on $(a, b)$. Then $y_1$ cannot be the zero function on $(a, b)$ (or $W$ is zero) and therefore there is a subinterval on which $y_1$ has no zeros. Then on $I$ we have as above $(y_2/y_1)' = W/y_1^2$. Since $W$ has no zeros on $I$ it follows that $y_2/y_1$ is not constant on $I$, so $y_2$ is not a multiple of $y_1$ on $I$, let alone $(a, b)$.

A similar argument, but where $(y_1/y_2)' = -W/y_2^2$, shows that $y_1$ is not a multiple of $y_2$ on $(a, b)$.

Hence $y_1, y_2$ are linearly independent □
Now we complete a proof of an earlier proposition, which states:

Suppose $p, q$ are continuous on $(a, b)$. Then a set $\{y_1, y_2\}$ of solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on $(a, b)$ is a fundamental set of solutions on $(a, b)$ if and only if $y_1, y_2$ are linearly independent on $(a, b)$.

Proof: From the last proposition two solutions $y_1, y_2$ are linearly independent if and only if $W$ has no zeros on $(a, b)$. In motivating the definition of the Wronskian, we saw that $W$ has no zeros on $(a, b)$ if and only if the IVP posed in the first proposition in this section has a solution that is a linear combination of $y_1$ and $y_2$. Putting these facts together, $\{y_1, y_2\}$ is a fundamental set of solutions if and only if $y_1, y_2$ are linearly independent □
Linear Second Order Constant-Coefficient Equations

For the moment let’s restrict our view to some very simple linear second order o.d.e. If \(a, b, c\) are real numbers and \(a \neq 0\) then the linear second order o.d.e.

\[ ay'' + by' + cy = f(x) \]

is said to be a constant coefficient equation. If \(f(x) = 0\) it is a homogeneous constant coefficient equation. Otherwise it is non-homogeneous.

Suppose we are trying to find the general solution to the linear homogeneous constant coefficient second order o.d.e. \(ay'' + by' + cy = 0\).

Suppose that \(r\) is a constant and \(y = e^{rx}\). Then \(y' = re^{rx} = ry\) and \(y'' = r^2e^{rx} = r^2y\).

By substitution, \(ay'' + by' + c = 0\) becomes \((ar^2 + br + c)e^{rx} = 0\), implying that if \(r\) satisfies \(ar^2 + br + c = 0\) then \(y = e^{rx}\) is a solution.

The quadratic \(p(r) = ar^2 + br + c\) is called the characteristic polynomial of \(ay'' + by' + c = 0\). The equation \(p(r) = 0\) is the characteristic equation.

By the quadratic formula we have the roots of the characteristic polynomial are

\[ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

So we consider three cases:

Case 1: \(b^2 - 4ac > 0\). In this case the characteristic polynomial has two distinct real roots.

Let \(r_1 \neq r_2\) be the real roots. Then \(y_1 = e^{r_1x}\) and \(y_2 = e^{r_2x}\) are both solutions of \(ay'' + by' + cy = 0\). Since \(y_2/y_1 = e^{(r_2-r_1)x}\) is non-constant as \(r_2 - r_1 \neq 0\) we have that \(y_1, y_2\) are linearly independent. Thus the general solution to \(ay'' + by' + c = 0\) is \(y = c_1 e^{r_1x} + c_2 e^{r_2x}\).
Case 2: $b^2 - 4ac = 0$. In this case the characteristic polynomial has one repeated real root.

Let $r_1$ be the repeated root. So $y_1 = e^{r_1x}$ is a solution of $ay'' + by' + cy = 0$. Furthermore, the characteristic polynomial is $a(x - r_1)^2$ so $b = -2ar_1$ and $c = ar_1^2$.

To find another solution, let’s try the variation of parameters method:

Set $y_2 = uy_1 = ue^{r_1x}$ and assume $y_2$ is a solution of $ay'' + by' + cy = 0$. Then

$$ay'' + by' + cy = 0 \rightarrow a(u''e^{r_1x} + 2r_1u'e^{r_1x} + r_1^2ue^{r_1x}) + b(u'e^{r_1x} + r_1ue^{r_1x}) + cue^{r_1x} = 0 \rightarrow$$

$$[au'' + 2ar_1u' + ar_1^2u + bu' + br_1u + cu]e^{r_1x} = 0 \rightarrow$$

$$[au'' + 2ar_1u' + (ar_1^2 + br_1 + c)u]e^{r_1x} = 0 \rightarrow$$

$$[au'' + (2ar_1 + b)u']e^{r_1x} = 0 \rightarrow$$

$$au'' + (2ar_1 + b)u' = 0 \rightarrow au'' = 0 \text{ since } b = -2ar_1.$$

Therefore $u = c_1 + c_2x$. In particular we can pick $u = x$. Then $y_2 = xe^{r_1x}$ is a solution of $ay'' + by' + cy = 0$.

Moreover, $y_2/y_1 = x$ is non-constant, so $y_1, y_2$ are linearly independent (on any interval). Hence the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{r_1x} + c_2xe^{r_1x} = (c_1 + c_2x)e^{r_1x}.$$
Case 3: \( b^2 - 4ac < 0 \). Then the roots of the characteristic polynomial are complex conjugates \( \lambda \pm \omega i \), where \( \lambda, \omega \) are real numbers and \( \omega \neq 0 \).

Then \( e^{(\lambda+\omega i)x} \) and \( e^{(\lambda-\omega i)x} \) are solutions of \( ay'' + by' + cy = 0 \). While any linear combination of these is a solution, some are not real-valued, even for real values of \( x \). We restrict our attention towards finding the general solution in the real number case.

Note: \( e^{(\lambda \pm \omega i)x} = e^{\lambda x}e^{\pm \omega ix} = e^{\lambda x}(\cos (\omega x) \pm i \sin (\omega x)) \) using Euler’s formula \( e^{i\theta} = \cos (\theta) + i \sin (\theta) \).

In particular, any linear combination of these are solutions, so we can take

\[
y_1 = \frac{1}{2} \left( e^{\lambda x}(\cos (\omega x) + i \sin (\omega x)) + e^{\lambda x}(\cos (\omega x) - i \sin (\omega x)) \right) = e^{\lambda x} \cos (\omega x) \quad \text{and} \quad y_2 = \frac{1}{2i} \left( e^{\lambda x}(\cos (\omega x) + i \sin (\omega x)) - e^{\lambda x}(\cos (\omega x) - i \sin (\omega x)) \right) = e^{\lambda x} \sin (\omega x).
\]

Since \( y_2/y_1 = \tan (\omega x) \) is non-constant \( y_1, y_2 \) are linearly independent, and hence the general solution to \( ay'' + by' + cy = 0 \) is

\[
y(t) = c_1 e^{\lambda x} \cos (\omega x) + c_2 e^{\lambda x} \sin (\omega x) = (c_1 \cos (\omega x) + c_2 \sin (\omega x))e^{\lambda x}.
\]

Since we are only interested in real solutions we assume \( c_1, c_2 \) are real. If we were interested in complex solutions we’d allow \( c_1, c_2, x \) to be complex numbers.

Examples:

1. Find the general solution of \( y'' + 5y' - 14y = 0 \).

   The characteristic polynomial is \( r^2 + 5r - 14 = (r + 7)(r - 2) \). The roots are \( r = -7, 2 \). So the general solution is

   \[
y = c_1 e^{-7x} + c_2 e^{2x}.
\]

2. Solve the IVP \( y'' - 6y' + 9y = 0 \), \( y(0) = 1 \), \( y'(0) = 5 \).

   The characteristic polynomial is \( r^2 - 6r + 9 = (r - 3)^2 \). The repeated root is \( r = 3 \). So the general solution is

   \[
y = c_1 e^{3x} + c_2 xe^{3x}.
\]

   Then \( y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 xe^{3x} \).

   So imposing the initial conditions yields

   \[
c_1 = 1 \quad \text{and} \quad 3c_1 + c_2 = 5 \rightarrow c_1 = 1 \quad \text{and} \quad c_2 = 2.
\]

   So the solution is

   \[
y = e^{3x} + 2xe^{3x}.
\]
The characteristic polynomial is $r^2 + 4$, which has complex roots $r = \pm 2i$. Since $\pm 2i = 0 \pm 2i$, the general solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Then $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$.

By imposing the initial conditions we get $c_1 = -1$ and $2c_2 = 2$, so $c_1 = -1$ and $c_2 = 1$.

So the solution is

$$y = -\cos(2x) + \sin(2x).$$

Let’s consider the non-homogenous case where $y'' + p(x)y' + q(x)y = f(x)$. As before we’ll call the equation with $f(x) = 0$ the complementary equation. The following proposition, whose proof is beyond the scope of our class, implies the existence and uniqueness of a solutions to an IVP where $y'' + p(x)y' + q(x)y = f(x)$.

**Proposition:** Suppose $p, q, f$ are continuous on $(a, b)$ and $x_0$ is a point in $(a, b)$. Then for any constants $k_0$ and $k_1$ the IVP

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_1) = k_1$$

has a unique solution on $(a, b)$.

The following proposition gives a method for solving $y'' + p(x)y' + q(x)y = f(x)$ for its general solution.

**Proposition:** Suppose $p, q, f$ are continuous on $(a, b)$, $y_p$ is a particular solution to

$$y'' + p(x)y' + q(x)y = f(x)$$

on $(a, b)$, and $y_1, y_2$ is a fundamental set of solutions to the complementary equation $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$. Then $y$ is a solution to $y'' + p(x)y' + q(x)y = f(x)$ on $(a, b)$ if and only if $y = y_p + c_1 y_1 + c_2 y_2$ for some constants $c_1, c_2$.

**Proof:** If $y = y_p + c_1 y_1 + c_2 y_2$ then $y' = y'_p + c_1 y'_1 + c_2 y'_2$ and $y'' = y''_p + c_1 y''_1 + c_2 y''_2$, so

$$y'' + p(x)y' + q(x)y = y''_p + p(x)y'_p + q(x)y_p + c_1(y''_1 + p(x)y'_1 + q(x)y_1) + c_2(y''_2 + p(x)y'_2 + q(x)y_2) = f(x).$$

So this proves that if $y = y_p + c_1 y_1 + c_2 y_2$ for some constants $c_1, c_2$ then $y$ is a solution to $y'' + p(x)y' + q(x)y = f(x)$ (the reverse direction).

Now we show the forward direction. Suppose $y$ is a solution to $y'' + p(x)y' + q(x)y = f(x)$ on $(a, b)$. Then consider $w = y - y_p$. It suffices to show that $w$ is a solution to the complementary equation $y'' + p(x)y' + q(x)y = 0$ on $(a, b)$.

Well,

$$w'' + p(x)w' + q(x)w = y'' - y''_p + p(x)(y' - y'_p) + q(x)(y - y_p) = y'' + p(x)y' + q(x)y - (y''_p + p(x)y'_p + q(x)y_p) = 0 \square$$
Let’s solve the following IVP: \( y'' + 9y = 9, \ y(0) = 0, \ y'(0) = 4. \)

The characteristic polynomial of the complementary equation \( y'' + 9y = 0 \) is \( r^2 + 9 \) which has roots \( r = \pm 3i \). So \( \{ \cos (3x), \sin (3x) \} \) are a fundamental set of solutions to \( y'' + 9y = 0 \).

Obviously \( y_1 = 1 \) is a particular solution of \( y'' + 9y = 9 \). Hence \( y = 1 + c_1 \cos (3x) + c_2 \sin (3x) \) is the general solution of \( y'' + 9y = 9 \).

Imposing the initial condition \( y(0) = 0 \) implies \( 0 = 1 + c_1 \), so \( c_1 = -1 \). Imposing the initial condition \( y'(0) = 4 \) implies \( 4 = 3c_2 \), so \( c_2 = \frac{4}{3} \).

So the solution to the IVP is

\[
y = 1 - \cos (3x) + \frac{4}{3} \sin (3x).
\]
Method of Undetermined Coefficients

The method of undetermined coefficients is a guess and check method that will only use for constant coefficient equations and is only feasible for certain forms of the forcing functions \( f(x) \).

The method of undetermined coefficients works if the forcing function is a linear combination of functions of the forms:

\[
1, \ x^n, \ n = 1, 2, 3, \ldots \\
\ e^{\alpha x}, \ x^n e^{\alpha x}, \ n = 1, 2, 3, \ldots \\
\cos (\beta x), \ \sin (\beta x), \ e^{\alpha x} \cos (\beta x), \ e^{\alpha x} \sin (\beta x) \\
x^n \cos (\beta t), \ x^n \sin (\beta x), \ n = 1, 2, 3, \ldots \\
x^n e^{\alpha x} \cos (\beta x), \ x^n e^{\alpha x} \sin (\beta x), \ n = 1, 2, 3, \ldots
\]

In a general sense (we will see an exception later) you guess at a form of the solution that is similar to the forcing function. You guess will contain unknown coefficients that you will need to solve for. If the term contains \( t^n \) you must include all lower order terms of the polynomial.

Example (without the exception): Let’s find the general solution to \( y'' - 4y' + 4y = 4x^2 + 2x \).

The characteristic polynomial of the complementary equation \( y'' - 4y' + 4y = 0 \) is \( r^2 - 4r + 4 = (r - 2)^2 \). So \( \{e^{2x}, xe^{2x}\} \) is a fundamental set of solutions to the complementary equation \( y'' - 4y' + 4y = 0 \).

We can find a particular solution by any means necessary. Suppose we can find \( y_p \) as a polynomial solution to \( y'' - 4y' + 4y = 4x^2 + 2x \). It would have to be degree two. So let \( y_p = ax^2 + bx + c \). Then \( y' = 2ax + b \) and \( y'' = 2a \).

Then \( 2a - 4(2ax + b) + 4(ax^2 + bx + c) = 4x^2 + 2x \) implies \( 4a = 4, \ -8a + 4b = 2, \) and \( 2a - 4b + 4c = 0 \). So \( a = 1, \ b = 2.5, \) and \( c = 2 \). So \( y_p = x^2 + 2.5x + 2 + (c_1 + c_2x)e^{2x} \).

Proposition (Superposition Principle): Suppose \( p, q, f_1, f_2 \) are continuous on \((a, b)\), \( y_{p_1} \) is a particular solution to \( y'' + p(x)y' + q(x)y = f_1(x) \) on \((a, b)\), and \( y_{p_2} \) is a particular solution to \( y'' + p(x)y' + q(x)y = f_2(x) \) on \((a, b)\). Then \( y = y_{p_1} + y_{p_2} \) is a solution to \( y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) \) on \((a, b)\).

The proof is straightforward and is an easy exercise left for you (ELFY).
Let’s find the general solution to $y'' - 7y' + 12y = 5e^{-2x} + x^2$.

Clearly the general solution to the complementary homogeneous equation is $y = c_1e^{3x} + c_2e^{4x}$.

We can try to find a pair of particular solutions to $y'' - 7y' + 12y = 5e^{-2x}$ and $y'' - 7y' + 12y = x^2$ respectively.

Let’s begin with the former. Let’s guess there is a solution of the form $y = Ae^{-2x}$ (why?) and let’s substitute it into $y'' - 7y' + 12y = 5e^{-2x}$:

$$4Ae^{-2x} - 7(-2Ae^{-2x}) + 12Ae^{-2x} = 5e^{-2x} \rightarrow$$

$$4A + 14A + 12A = 5 \rightarrow 30A = 5 \rightarrow A = \frac{1}{6}.$$ 

So $y_{P_1} = \frac{1}{6}e^{-2x}$ is a particular solution.

Now we proceed with the latter. Let’s guess there is a solution of the form $y = ax^2 + bx + c$ and let’s substitute it into $y'' - 7y' + 12y = x^2$:

$$2a - 7(2ax + b) + 12(ax^2 + bx + c) = x^2 \rightarrow$$

$$2a - 7b + 12c = 0, \quad 12b - 14a = 0, \quad \text{and} \quad 12a = 1 \rightarrow a = \frac{1}{12} \rightarrow b = \frac{7}{72} \rightarrow c = \frac{37}{864}.$$ 

So $y_{P_2} = \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864}$ is a particular solution.

By superposition, $y_P = \frac{1}{6}e^{-2x} + \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864}$, is a particular solution of $y'' - 7y' + 12y = 5e^{-2x} + x^2$.

So the general solution is $y = \frac{1}{6}e^{-2x} + \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864} + c_1e^{3x} + c_2e^{4x}$. 
The exception: What if we are trying to solve $y'' - 7y' + 12y = 5e^{4x}$?

We may try as before: We know that $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions to the complementary homogeneous o.d.e. Then guess $y_P = Ae^{4x}$ is a particular solution.

Then by substitution, $y''_P - 7y'_P + 12y_P = 0$, as $y_P$ is a linear combination of the fundamental set of solutions. So it cannot be a particular solution!

Instead we will try $y_P = ue^{4x}$ where $u$ is some undetermined (non-constant) function of $x$:

Then 
\[
y''_P - 7y'_P + 12y_P = (ue^{4x})'' - 7(ue^{4x})' + 12ue^{4x}
\]
\[
= (u'e^{4x} + 4ue^{4x})' - 7(u'e^{4x} + 4ue^{4x}) + 12ue^{4x}
\]
\[
= (u''e^{4x} + 8u'e^{4x} + 16ue^{4x}) - 7u'e^{4x} - 28ue^{4x} + 12ue^{4x}
\]
\[
= u''e^{4x} + u'e^{4x}
\]
\[
= 5e^{4x}, \text{ which implies } u'' + u' = 5.
\]

By inspection, we can pick $u = 5x$.

So $y_P = 5xe^{4x}$ is a particular solution, and the general solution is $y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$.

Let’s solve the IVP 
\[
y'' - 4y' + 4y = (1 - 3x + 6x^2)e^{2x}, \ y(0) = 1, \ y'(0) = 4.
\]

A fundamental set of solutions to the complementary homogeneous o.d.e. is $\{e^{2x}, xe^{2x}\}$.

Let’s use the method of undetermined coefficients to find a particular solution:

Let $y = ue^{2x}$. Then $y' = u'e^{2x} + 2ue^{2x}$ and $y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$. Then

\[
u''e^{2x} + 4u'e^{2x} + 4ue^{2x} - 4(u'e^{2x} + 2ue^{2x}) + 4(ue^{2x}) = (1 - 3x + 6x^2)e^{2x} \text{ implies}
\]
\[
u'' = 1 - 3x + 6x^2.
\]
By integrating twice (choosing 0 for the constant of integration) we get

$$u = 0.5x^2 - 0.5x^3 + 0.5x^4 = \frac{1}{2}x^2(1 - x + x^2).$$

So a particular solution is

$$y_p = \frac{1}{2}x^2(1 - x + x^2)e^{2x}.$$  

the general solution is

$$y = \frac{x^2e^{2x}}{2}(1 - x + x^2) + (c_1 + c_2x)e^{2x}. $$

$$y' = xe^{2x}(1 - x + x^2) + x^2e^{2x}(1 - x + x^2) + \frac{x^2e^{2x}}{2}(-1 + 2x) + c_2e^{2x} + 2(c_1 + c_2x)e^{2x}. $$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 4$ yields:

$$1 = c_1$$  

and

$$4 = c_2 + 2c_1 = c_2 + 2 \rightarrow c_1 = 1$$  

and $c_2 = 2$.

So the solution to the IVP is

$$y = \frac{x^2e^{2x}}{2}(1 - x + x^2) + (1 + 2x)e^{2x}.$$

Now let’s find the general solution for $y'' + y = 2xe^{-x} + (1 + 10x^2)e^{3x}$ using both the method of undetermined coefficients and superposition:

The complementary homogeneous equation has the general solution $y = c_1 \cos(x) + c_2 \sin(x)$.

Consider $y'' + y = 2xe^{-x}$. Suppose $y = ue^{-x}$ is a particular solution. Then

$$u''e^{-x} - 2u' e^{-x} + ue^{-x} + ue^{-x} = 2xe^{-x} \text{ implies }$$

$$u'' - 2u' + 2u = 2x.$$  

Let’s find a particular solution. Suppose that $u = ax + b$. Then $u' = a$ and $u'' = 0$, and thus

$$-2a + 2(ax + b) = 2x \text{ requires that } 2a = 2 \text{ and } 2b - 2a = 0.$$  

So use $a = 1$ and $b = 1$.

So $u = x + 1$ is a particular solution. Then $y = (x + 1)e^{-x}$ is a particular solution of

$$y'' + y = 2xe^{-x}.$$
Now consider \( y'' + y = (1 + 10x^2)e^{3x} \). Suppose \( y = ue^{3x} \) is a particular solution. Then

\[
y'' + y = u''e^{3x} + 6ue^{3x} + 9ue^{3x} + ue^{3x} = (1 + 10x^2)e^{3x}
\]

implies

\[
u'' + 6u' + 10u = 1 + 10x^2
\]

Suppose \( u = ax^2 + bx + c \). Then \( u' = 2ax + b \) and \( u'' = 2a \), so

\[
2a + 6(2ax + b) + 10(ax^2 + bx + c) = 1 + 10x^2 \quad \rightarrow
\]

\[
2a + 6b + 10c = 1
12a + 10b = 0
10a = 10
\]

So \( a = 1, b = -1.2 \) and \( c = 0.62 \). So \( y = (x^2 - 1.2x + 0.62)e^{3x} \) is a particular solution of \( y'' + y = (1 + 10x^2)e^{3x} \).

Putting all this together, with superposition, the general solution to the o.d.e. \( y'' + y = 2xe^{-x} + (1 + 10x^2)e^{3x} \) is

\[
y = (x^2 - 1.2x + 0.62)e^{3x} + (x + 1)e^{-x} + c_1 \cos(x) + c_2 \sin(x).
\]

Now let’s consider finding a general solution to \( y'' + 3y' + 2y = 7 \cos(x) - \sin(x) \).

The complementary homogenous equation has a general solution of \( y = c_1 e^{-x} + e^{-2x} \).

Let’s find a particular solution. Let’s suppose that we can find one of the form \( y = A \cos(x) + B \sin(x) \) where \( A, B \) are constants.

Then \( y' = -A \sin(x) + B \cos(x) \) and \( y'' = -A \cos(x) - B \sin(x) \). By substitution, this yields

\[
-A \cos(x) - B \sin(x) + 3(-A \sin(x) + B \cos(x)) + 2(A \cos(x) + B \sin(x)) = 7 \cos(x) - \sin(x)
\]

which requires

\[
A + 3B = 7
-3A + B = -1.
\]

This requires \( A = 1 \) and \( B = 2 \). So \( y = \cos(x) + 2 \sin(x) \) is a particular solution.

Thus the general solution is \( y = \cos(x) + 2 \sin(x) + c_1 e^{-x} + c_2 e^{-2x} \).
What if we try to find the general solution to \( y'' + 4y = 4 \cos(2x) + \sin(2x) \)?

Well, let’s see. The general solution to the complementary homogeneous equation is \( y = c_1 \cos(2x) + c_2 \sin(2x) \). But if we assume there is a particular solution of the form \( y_p = A \cos(2x) + B \sin(2x) \) we get \( y''_p + 4y_p = 0 \) as \( y_p \) is a linear combination of the fundamental set of solutions to the complementary o.d.e.

So instead let’s try for a particular solution of the form \( y = x(A \cos(2x) + B \sin(2x)) \).

Then \( y' = (A \cos(2x) + B \sin(2x)) + x(-2A \sin(2x) + 2B \cos(2x)) \) and
\[
 y'' = -2A \sin(2x) + 2B \cos(2x) + (-2A \sin(2x) + 2B \cos(2x)) + x(-4A \cos(2x) - 4B \sin(2x)).
\]

We can simplify to get \( y'' = -4A \sin(2x) + 4B \cos(2x) - 4x(A \cos(2x) + B \sin(2x)). \)

Then \( y'' + 4y = -4A \sin(2x) + 4B \cos(2x) - 4x(A \cos(2x) + B \sin(2x)) + 4x(A \cos(2x) + B \sin(2x)) = -4A \sin(2x) + 4B \cos(2x). \)

So we set this equal to \( 4 \cos(2x) + \sin(2x) \) to get \( A = -\frac{1}{4} \) and \( B = 1 \).

So \( y = x \left( -\frac{1}{4} \cos(2x) + \sin(2x) \right) \) is a particular solution, and hence, a general solution is
\[
 y = x \left( -\frac{1}{4} \cos(2x) + \sin(2x) \right) + c_1 \cos(2x) + c_2 \sin(2x).
\]

One more example and we’ll summarize what to do when trying to solve
\[
 ay'' + by' + cy = p(x) \cos(\omega x) + q(x) \sin(\omega x)
\]
where \( p(x), q(x) \) are polynomials.

Let’s find a general solution for \( y'' + 2y' + y = 4x \cos(x) + (1 + x) \sin(x) \).

A general solution for the complementary homogeneous equation is \( y = (c_1 + c_2 x) e^{-x} \).

Suppose there is a particular solution of the form \( y = (ax + b) \cos(x) + (cx + d) \sin(x) \).

Then \( y' = a \cos(x) - (ax + b) \sin(x) + c \sin(x) + (cx + d) \cos(x) \) and
\[
 y'' = -a \sin(x) - a \sin(x) - (ax + b) \cos(x) + c \cos(x) + c \cos(x) - (cx + d) \sin(x).
\]

With a little bit of simplification, we get \( y' = (a + d) \cos(x) + (-b + c) \sin(x) - ax \sin(x) + cx \cos(x) \) and
\[
 y'' = (-b + 2c) \cos(x) + (-2a - d) \sin(x) - cx \sin(x) - ax \cos(x).
\]

Then \( y'' + 2y' + y \) becomes
\[
 2(a + c + d) \cos(x) - 2(a + b - c) \sin(x) - 2ax \sin(x) + 2cx \cos(x).
\]
Setting this equal to $4x \cos(x) + (1 + x) \sin(x)$ yields

\begin{align*}
2a + 2c + 2d &= 0 \\
-2a - 2b + 2c &= 1 \\
-2a &= 1 \\
2c &= 4.
\end{align*}

From this we get $a = -0.5$, $b = 2$, $c = 2$, and $d = -1.5$.

So $y = (-0.5x + 2) \cos(x) + (2x - 1.5) \sin(x)$ is a particular solution of $y'' + 2y' + y = 4x \cos(x) + (1 + x) \sin(x)$.

And the general solution is $y = (-0.5x + 2) \cos(x) + (2x - 1.5) \sin(x) + (c_1 + c_2x)e^{-x}$.

(ELFY) Check that this method (assuming that $y = (ax + b) \cos(x) + (cx + d) \sin(x)$ would be a particular solution) does not work for $y'' + y = 4x \cos(x) + (1 + x) \sin(x)$.

**SUMMARY** (skipping the proofs): Let $ay'' + by' + c = p(x) \cos(\omega x) + q(x) \sin(\omega x)$ where $p(x), q(x)$ are polynomials and $a, b, c, \omega$ are real numbers. Let $k$ be the maximum degree between $p(x)$ and $q(x)$.

1. If $\cos(\omega x)$ and $\sin(\omega x)$ are not solutions of the complementary homogeneous equation then there is a particular solution of the form

   \[y = (a_0 + a_1x + \cdots + a_kx^k) \cos(\omega x) + (b_0 + b_1x + \cdots + b_kx^k) \sin(\omega x).\]

2. If $\cos(\omega x)$ and $\sin(\omega x)$ are solutions of the complementary homogeneous equation then there is a particular solution of the form

   \[y = (a_1x + \cdots + a_{k+1}x^{k+1}) \cos(\omega x) + (b_1x + \cdots + b_{k+1}x^{k+1}) \sin(\omega x).\]
Now let’s try to find a general solution of
\[ y'' - 5y' + 6y = e^{-x}(4x \cos (x) + (-1 + 18x) \sin (x)) . \]

The general solution to the complementary homogeneous equation is
\[ y = c_1 e^{2x} + c_2 e^{3x} . \]

Let’s assume that we can find a particular solution of the form
\[ y = u e^{-x} . \]

Then
\[ y' = u' e^{-x} - u e^{-x} \]
and
\[ y'' = u'' e^{-x} - 2u' e^{-x} + u e^{-x} . \]

So
\[ u'' e^{-x} - 2u' e^{-x} + u e^{-x} - 5(u' e^{-x} - u e^{-x}) + 6ue^{-x} = e^{-x}(x \cos (x) - \sin (x)) \]
which implies
\[ u'' - 7u' + 12u = 4x \cos (x) + (-1 + 18x) \sin (x) . \]

Suppose this has a particular solution of the form
\[ u = (ax + b) \cos (x) + (cx + d) \sin (x) . \]

Then
\[ u' = a \cos (x) - (ax + b) \sin (x) + c \sin (x) + (cx + d) \cos (x) \]
and
\[ u'' = -a \sin (x) - a \sin (x) - (ax + b) \cos (x) + c \cos (x) + c \cos (x) - (cx + d) \sin (x) . \]

This simplifies to
\[ u' = (a + d) \cos (x) + (-b + c) \sin (x) - ax \sin (x) + cx \cos (x) \]
and
\[ u'' = (-b + 2c) \cos (x) + (-2a - d) \sin (x) - cx \sin (x) - ax \cos (x) . \]

Then
\[ u'' - 7u' + 12u \]
becomes
\[ (-7a + 11b + 2c - 7d) \cos (x) + (-2a + 7b - 7c + 11d) \sin (x) + (7a + 11c)x \sin (x) + (11a - 7c)x \cos (x) . \]

Setting this equal to
\[ 4x \cos (x) + (-1 + 18x) \sin (x) \]
requires
\[
\begin{align*}
-7a + 11b + 2c - 7d &= 0 \\
-2a + 7b - 7c + 11d &= -1 \\
7a + 11c &= 18 \\
11a - 7c &= 4 .
\end{align*}
\]

By starting with the final two equations, we get
\[ a = 1 \]
and
\[ c = 1 . \]

Then the first two equations reduce to
\[
\begin{align*}
11b - 7d &= 5 \\
7b + 11d &= 8 .
\end{align*}
\]

Then
\[ b = \frac{111}{170} \]
and
\[ d = \frac{53}{170} . \]
So

\[ y = u e^{-x} = \left( \left( x + \frac{111}{170} \right) \cos(x) + \left( x + \frac{53}{170} \right) \sin(x) \right) e^{-x} \]

is a particular solution.

Then a general solution is

\[ y = \left( \left( x + \frac{111}{170} \right) \cos(x) + \left( x + \frac{53}{170} \right) \sin(x) \right) e^{-x} + c_1 e^{2x} + c_2 e^{3x}. \]
Reduction of Order

Suppose we want to find a general solution to the linear second order o.d.e. $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$ given a non-trivial solution $y_1$ of the complementary homogeneous equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$.

Let’s look for a solution of the form $y = uy_1$. Then $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$. By substitution we get

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)(uy_1) = F(x)$$

which reduces to

$$P_0(x)y_1u'' + (2P_0(x)y_1' + P_1(x)y_1)u' = F(x).$$

This can be rewritten as

$$Q_0(x)u'' + Q_1(x)u' = F(x),$$

where $Q_0(x) = P_0(x)y_1$ and $Q_1(x) = 2P_0(x)y_1' + P_1(x)y_1$.

This is a first order equation in terms of $z = u'$. So we can solve for $z = u'$ using first order methods. Then integrate to get $u$.

The method illustrated above is called reduction of order. This method does require that we already have a non-trivial solution the complementary o.d.e.
Example: Let’s find the general solution to \( x^2y'' + xy' - y = 4x^{-2} \) given that \( y = x \) is a solution to the complementary equation.

Setting \( y = ux \) yields \( y' = u'x + u \) and \( y'' = u''x + 2u' \). By substitution, we get

\[
x^2(u''x + 2u') + x(u'x + u) - ux = 4x^{-2}
\]

which implies

\[
x^3u'' + 3x^2u' = 4x^{-2}.
\]

In terms of \( z = u' \) this is equivalent to

\[
z' + \frac{3}{x}z = \frac{4}{x^3}.
\]

Let’s find an integrating factor: \( \mu(x) = e^{\int \frac{3}{x} \, dx} = e^{3\ln|x|} = \pm x^3 \).

So let’s multiply both sides of \( z' + \frac{3}{x}z = \frac{4}{x^3} \) by \( x^3 \):

\[
x^3z' + 3x^2z = \frac{4}{x^2}.
\]

Thus \( (x^3z)' = \frac{4}{x^2} \), and by integrating both sides, this becomes

\[
x^3z = -\frac{4}{x} + c = \frac{cx - 4}{x}.
\]

So

\[
z = \frac{cx - 4}{x^4} = \frac{c}{x^3} - \frac{4}{x^4}.
\]

Integrating both sides of this yields

\[
u = \frac{c_1}{x^2} + \frac{4}{3x^3} + c_2.
\]

Finally we get a general solution

\[
y = ux = \frac{c_1}{x} + \frac{4}{3x^2} + c_2x.
\]

Incidentally we see from the above that \( yp = \frac{4}{3x^2} \) is a particular solution and that \( \left\{ x, \frac{1}{x} \right\} \) is a fundamental set of solutions to the complementary homogeneous o.d.e. That’s what justifies that it’s a general solution.
Let’s solve the IVP \((x^2 - 4)y'' + 4xy' + 2y = x + 2, \ y(0) = -1, \ y'(0) = -3\), given that \(y = \frac{1}{x - 2}\) is a solution to the complementary homogeneous o.d.e.

Let \(y = u(x - 2)^{-1}\) be a solution. Then \(y' = u'(x - 2)^{-2} - u(x - 2)^{-2}\) and \(y'' = u''(x - 2)^{-1} - 2u'(x - 2)^{-2} + 2u(x - 2)^{-3}\).

By substitution we get
\[
(x^2 - 4)(u''(x-2)^{-1} - 2u'(x-2)^{-2} + 2u(x-2)^{-3}) + 4x(u'(x-2)^{-1} - u(x-2)^{-2}) + 2u(x-2)^{-1} = x + 2,
\]
so
\[
(x + 2)u'' - 2(x + 2)(x - 2)^{-1}u' - 2(x + 2)(x - 2)^{-2}u + 4x(x - 2)^{-1}u' - 4x(x - 2)^{-2}u + 2(x - 2)^{-1}u = x + 2,
\]
so
\[
(x + 2)u'' + (-2(x + 2)(x - 2)^{-1} + 4x(x - 2)^{-1})u' + (2(x + 2)(x - 2)^{-2} - 4x(x - 2)^{-2} + 2(x - 2)^{-1})u = x + 2,
\]
so
\[
(x + 2)u'' + 2u' = x + 2.
\]

Letting \(z = u'\) and dividing through by \(x + 2\) yields \(z' + 2(x + 2)^{-1}z = 1\).

So we can easily find an integrating factor: \(\mu = e^{\frac{2}{x + 2} dx} = e^{2 \ln |x + 2|} = (x + 2)^2\).

Let’s solve \((x + 2)^2z' + 2(x + 2)z = (x + 2)^2\), which is equivalent to
\[
((x + 2)^2z)' = x^2 + 4x + 4.
\]

Integrating both sides gives \((x + 2)^2z = \frac{1}{3}x^3 + 2x^2 + 4x + c\).

So
\[
z = \frac{x^3}{3(x + 2)^2} + \frac{2x^2}{(x + 2)^2} + \frac{4x}{(x + 2)^2} + \frac{c}{(x + 2)^2} = \frac{x^3}{3(x + 2)^2} + \frac{2x}{x + 2} + \frac{c}{(x + 2)^2}.
\]

We can integrate each term (using the basic substitution \(t = x + 2\)) and simplify. This is left to you (ELFY). You should get...

\[
u = \frac{c_1}{x + 2} + \frac{(x + 2)^2 + c_2}{6}.
\]
Finally we get a general solution by multiplying by \((x - 2)^{-1}\):

\[
y = u(x - 2)^{-1} = \frac{c_1}{x^2 - 4} + \frac{(x + 2)^2 + c_2}{6(x - 2)}.
\]

Now we may impose the initial conditions:

\[
y' = \frac{x + 2}{3(x - 2)} - \frac{(x + 2)^2 + c_2}{6(x - 2)^2} - \frac{2c_1x}{(x^2 - 4)^2}.
\]

Then \(y(0) = -1\) implies \(-1 = -\frac{c_1}{4} - \frac{4 + c_2}{12}\) and \(y'(0) = -3\) implies \(-3 = -\frac{1}{3} + \frac{4 + c_2}{24}\).

Then \(c_2 = -68\). Then \(c_1 = \frac{76}{3}\).

So the solution to the IVP is

\[
y = \frac{76}{3(x^2 - 4)} + \frac{(x + 2)^2 - 68}{6(x - 2)}.
\]
Euler Equations

A second order differential equation that can be written in the form

\[ ax^2y'' + bxy' + cy = g(x) \]

where \(a\), \(b\), and \(c\) are real constants with \(a \neq 0\) is called an Euler equation.

By the existence and uniqueness theorem we see that if \(g\) is continuous everywhere, except possibly at 0 that an Euler equation has solutions defined on \((-\infty, 0)\) and \((0, \infty)\). To see this put the differential equation into standard form and observe that we have a discontinuity at 0 for both \(p(x)\) and \(q(x)\).

We will restrict our attention to the interval \((0, \infty)\).

To find the general solution of the homogeneous Euler equation

\[ ax^2y'' + bxy' + cy = 0 \]

we will assume that there is a solution of the form \(y = x^m\). Then \(y' = mx^{m-1}\) and \(y'' = m(m-1)x^{m-2}\). Substituting into the differential equation:

\[ ax^2m(m-1)x^{m-2} + bmx^{m-1} + cx^m = 0 \]

\[ \Rightarrow am(m-1)x^m + bm x^m + cx^m = 0 \]

Since we have \(x > 0\) we can divide by \(x^m\) to get the characteristic equation

\[ am(m-1) + bm + c = 0 \]

\[ am^2 + (b-a)m + c = 0 \]

As with constant coefficient equations we have three cases.

Case 1: Two real solutions, \(m_1\) and \(m_2\). The general solution is

\[ y = c_1x^{m_1} + c_2x^{m_2}. \]
Example: Find the general solution of $6x^2y'' + 17xy' + 3y = 0$ on $(0, \infty)$.

The characteristic equation is

$$6m^2 + (17 - 6)m + 3 = 0$$

$$6m^2 + 11m + 3 = 0$$

$$(3m + 1)(2m + 3) = 0$$

Then $y_1 = x^{-\frac{1}{3}}$ and $y_2 = x^{-\frac{3}{2}}$ and the general solution is $y = c_1x^{-\frac{1}{3}} + c_2x^{-\frac{3}{2}}$.

Case 2: One real solution, $m_1$. One solution is $y_1 = x^{m_1}$. Using reduction of order and that the polynomial must be $a(m - m_1)^2$ we get a second linearly independent solution given by $y_2 = x^{m_1} \ln x$ (ELFY). The general solution is then

$$y = c_1x^{m_1} + c_2x^{m_1} \ln x.$$ 

Example: Find the general solution of $4x^2y'' + 8xy' + y = 0$ on $(0, \infty)$.

The characteristic equation is

$$4m(m - 1) + 8m + 1 = 0$$

$$4m^2 + 4m + 1 = 0$$

$$(2m + 1)^2 = 0$$

Then $y_1 = x^{-\frac{1}{2}}$ and $y_2 = x^{-\frac{1}{2}} \ln x$. The general solution is $y = \frac{c_1 + c_2 \ln x}{\sqrt{x}}$. 

Case 3: Two complex solutions $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

\[ y = x^{\alpha+i\beta} \]

\[ y = x^{\alpha} e^{i\beta} \]

To deal with $x^{i\beta}$ we use Euler’s identity:

\[ x^{i\beta} = e^{i\beta \ln x} = \cos (\beta \ln x) + i \sin (\beta \ln x) \]

From this we can see that the general solution is given by

\[ y = x^{\alpha} (c_1 \cos (\beta \ln x) + c_2 \sin (\beta \ln x)) \]

Example: Find the general solution for $x^2 y'' + 3xy' + 10y = 0$ on $(0, \infty)$.

The characteristic equation is $m^2 + 2m + 10 = 0$. From the quadratic formula

\[ m = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2} = -1 \pm 3i \]

So the general solution is then given by $y = \frac{c_1 \cos (3 \ln x) + c_2 \sin (3 \ln x)}{x}$