Linear Second Order o.d.e.

A linear second order o.d.e. can be written in the form $P_0(x)y'' + P_1(x)y' + P_2(x)y = f(x)$ where $P_0(x) \neq 0$ (as a function). $f(x)$ is called the forcing function.

As before, if $f(x) = 0$ then the o.d.e. is homogeneous. Otherwise it's non-homogeneous. If P_0, P_1, P_2 are constants, then the equation is called a constant coefficient equation.

We begin by considering the homogeneous case. In the homogeneous case we may write the equation in the form $y'' + p(x)y' + q(x)y = 0$.

This theorem gives sufficient conditions for existence and uniqueness of a solution to a linear homogeneous second order IVP. We omit the proof.

Proposition: Suppose p, q are continuous functions on (a, b), x_0 is in (a, b) and k_0, k_1 are arbitrary real numbers. Then the IVP

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = k_0,\ y'(x_0) = k_1,$$

has a unique solution on (a, b).

As before $y = 0$ will be called the trivial solution.

Consider the linear homogeneous second order o.d.e. $y'' + \dfrac{4x}{x^2-1}y' + \dfrac{2}{x^2-1}y = 0$.

Let's verify that $y = \dfrac{1}{x+1}$ is a solution:

Starting with $y = \dfrac{1}{x+1}$. $y' = -\dfrac{1}{(x+1)^2}$ and $y'' = \dfrac{2}{(x+1)^3}$. Then

$$\begin{aligned}
y'' + \frac{4x}{x^2-1}y' + \frac{2}{x^2-1}y &= \frac{2}{(x+1)^3} + \frac{4x}{x^2-1}\left(-\frac{1}{(x+1)^2}\right) + \frac{2}{x^2-1}\left(\frac{1}{x+1}\right) \\
&= \frac{2}{(x+1)^3} - \frac{4x}{(x-1)(x+1)^3} + \frac{2}{(x-1)(x+1)^2} \\
&= \frac{2(x-1) - 4x + 2(x+1)}{(x-1)(x+1)^3} \\
&= 0.
\end{aligned}$$

Exercise Left For You (ELFY): Show that $y = \dfrac{1}{x-1}$ is also a solution.

Proposition: If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ on (a, b) then any linear combination $y = c_1y_1 + c_2y_2$ is also a solution on (a, b).

Proof: Assume y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ on (a, b). Let c_1, c_2 be constants and $y = c_1y_1 + c_2y_2$.

Then

$$\begin{aligned} y'' + p(x)y' + q(x) &= (c_1y_1'' + c_2y_2'') + p(x)(c_1y_1' + c_2y_2') + q(x)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ &= 0. \end{aligned}$$

That completes the proof □

So $y = \dfrac{c_1}{x+1} + \dfrac{c_1}{x-1}$ are all solutions of $y'' + \dfrac{4x}{x^2-1}y' + \dfrac{2}{x^2-1}y = 0$.

Assume y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ on (a, b). $\{y_1, y_2\}$ is called a fundamental set of solutions on (a, b) if every solution can be expressed as $c_1y_1 + c_2y_2$ for an appropriate choice of constants.
When $\{y_1, y_2\}$ is a fundamental set of solutions on (a, b), $c_1y_1 + c_2y_2$ is said to be the general solution on (a, b).

Two functions y_1, y_2 defined on (a, b) are linearly independent on (a, b) if neither is a constant multiple of the other on (a, b). Otherwise they are said to be linearly dependent.

Proposition: Suppose p, q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is a fundamental set of solutions on (a, b) if and only if y_1, y_2 are linearly independent on (a, b).

For now we postpone the proof. We begin by understanding what is must be shown in order to prove $\{y_1, y_2\}$ is a fundamental set...

$\{y_1, y_2\}$ is a fundamental set if and only if every IVP has a solution of the form $c_1y_1 + c_2y_2$.

That equivalent to always being able to solve the system of equations for c_1, c_2

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= k_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= k_1 \end{aligned}$$

for any choice of k_0, k_1.

This system has a solution for all k_0, k_1 if and only if $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$ (forces the lines to not be parallel in the c_1c_2-plane).

In the case where $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$, Cramer's Rule gives us

$$c_1 = \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \text{ and } c_2 = \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}.$$

We define the Wronskian of set of two functions $\{y_1, y_2\}$ as the function $W = y_1y_2' - y_1'y_2$, which can be written as the determinant:

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Proposition: Suppose p, q are continuous on (a, b) and $\{y_1, y_2\}$ are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b). Let x_0 be any point in (a, b). Then the Wronskian of $\{y_1, y_2\}$ satisfies

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt} \text{ (called Abel's formula).}$$

As a consequence, either $W = 0$ on (a, b) or W has no zeros on (a, b).

Proof: $W = y_1y_2' - y_1'y_2$ so $W' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' = y_1y_2'' - y_1''y_2$.

We can replace $y_i'' = -p(x)y_i' - q(x)y_i$ for $i = 1, 2$ to get

$$W' = y_1(-p(x)y_2' - q(x)y_2) - (-p(x)y_1' - q(x)y_1)y_2 = -p(x)y_1y_2' + p(x)y_1'y_2 = -p(x)W.$$

So W is a solution to the IVP $W' + p(x)W = 0$. We know that for this homogeneous first order o.d.e. either $W = 0$ on (a, b) or $W = ce^{-\int p(x)\,dx}$ with $c \neq 0$ on (a, b). In the latter case $W \neq 0$ for all points on (a, b).

So if $W(x_0) = 0$ then $W = 0$. If $W(x_0) \neq 0$ then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt} \text{ which actually works in both cases } \square$$

Let's verify Abel's formula works for solutions $y_1 = x^2$ and $y_2 = x^{-2}$ of $x^2y'' + xy' - 4y = 0$.

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{pmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1} = -\frac{4}{x}.$$

By rewriting the equation in the form $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$ we see that $p(x) = \frac{1}{x}$. Then on an interval avoiding $x = 0$,

$$e^{-\int_{x_0}^{x} p(t)\,dt} = e^{-\int_{x_0}^{x} \frac{1}{t}\,dt} = e^{-\ln(x/x_0)} = \frac{x_0}{x}.$$

Thus $W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt} = -\frac{4}{x_0} \cdot \frac{x_0}{x} = -\frac{4}{x} = W(x)$.

Proposition: Suppose p, q are continuous on (a, b) and $\{y_1, y_2\}$ are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b). Then y_1, y_2 are linearly independent on (a, b) if and only if the Wronskian $W = y_1y_2' - y_1'y_2$ has no zeros on (a, b).

Proof: First we assume $W(x_0) = 0$ for some x_0 on (a, b) and show that it must be the case that y_1, y_2 are linearly dependent on (a, b). Clearly y_1, y_2 are linearly dependent on (a, b) if $y_1 = 0$ on (a, b). Assume otherwise. Then y_1 is non-zero on some subinterval I of (a, b) as it's continuous (because it's differentiable).

Then on I, (y_2/y_1) is defined and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}.$$

Since $W(x_0) = 0$ a previous proposition implies that $W = 0$ on I. Hence $y_2/y_1 = c$ on I. So $y_2 = cy_1$ on I. We'd like to show this is the case on (a, b). Let $Y = cy_1 - y_2$. Since Y is a linear combination of y_1 and y_2, it is a solution of the IVP $y'' + p(x)y' + q(x)y = 0$, with $y(x_0) = y'(x_0) = 0$ for some x_0 in I.

Since the trivial solution $Y = 0$ works and a proposition guarantees the IVP has a unique solution, it must be the case that $Y = 0$ and hence $y_2 = cy_1$ on (a, b).

Now assume W has no zeros on (a, b). Then y_1 cannot be the zero function on (a, b) (or W is zero) and therefore there is a subinterval on which y_1 has no zeros. Then on I we have as above $(y_2/y_1)' = W/y_1^2$. Since W has no zeros on I it follows that y_2/y_1 is not constant on I, so y_2 is not a multiple of y_1 on I, let alone (a, b).

A similar argument, but where $(y_1/y_2)' = -W/y_2^2$, shows that y_1 is not a multiple of y_2 on (a, b).

Hence y_1, y_2 are linearly independent $\square$

Now we complete a proof of an earlier proposition, which states:

> Suppose p, q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions to
> $$y'' + p(x)y' + q(x)y = 0$$
> on (a, b) is a fundamental set of solutions on (a, b) if and only if y_1, y_2 are linearly independent on (a, b).

Proof: From the last proposition two solutions y_1, y_2 are linearly independent if and only if W has no zeros on (a, b). In motivating the definition of the Wronskian, we saw that W has no zeros on (a, b) if and only if the IVP posed in the first proposition in this section has a solution that is a linear combination of y_1 and y_2. Putting these facts together, $\{y_1, y_2\}$ is a fundamental set of solutions if and only if y_1, y_2 are linearly independent $\square$

Linear Second Order Constant-Coefficient Equations

For the moment let's restrict our view to some very simple linear second order o.d.e. If a, b, c are real numbers and $a \neq 0$ then the linear second order o.d.e.

$$ay'' + by' + cy = f(x)$$

is said to be a constant coefficient equation. If $f(x) = 0$ it is a homogeneous constant coefficient equation. Otherwise it is non-homogeneous.

Suppose we are trying to find the general solution to the linear homogeneous constant coefficient second order o.d.e. $ay'' + by' + cy = 0$.

Suppose that r is a constant and $y = e^{rx}$. Then $y' = re^{rx} = ry$ and $y'' = r^2e^{rx} = r^2y$.

By substitution, $ay'' + by' + c = 0$ becomes $(ar^2 + br + c)e^{rx} = 0$, implying that if r satisfies $ar^2 + br + c = 0$ then $y = e^{rx}$ is a solution.

The quadratic $p(r) = ar^2 + br + c$ is called the characteristic polynomial of $ay'' + by' + c = 0$. The equation $p(r) = 0$ is the characteristic equation.

By the quadratic formula we have the the roots of the characteristic polynomial are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So we consider three cases:

Case 1: $b^2 - 4ac > 0$. In this case the characteristic polynomial has two distinct real roots.

Let $r_1 \neq r_2$ be the real roots. Then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are both solutions of $ay'' + by' + cy = 0$. Since $y_2/y_1 = e^{(r_2-r_1)x}$ is non-constant as $r_2 - r_1 \neq 0$ we have that y_1, y_2 are linearly independent. Thus the general solution to $ay'' + by' + c = 0$ is $y = c_1e^{r_1x} + c_2e^{r_2x}$.

Case 2: $b^2 - 4ac = 0$. In this case the characteristic polynomial has one repeated real root.

Let r_1 be the repeated root. So $y_1 = e^{r_1 x}$ is a solution of $ay'' + by' + cy = 0$. Furthermore, the characteristic polynomial is $a(x - r_1)^2$ so $b = -2ar_1$ and $c = ar_1^2$.

To find another solution, let's try the variation of parameters method:

Set $y_2 = uy_1 = ue^{r_1 x}$ and assume y_2 is a solution of $ay'' + by' + cy = 0$. Then

$$ay_2'' + by_2' + cuy_2 = 0 \rightarrow$$

$$a(u'e^{r_1 x} + r_1 ue^{r_1 x})' + b(u'e^{r_1 x} + r_1 ue^{r_1 x}) + cue^{r_1 x} = 0 \rightarrow$$

$$a(u''e^{r_1 x} + 2r_1 u'e^{r_1 x} + r_1^2 ue^{r_1 x}) + b(u'e^{r_1 x} + r_1 ue^{r_1 x}) + cue^{r_1 x} = 0 \rightarrow$$

$$[a(u'' + 2r_1 u' + r_1^2 u) + b(u' + r_1 u) + cu]e^{r_1 x} = 0 \rightarrow$$

$$[au'' + 2ar_1 u' + ar_1^2 u + bu' + br_1 u + cu]e^{r_1 x} = 0 \rightarrow$$

$$[au'' + 2ar_1 u' + bu' + (ar_1^2 + br_1 + c)u]e^{r_1 x} = 0 \rightarrow$$

$$[au'' + (2ar_1 + b)u']e^{r_1 x} = 0 \rightarrow$$

$$au'' + (2ar_1 + b)u' = 0 \rightarrow au'' = 0 \text{ since } b = -2ar_1.$$

Therefore $u = c_1 + c_2 x$. In particular we can pick $u = x$. Then $y_2 = xe^{r_1 x}$ is a solution of $ay'' + by' + cy = 0$.

Moreover, $y_2/y_1 = x$ is non-constant, so y_1, y_2 are linearly independent (on any interval). Hence the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 xe^{r_1 x} = (c_1 + c_2 x)e^{r_1 x}.$$

Case 3: $b^2 - 4ac < 0$. Then the roots of the characteristic polynomial are complex conjugates $\lambda \pm \omega i$, where λ, ω are real numbers and $\omega \neq 0$.

Then $e^{(\lambda+\omega i)x}$ and $e^{(\lambda-\omega i)x}$ are solutions of $ay'' + by' + cy = 0$. While any linear combination of these is a solution, some are not real-valued, even for real values of x. We restrict our attention towards finding the general solution in the real number case.

Note: $e^{(\lambda\pm\omega i)x} = e^{\lambda x}e^{\pm\omega ix} = e^{\lambda x}(\cos(\omega x) \pm i\sin(\omega x))$ using Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

In particular, any linear combination of these are solutions, so we can take

$$y_1 = \frac{1}{2}\left(e^{\lambda x}(\cos(\omega x) + i\sin(\omega x)) + e^{\lambda x}(\cos(\omega x) - i\sin(\omega x))\right) = e^{\lambda x}\cos(\omega x) \text{ and}$$

$$y_2 = \frac{1}{2i}\left(e^{\lambda x}(\cos(\omega x) + i\sin(\omega x)) - e^{\lambda x}(\cos(\omega x) - i\sin(\omega x))\right) = e^{\lambda x}\sin(\omega x).$$

Since $y_2/y_1 = \tan(\omega x)$ is non-constant y_1, y_2 are linearly independent, and hence the general solution to $ay'' + by' + cy = 0$ is

$$y(t) = c_1 e^{\lambda x}\cos(\omega x) + c_2 e^{\lambda x}\sin(\omega x) = (c_1\cos(\omega x) + c_2\sin(\omega x))e^{\lambda x}.$$

Since we are only interested in real solutions we assume c_1, c_2 are real. If we were interested in complex solutions we'd allow c_1, c_2, x to be complex numbers.

Examples:

(1) Find the general solution of $y'' + 5y' - 14y = 0$.

The characteristic polynomial is $r^2 + 5r - 14 = (r + 7)(r - 2)$. The roots are $r = -7, 2$. So the general solution is

$$y = c_1 e^{-7x} + c_2 e^{2x}.$$

(2) Solve the IVP $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 5$.

The characteristic polynomial is $r^2 - 6r + 9 = (r - 3)^2$. The repeated root is $r = 3$. So the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x}.$$

Then $y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x}$.

So imposing the initial conditions yields

$$c_1 = 1 \text{ and } 3c_1 + c_2 = 5 \rightarrow c_1 = 1 \text{ and } c_2 = 2.$$

So the solution is

$$y = e^{3x} + 2xe^{3x}.$$

(3) Solve the IVP $y'' + 4y = 0$, $y(0) = -1$, $y'(0) = 2$.

The characteristic polynomial is $r^2 + 4$, which has complex roots $r = \pm 2i$. Since $\pm 2i = 0 \pm 2i$, the general solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Then $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$.

By imposing the initial conditions we get $c_1 = -1$ and $2c_2 = 2$, so $c_1 = -1$ and $c_2 = 1$.

So the solution is

$$y = -\cos(2x) + \sin(2x).$$

Let's consider the non-homogenous case where $y''+p(x)y'+q(x)y = f(x)$. As before we'll call the equation with $f(x) = 0$ the complementary equation. The following proposition, whose proof is beyond the scope of our class, implies the existence and uniqueness of a solutions to an IVP where $y'' + p(x)y' + q(x)y = f(x)$.

Proposition: Suppose p, q, f are continuous on (a, b) and x_0 is a point in (a, b). Then for any constants k_0 and k_1 the IVP

$$y'' + p(x)y' + q(x) = f(x), \; y(x_0) = k_0, \; y'(x_1) = k_1$$

has a unique solution on (a, b).

The following proposition gives a method for solving $y''+p(x)y'+q(x) = f(x)$ for its general solution.

Proposition: Suppose p, q, f are continuous on (a, b), y_p is a particular solution to

$$y'' + p(x)y' + q(x)y = f(x)$$

on (a, b), and y_1, y_2 is a fundamental set of solutions to the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b). Then y is a solution to $y'' + p(x)y' + q(x)y = f(x)$ on (a, b) if and only if $y = y_p + c_1y_1 + c_2y_2$ for some constants c_1, c_2.

Proof: If $y = y_p + c_1y_1 + c_2y_2$ then $y' = y_p' + c_1y_1' + c_2y_2'$ and $y'' = y_p'' + c_1y_1'' + c_2y_2''$, so

$$y''+p(x)y'+q(x)y = y_p''+p(x)y_p'+q(x)y_p+c_1(y_1''+p(x)y_1'+q(x)y_1)+c_2(y_2''+p(x)y_2'+q(x)y_2) = f(x).$$

So this proves that if $y = y_p + c_1y_1 + c_2y_2$ for some constants c_1, c_2 then y is a solution to $y'' + p(x)y' + q(x)y = f(x)$ (the reverse direction).

Now we show the forward direction. Suppose y is a solution to $y''+p(x)y'+q(x)y = f(x)$ on (a, b). Then consider $w = y - y_p$. It suffices to show that w is a solution to the complementary equation $y'' + p(x)y' + q(x)y = 0$ on (a, b).

Well,

$$w''+p(x)w'+q(x)y = y''-y_p''+p(x)(y'-y_p')+q(x)(y-y_p) = y''+p(x)y'+q(x)y-(y_p''+p(x)y_p'+q(x)y_p) = 0 \quad \square$$

Let's solve the following IVP: $y'' + 9y = 9$, $y(0) = 0$, $y'(0) = 4$.

The characteristic polynomial of the complementary equation $y'' + 9y = 0$ is $r^2 + 9$ which has roots $r = \pm 3i$. So $\{\cos(3x), \sin(3x)\}$ are a fundamental set of solutions to $y'' + 9y = 0$.

Obviously $y_1 = 1$ is a particular solution of $y'' + 9y = 9$. Hence $y = 1 + c_1 \cos(3x) + c_2 \sin(3x)$ is the general solution of $y'' + 9y = 9$.

Imposing the initial condition $y(0) = 0$ implies $0 = 1 + c_1$, so $c_1 = -1$. Imposing the initial condition $y'(0) = 4$ implies $4 = 3c_2$, so $c_2 = \frac{4}{3}$.

So the solution to the IVP is

$$y = 1 - \cos(3x) + \frac{4}{3} \sin(3x).$$

Method of Undetermined Coefficients

The method of undetermined coefficients is a guess and check method that will only use for constant coefficient equations and is only feasible for certain forms of the forcing functions $f(x)$.

The method of undetermined coefficients works if the forcing function is a linear combination of functions of the forms:

$1,\ x^n,\ n = 1, 2, 3, \cdots,$

$e^{ax},\ x^n e^{ax},\ \ n = 1, 2, 3, \cdots$

$\cos(\beta x),\ \sin(\beta x),\ e^{ax}\cos(\beta x),\ e^{ax}\sin(\beta x)$

$x^n\cos(\beta t),\ x^n\sin(\beta x),\ \ n = 1, 2, 3, \cdots$

$x^n e^{ax}\cos(\beta x),\ x^n e^{ax}\sin(\beta x),\ \ n = 1, 2, 3, \cdots$

In a general sense (we will see an exception later) you guess at a form of the solution that is similar to the forcing function. You guess will contain unknown coefficients that you will need to solve for. If the term contains t^n you must include all lower order terms of the polynomial.

Example (without the exception): Let's find the general solution to $y'' - 4y' + 4y = 4x^2 + 2x$.

The characteristic polynomial of the complementary equation $y'' - 4y' + 4y = 0$ is $r^2 - 4r + 4 = (r - 2)^2$. So $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions to the complementary equation $y'' - 4y' + 4y = 0$.

We can find a particular solution by any means necessary. Suppose we can find y_p as a polynomial solution to $y'' - 4y' + 4y = 4x^2 + 2x$. It would have to be degree two. So let $y_p = ax^2 + bx + c$. Then $y' = 2ax + b$ and $y'' = 2a$.

Then $2a - 4(2ax + b) + 4(ax^2 + bx + c) = 4x^2 + 2x$ implies $4a = 4$, $-8a + 4b = 2$, and $2a - 4b + 4c = 0$. So $a = 1$, $b = 2.5$, and $c = 2$. So $y_p = x^2 + 2.5x + 2$ is a particular solution. So the general solution is $y = x^2 + 2.5x + 2 + (c_1 + c_2 x)e^{2x}$.

Proposition (Superposition Principle): Suppose p, q, f_1, f_2 are continuous on (a, b), y_{p_1} is a particular solution to $y'' + p(x)y' + q(x)y = f_1(x)$ on (a, b), and y_{p_2} is a particular solution to $y'' + p(x)y' + q(x)y = f_2(x)$ on (a, b). Then $y = y_{p_1} + y_{p_2}$ is a solution to $y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$ on (a, b).

The proof is straightforward and is an easy exercise left for you (ELFY).

Let's find the general solution to $y'' - 7y' + 12y = 5e^{-2x} + x^2$.

Clearly the general solution to the complementary homogeneous equation is $y = c_1 e^{3x} + c_2 e^{4x}$.

We can try to find a pair of particular solutions to $y'' - 7y' + 12y = 5e^{-2x}$ and $y'' - 7y' + 12y = x^2$ respectively.

Let's begin with the former. Let's guess there is a solution of the form $y = Ae^{-2x}$ (why?) and let's substitute it into $y'' - 7y' + 12y = 5e^{-2x}$:

$$4Ae^{-2x} - 7(-2Ae^{-2x}) + 12Ae^{-2x} = 5e^{-2x} \rightarrow$$

$$4A + 14A + 12A = 5 \rightarrow 30A = 5 \rightarrow A = \frac{1}{6}.$$

So $y_{P_1} = \frac{1}{6}e^{-2x}$ is a particular solution.

Now we proceed with the latter. Let's guess there is a solution of the form $y = ax^2 + bx + c$ and let's substitute it into $y'' - 7y' + 12y = x^2$:

$$2a - 7(2ax + b) + 12(ax^2 + bx + c) = x^2 \rightarrow$$

$$2a - 7b + 12c = 0, \ 12b - 14a = 0, \text{ and } 12a = 1 \rightarrow a = \frac{1}{12} \rightarrow b = \frac{7}{72} \rightarrow c = \frac{37}{864}.$$

So $y_{P_2} = \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864}$ is a particular solution.

By superposition, $y_P = \frac{1}{6}e^{-2x} + \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864}$, is a particular solution of $y'' - 7y' + 12y = 5e^{-2x} + x^2$.

So the general solution is $y = \frac{1}{6}e^{-2x} + \frac{1}{12}x^2 + \frac{7}{72}x + \frac{37}{864} + c_1 e^{3x} + c_2 e^{4x}$.

The exception: What if we are trying to solve $y'' - 7y' + 12y = 5e^{4x}$?

We may try as before: We know that $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions to the complementary homogeneous o.d.e. Then guess $y_P = Ae^{4x}$ is a particular solution.

Then by substitution, $y_P'' - 7y_P' + 12y_P = 0$, as y_P is a linear combination of the fundamental set of solutions. So it cannot be a particular solution!

Instead we will try $y_P = ue^{4x}$ where u is some undetermined (non-constant) function of x:

Then

$$\begin{aligned} y_P'' - 7y_P' + 12y_P &= (ue^{4x})'' - 7(ue^{4x})' + 12ue^{4x} \\ &= (u'e^{4x} + 4ue^{4x})' - 7(u'e^{4x} + 4ue^{4x}) + 12ue^{4x} \\ &= (u''e^{4x} + 8u'e^{4x} + 16ue^{4x}) - 7u'e^{4x} - 28ue^{4x} + 12ue^{4x} \\ &= u''e^{4x} + u'e^{4x} \\ &= 5e^{4x}, \text{ which implies } u'' + u' = 5. \end{aligned}$$

By inspection, we can pick $u = 5x$.

So $y_P = 5xe^{4x}$ is a particular solution, and the general solution is $y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$.

Let's solve the IVP

$$y'' - 4y' + 4y = (1 - 3x + 6x^2)e^{2x},\ y(0) = 1,\ y'(0) = 4.$$

A fundamental set of solutions to the complementary homogeneous o.d.e. is $\{e^{2x}, xe^{2x}\}$.

Let's use the method of undetermined coefficients to find a particular solution:

Let $y = ue^{2x}$. Then $y' = u'e^{2x} + 2ue^{2x}$ and $y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$. Then

$$u''e^{2x} + 4u'e^{2x} + 4ue^{2x} - 4(u'e^{2x} + 2ue^{2x}) + 4(ue^{2x}) = (1 - 3x + 6x^2)e^{2x} \text{ implies}$$

$$u'' = 1 - 3x + 6x^2.$$

By integrating twice (choosing 0 for the constant of integration) we get

$$u = 0.5x^2 - 0.5x^3 + 0.5x^4 = \frac{1}{2}x^2(1 - x + x^2).$$

So a particular solution is $y_P = \frac{1}{2}x^2(1 - x + x^2)e^{2x}$ the general solution is

$$y = \frac{x^2e^{2x}}{2}(1 - x + x^2) + (c_1 + c_2x)e^{2x}.$$

$$y' = xe^{2x}(1 - x + x^2) + x^2e^{2x}(1 - x + x^2) + \frac{x^2e^{2x}}{2}(-1 + 2x) + c_2e^{2x} + 2(c_1 + c_2x)e^{2x}.$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 4$ yields:

$$1 = c_1 \text{ and } 4 = c_2 + 2c_1 = c_2 + 2 \rightarrow c_1 = 1 \text{ and } c_2 = 2.$$

So the solution to the IVP is

$$y = \frac{x^2e^{2x}}{2}(1 - x + x^2) + (1 + 2x)e^{2x}.$$

Now let's find the general solution for $y'' + y = 2xe^{-x} + (1 + 10x^2)e^{3x}$ using both the method of undetermined coefficients and superposition:

The complementary homogeneous equation has the general solution $y = c_1 \cos(x) + c_2 \sin(x)$.

Consider $y'' + y = 2xe^{-x}$. Suppose $y = ue^{-x}$ is a particular solution. Then

$$u''e^{-x} - 2u'e^{-x} + ue^{-x} + ue^{-x} = 2xe^{-x} \text{ implies}$$

$$u'' - 2u' + 2u = 2x.$$

Let's find a particular solution. Suppose that $u = ax + b$. Then $u' = a$ and $u'' = 0$, and thus $-2a + 2(ax + b) = 2x$ requires that $2a = 2$ and $2b - 2a = 0$. So use $a = 1$ and $b = 1$.

So $u = x + 1$ is a particular solution. Then $y = (x + 1)e^{-x}$ is a particular solution of $y'' + y = 2xe^{-x}$.

Now consider $y'' + y = (1 + 10x^2)e^{3x}$. Suppose $y = ue^{3x}$ is a particular solution. Then

Then $y'' + y = u''e^{3x} + 6u'e^{3x} + 9ue^{3x} + ue^{3x} = (1 + 10x^2)e^{3x}$ implies

$$u'' + 6u' + 10u = 1 + 10x^2$$

Suppose $u = ax^2 + bx + c$. Then $u' = 2ax + b$ and $u'' = 2a$, so

$$2a + 6(2ax + b) + 10(ax^2 + bx + c) = 1 + 10x^2 \rightarrow$$

$$\begin{aligned} 2a + 6b + 10c &= 1 \\ 12a + 10b &= 0 \\ 10a &= 10 \end{aligned}$$

So $a = 1$, $b = -1.2$ and $c = 0.62$. So $y = (x^2 - 1.2x + 0.62)e^{3x}$ is a particular solution of $y'' + y = (1 + 10x^2)e^{3x}$.

Putting all this together, with superposition, the general solution to the o.d.e. $y'' + y = 2xe^{-x} + (1 + 10x^2)e^{3x}$ is

$$y = (x^2 - 1.2x + 0.62)e^{3x} + (x + 1)e^{-x} + c_1 \cos(x) + c_2 \sin(x).$$

Now let's consider finding a general solution to $y'' + 3y' + 2y = 7\cos(x) - \sin(x)$.

The complementary homogenous equation has a general solution of $y = c_1 e^{-x} + c_2 e^{-2x}$.

Let's find a particular solution. Let's suppose that we can find one of the form $y = A\cos(x) + B\sin(x)$ where A, B are constants.

Then $y' = -A\sin(x) + B\cos(x)$ and $y'' = -A\cos(x) - B\sin(x)$. By substitution, this yields

$-A\cos(x) - B\sin(x) + 3(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) = 7\cos x - \sin(x)$ which requires

$$\begin{aligned} A + 3B &= 7 \\ -3A + B &= -1. \end{aligned}$$

This requires $A = 1$ and $B = 2$. So $y = \cos(x) + 2\sin(x)$ is a particular solution.

Thus the general solution is $y = \cos(x) + 2\sin(x) + c_1 e^{-x} + c_2 e^{-2x}$.

What if we try to find the general solution to $y'' + 4y = 4\cos(2x) + \sin(2x)$?

Well, let's see. The general solution to the complementary homogeneous equation is $y = c_1\cos(2x) + c_2\sin(2x)$. But if we assume there is a particular solution of the form $y_P = A\cos(2x) + B\sin(2x)$ we get $y_P'' + 4y_P = 0$ as y_P is a linear combination of the fundamental set of solutions to the complementary o.d.e.

So instead let's try for a particular solution of the form $y = x(A\cos(2x) + B\sin(2x))$.

Then $y' = (A\cos(2x) + B\sin(2x)) + x(-2A\sin(2x) + 2B\cos(2x))$ and
$y'' = -2A\sin(2x) + 2B\cos(2x) + (-2A\sin(2x) + 2B\cos(2x)) + x(-4A\cos(2x) - 4B\sin(2x))$.

We can simplify to get $y'' = -4A\sin(2x) + 4B\cos(2x) - 4x(A\cos(2x) + B\sin(2x))$.

Then $y'' + 4y = -4A\sin(2x) + 4B\cos(2x) - 4x(A\cos(2x) + B\sin(2x)) + 4x(A\cos(2x) + B\sin(2x)) = -4A\sin(2x) + 4B\cos(2x)$.

So we set this equal to $4\cos(2x) + \sin(2x)$ to get $A = -\dfrac{1}{4}$ and $B = 1$.

So $y = x\left(-\frac{1}{4}\cos(2x) + \sin(2x)\right)$ is a particular solution, and hence, a general solution is

$$y = x\left(-\frac{1}{4}\cos(2x) + \sin(2x)\right) + c_1\cos(2x) + c_2\sin(2x).$$

One more example and we'll summarize what to do when trying to solve

$$ay'' + by' + cy = p(x)\cos(\omega x) + q(x)\sin(\omega x)$$

where $p(x), q(x)$ are polynomials.

Let's find a general solution for $y'' + 2y' + y = 4x\cos(x) + (1 + x)\sin(x)$.

A general solution for the complementary homogeneous equation is $y = (c_1 + c_2 x)e^{-x}$.

Suppose there is a particular solution of the form $y = (ax + b)\cos(x) + (cx + d)\sin(x)$.

Then $y' = a\cos(x) - (ax + b)\sin(x) + c\sin(x) + (cx + d)\cos(x)$ and

$$y'' = -a\sin(x) - a\sin(x) - (ax + b)\cos(x) + c\cos(x) + c\cos(x) - (cx + d)\sin(x).$$

With a little bit of simplification, we get $y' = (a + d)\cos(x) + (-b + c)\sin(x) - ax\sin(x) + cx\cos(x)$ and

$$y'' = (-b + 2c)\cos(x) + (-2a - d)\sin(x) - cx\sin(x) - ax\cos(x).$$

Then $y'' + 2y' + y$ becomes

$$2(a + c + d)\cos(x) - 2(a + b - c)\sin(x) - 2ax\sin(x) + 2cx\cos(x).$$

Setting this equal to $4x\cos(x) + (1+x)\sin(x)$ yields

$$\begin{aligned} 2a + 2c + 2d &= 0 \\ -2a - 2b + 2c &= 1 \\ -2a &= 1 \\ 2c &= 4. \end{aligned}$$

From this we get $a = -0.5$, $b = 2$, $c = 2$, and $d = -1.5$.

So $y = (-0.5x + 2)\cos(x) + (2x - 1.5)\sin(x)$ is a particular solution of $y'' + 2y' + y = 4x\cos(x) + (1+x)\sin(x)$.

And the general solution is $y = (-0.5x + 2)\cos(x) + (2x - 1.5)\sin(x) + (c_1 + c_2 x)e^{-x}$.

(ELFY) Check that this method (assuming that $y = (ax+b)\cos(x) + (cx+d)\sin(x)$ would be a particular solution) does not work for $y'' + y = 4x\cos(x) + (1+x)\sin(x)$.

SUMMARY (skipping the proofs): Let $ay'' + by' + c = p(x)\cos(\omega x) + q(x)\sin(\omega x)$ where $p(x), q(x)$ are polynomials and a, b, c, ω are real numbers. Let k be the maximum degree between $p(x)$ and $q(x)$.

(1) If $\cos(\omega x)$ and $\sin(\omega x)$ are not solutions of the complementary homogeneous equation then there is a particular solution of the form
$$y = (a_0 + a_1 x + \cdots a_k x^k)\cos(\omega x) + (b_0 + b_1 x + \cdots + b_k x^k)\sin(\omega x).$$

(2) If $\cos(\omega x)$ and $\sin(\omega x)$ are solutions of the complementary homogeneous equation then there is a particular solution of the form
$$y = (a_1 x + \cdots a_{k+1} x^{k+1})\cos(\omega x) + (b_1 x + \cdots + b_{k+1} x^{k+1})\sin(\omega x).$$

Now let's try to find a general solution of

$$y'' - 5y' + 6y = e^{-x}\left(4x\cos(x) + (-1+18x)\sin(x)\right).$$

The general solution to the complementary homogeneous equation is $y = c_1e^{2x} + c_2e^{3x}$.

Let's assume that we can find a particular solution of the form $y = ue^{-x}$.

Then $y' = u'e^{-x} - ue^{-x}$ and $y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}$.

So

$u''e^{-x} - 2u'e^{-x} + ue^{-x} - 5(u'e^{-x} - ue^{-x}) + 6ue^{-x} = e^{-x}\left(x\cos(x) - \sin(x)\right)$ which implies

$$u'' - 7u' + 12u = 4x\cos(x) + (-1+18x)\sin(x).$$

Suppose this has a particular solution of the form $u = (ax+b)\cos(x) + (cx+d)\sin(x)$.

Then $u' = a\cos(x) - (ax+b)\sin(x) + c\sin(x) + (cx+d)\cos(x)$ and

$$u'' = -a\sin(x) - a\sin(x) - (ax+b)\cos(x) + c\cos(x) + c\cos(x) - (cx+d)\sin(x).$$

This simplifies to $u' = (a+d)\cos(x) + (-b+c)\sin(x) - ax\sin(x) + cx\cos(x)$ and

$$u'' = (-b+2c)\cos(x) + (-2a-d)\sin(x) - cx\sin(x) - ax\cos(x).$$

Then $u'' - 7u' + 12u$ becomes

$(-7a+11b+2c-7d)\cos(x)+(-2a+7b-7c+11d)\sin(x)+(7a+11c)x\sin(x)+(11a-7c)x\cos(x)$.

Setting this equal to $4x\cos(x) + (-1+18x)\sin(x)$ requires

$$\begin{aligned} -7a + 11b + 2c - 7d &= 0 \\ -2a + 7b - 7c + 11d &= -1 \\ 7a + 11c &= 18 \\ 11a - 7c &= 4. \end{aligned}$$

By starting with the final two equations, we get $a = 1$ and $c = 1$. Then the first two equations reduce to

$$\begin{aligned} 11b - 7d &= 5 \\ 7b + 11d &= 8. \end{aligned}$$

Then $b = \dfrac{111}{170}$ and $d = \dfrac{53}{170}$.

So

$$y = ue^{-x} = \left(\left(x + \frac{111}{170} \right) \cos(x) + \left(x + \frac{53}{170} \right) \sin(x) \right) e^{-x}$$

is a particular solution.

Then a general solution is

$$y = \left(\left(x + \frac{111}{170} \right) \cos(x) + \left(x + \frac{53}{170} \right) \sin(x) \right) e^{-x} + c_1 e^{2x} + c_2 e^{3x}.$$

Reduction of Order

Suppose we want to find a general solution to the linear second order o.d.e. $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$ given a non-trivial solution y_1 of the complementary homogeneous equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$.

Let's look for a solution of the form $y = uy_1$. Then $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$. By substitution we get

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)(uy_1) = F(x) \text{ which reduces to}$$

$$P_0(x)y_1u'' + (2P_0(x)y_1' + P_1(x)y_1)u' = F(x).$$

This can be rewritten as

$$Q_0(x)u'' + Q_1(x)u' = F(x),$$

where $Q_0(x) = P_0(x)y_1$ and $Q_1(x) = 2P_0(x)y_1' + P_1(x)y_1$.

This is a first order equation in terms of $z = u'$. So we can solve for $z = u'$ using first order methods. Then integrate to get u.

The method illustrated above is called *reduction of order*. This method does require that we already have a non-trivial solution the complementary o.d.e.

Example: Let's find the general solution to $x^2y'' + xy' - y = 4x^{-2}$ given that $y = x$ is a solution to the complementary equation.

Setting $y = ux$ yields $y' = u'x + u$ and $y'' = u''x + 2u'$. By substitution, we get

$$x^2(u''x + 2u') + x(u'x + u) - ux = 4x^{-2} \text{ which implies}$$

$$x^3u'' + 3x^2u' = 4x^{-2}.$$

In terms of $z = u'$ this is equivalent to $z' + \frac{3}{x}z = \frac{4}{x^5}$.

Let's find an integrating factor: $\mu(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|} = \pm x^3$.

So let's multiply both sides of $z' + \frac{3}{x}z = \frac{4}{x^5}$ by x^3: $\qquad x^3z' + 3x^2z = \frac{4}{x^2}$.

Thus $(x^3z)' = \frac{4}{x^2}$, and by integrating both sides, this becomes $x^3z = -\frac{4}{x} + c = \frac{cx - 4}{x}$.

So $z = \frac{cx-4}{x^4} = \frac{c}{x^3} - \frac{4}{x^4}$. Integrating both sides of this yields $u = \frac{c_1}{x^2} + \frac{4}{3x^3} + c_2$.

Finally we get a general solution $y = ux = \frac{c_1}{x} + \frac{4}{3x^2} + c_2x$.

Incidentally we see from the above that $y_p = \frac{4}{3x^2}$ is a particular solution and that $\left\{x, \frac{1}{x}\right\}$ is a fundamental set of solutions to the complementary homogeneous o.d.e. That's what justifies that it's a general solution.

Let's solve the IVP $(x^2-4)y''+4xy'+2y=x+2,\ y(0)=-1,\ y'(0)=-3$, given that $y=\dfrac{1}{x-2}$ is a solution to the complementary homogeneous o.d.e.

Let $y=u(x-2)^{-1}$ be a solution. Then

$$y'=u'(x-2)^{-1}-u(x-2)^{-2} \text{ and } y''=u''(x-2)^{-1}-2u'(x-2)^{-2}+2u(x-2)^{-3}.$$

By substitution we get

$$(x^2-4)(u''(x-2)^{-1}-2u'(x-2)^{-2}+2u(x-2)^{-3})+4x(u'(x-2)^{-1}-u(x-2)^{-2})+2u(x-2)^{-1}=x+2, \text{ so}$$

$$(x+2)u''-2(x+2)(x-2)^{-1}u'+2(x+2)(x-2)^{-2}u+4x(x-2)^{-1}u'-4x(x-2)^{-2}u+2(x-2)^{-1}u=x+2, \text{ so}$$

$$(x+2)u''+(-2(x+2)(x-2)^{-1}+4x(x-2)^{-1})u'+(2(x+2)(x-2)^{-2}-4x(x-2)^{-2}+2(x-2)^{-1})u=x+2, \text{ so}$$

$$(x+2)u''+2u'=x+2.$$

Letting $z=u'$ and dividing through by $x+2$ yields $z'+2(x+2)^{-1}z=1$.

So we can easily find an integrating factor: $\mu=e^{\int 2(x+2)^{-1}\,dx}=e^{2\ln|x+2|}=(x+2)^2$.

Let's solve $(x+2)^2z'+2(x+2)z=(x+2)^2$, which is equivalent to

$$((x+2)^2z)'=x^2+4x+4.$$

Integrating both sides gives $(x+2)^2z=\dfrac{1}{3}x^3+2x^2+4x+c$.

So

$$z=\frac{x^3}{3(x+2)^2}+\frac{2x^2}{(x+2)^2}+\frac{4x}{(x+2)^2}+\frac{c}{(x+2)^2}=\frac{x^3}{3(x+2)^2}+\frac{2x}{x+2}+\frac{c}{(x+2)^2}.$$

We can integrate each term (using the basic substitution $t=x+2$) and simplify. This is left to you (ELFY). You should get...

$$u=\frac{c_1}{x+2}+\frac{(x+2)^2+c_2}{6}.$$

Finally we get a general solution by multiplying by $(x-2)^{-1}$:

$$y = u(x-2)^{-1} = \frac{c_1}{x^2-4} + \frac{(x+2)^2+c_2}{6(x-2)}.$$

Now we may impose the initial conditions:

$$y' = \frac{x+2}{3(x-2)} - \frac{(x+2)^2+c_2}{6(x-2)^2} - \frac{2c_1 x}{(x^2-4)^2}.$$

Then $y(0) = -1$ implies $-1 = -\frac{c_1}{4} - \frac{4+c_2}{12}$ and $y'(0) = -3$ implies $-3 = -\frac{1}{3} + \frac{4+c_2}{24}$.

Then $c_2 = -68$. Then $c_1 = \frac{76}{3}$.

So the solution to the IVP is

$$y = \frac{76}{3(x^2-4)} + \frac{(x+2)^2-68}{6(x-2)}.$$

Euler Equations

A second order differential equation that can be written in the form

$$ax^2y'' + bxy' + cy = g(x)$$

where $a,\ b,$ and c are real constants with $a \neq 0$ is called an **Euler equation**.

By the existence and uniqueness theorem we see that if g is continuous everywhere, except possibly at 0 that an Euler equation has solutions defined on $(-\infty, 0)$ and $(0, \infty)$. To see this put the differential equation into standard form and observe that we have a discontinuity at 0 for both $p(x)$ and $q(x)$.

We will restrict our attention to the interval $(0, \infty)$.

To find the general solution of the homogeneous Euler equation

$$ax^2y'' + bxy' + cy = 0$$

we will assume that there is a solution of the form $y = x^m$. Then $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substituting into the differential equation:

$$ax^2m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0$$

$$am(m-1)x^m + bmx^m + cx^m = 0$$

Since we have $x > 0$ we can divide by x^m to get the characteristic equation

$$am(m-1) + bm + c = 0$$

$$am^2 + (b-a)m + c = 0$$

As with constant coefficient equations we have three cases.

Case 1: Two real solutions, m_1 and m_2. The general solution is

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

Example: Find the general solution of $6x^2y'' + 17xy' + 3y = 0$ on $(0, \infty)$.

The characteristic equation is

$$6m^2 + (17 - 6)\, m + 3 = 0$$

$$6m^2 + 11m + 3 = 0$$

$$(3m + 1)\,(2m + 3) = 0$$

Then $y_1 = x^{-\frac{1}{3}}$ and $y_2 = x^{-\frac{3}{2}}$ and the general solution is $y = c_1x^{-\frac{1}{3}} + c_2x^{-\frac{3}{2}}$

Case 2: One real solution, m_1. One solution is $y_1 = x^{m_1}$. Using reduction of order and that the polynomial must be $a(m - m_1)^2$ we get a second linearly independent solution given by $y_2 = x^{m_1} \ln x$ (ELFY). The general solution is then

$$y = c_1x^{m_1} + c_2x^{m_1} \ln x.$$

Example: Find the general solution of $4x^2y'' + 8xy' + y = 0$ on $(0, \infty)$.

The characteristic equation is

$$4m\,(m - 1) + 8m + 1 = 0$$

$$4m^2 + 4m + 1 = 0$$

$$(2m + 1)^2 = 0$$

Then $y_1 = x^{-\frac{1}{2}}$ and $y_2 = x^{-\frac{1}{2}} \ln x$. The general solution is $y = \dfrac{c_1 + c_2 \ln x}{\sqrt{x}}$.

Case 3: Two complex solutions $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

$$y = x^{\alpha + i\beta}$$

$$y = x^{\alpha} x^{i\beta}$$

To deal with $x^{i\beta}$ we use Euler's identity:

$$x^{i\beta} = e^{i\beta \ln x} = \cos(\beta \ln x) + i \sin(\beta \ln x)$$

From this we can see that the general solution is given by

$$y = x^{\alpha} \left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right)$$

.

Example: Find the general solution for $x^2 y'' + 3xy' + 10y = 0$ on $(0, \infty)$.

The characteristic equation is $m^2 + 2m + 10 = 0$. From the quadratic formula

$$m = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2} = -1 \pm 3i$$

So the general solution is then given by $y = \dfrac{c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)}{x}$