Variation of Parameters

In this lesson we cover a method called **variation of parameters** for finding a particular solution of $y'' + p(t)y' + q(t)y = f(t)$.

The first step is to find the general solution $y_h = c_1y_1 + c_2y_2$ to the complementary equation $y'' + p(t)y' + q(t)y = 0$, which we will call the **homogenous solution**.

We assume there is a particular solution of the form $y_p = uy_1 + vy_2$.

Then $y_p' = uy_1' + vy_2'$ with the "simplifying assumption" that $u'y_1 + v'y_2 = 0$ (which we are free to assume).

So $y_p' = u'y_1' + uy_1'' + v'y_2' + vy_2''$. Plugging this into $y'' + p(t)y' + q(t)y = f(t)$ yields $u'y_1' + v'y_2' = f(t)$.

Using elimination (multiply $u'y_1 + v'y_2 = 0$ by y_2' and $u'y_1' + v'y_2' = f(t)$ by $-y_2$) we can get $u' = \dfrac{-fy_2}{y_1y_2' - y_1'y_2}$.

Similarly $v' = \dfrac{fy_1}{y_1y_2' - y_1'y_2}$.

(Note: All this requires that the differential equation is in the standard form $y'' + p(t)y' + q(t)y = f(t)$.)

The first thing to notice is that in each denominator is the Wronskian $W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ of y_1 and y_2.

If we define $W_1 = \det\begin{pmatrix} 0 & y_2 \\ 1 & y_2' \end{pmatrix}$ and $W_2 = \det\begin{pmatrix} y_1 & 0 \\ y_1' & 1 \end{pmatrix}$ then the above equations become

$$u' = \frac{fW_1}{W}$$

$$v' = \frac{fW_2}{W}$$

Examples:

Find the general solution to $x^2y'' + 8xy' + 10y = 4x^{-2}$.

The reader is left to check that the homogeneous solution is $y_h = c_1x^{-2} + c_2x^{-5}$. We will use $y_1 = x^{-2}$ and $y_2 = x^{-5}$.

The Wronskian is $W = \det \begin{pmatrix} x^{-2} & x^{-5} \\ -2x^{-3} & -5x^{-6} \end{pmatrix} = -3x^{-8}$.

First we put the equation into standard form $y'' + 8x^{-1}y' + 10x^{-2}y = 4x^{-4}$ to identify $f(x) = 4x^{-4}$.

Then

$$u' = \frac{fW_1}{W} = -\frac{fy_2}{W} = -\frac{4x^{-4}(x^{-5})}{-3x^{-8}} = -\frac{4x^{-9}}{-3x^{-8}} = \frac{4}{3}x^{-1}$$

$$v' = \frac{fW_2}{W} = \frac{fy_1}{W} = \frac{4x^{-4}(x^{-2})}{-3x^{-8}} = \frac{4x^{-6}}{-3x^{-8}} = -\frac{4}{3}x^2$$

So we can choose $u = \frac{4}{3}\ln x$ and $v = -\frac{4}{9}x^3$.

Then $y_p = uy_1 + vy_2 = \frac{4}{3}x^{-2}\ln x - \frac{4}{9}x^{-2}$ is a particular solution of $x^2y'' + 8xy' + 10y = 4x^{-2}$. Since $-\frac{4}{9}x^{-2}$ is a solution to the homogeneous equation we can simply take $y_p = \frac{4}{3}x^{-2}\ln x$ as our particular solution.

So the general solution to $x^2y'' + 8xy' + 10y = 4x^{-2}$ is

$$y = \frac{c_1}{x^2} + \frac{c_2}{x^5} + \frac{4\ln x}{3x^2}.$$

Consider the linear second order o.d.e. $y'' - 3y' + 2y = \frac{4}{1+e^{-x}}$. Clearly we have that $\{e^x, e^{2x}\}$ is a fundamental set of solutions to $y'' - 3y' + 2y = 0$.

The Wronskian is $W = \det \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} = e^{3x}$.

Let's look for a particular solution of the form $y = ue^x + ve^{2x}$.

Then $u' = \frac{fW_1}{W} = -\frac{fy_2}{W} = -\frac{\left(\frac{4}{1+e^{-x}}\right)e^{2x}}{e^{3x}} = \frac{-4e^{-x}}{1+e^{-x}}$.

Then $u = 4\ln(1+e^{-x})$ choosing zero for the constant.

Next $v' = \frac{fW_2}{W} = \frac{fy_1}{W} = \frac{\left(\frac{4}{1+e^{-x}}\right)e^{x}}{e^{3x}} = \frac{4e^{-2x}}{1+e^{-x}}$.

Using the substitution $z = 1 + e^{-x}$ we get $v = \displaystyle\int \frac{4(1-z)}{z} dz$ and hence it follows that $v = 4(\ln|z| - z) + 4 = 4(\ln(1+e^{-x}) - e^{-x})$ (choosing 4 as the constant).

So a particular solution to $y'' - 3y' + 2y = \dfrac{4}{1+e^{-x}}$ is

$$y = ue^x + ve^{2x} = 4e^x \ln(1+e^{-x}) + 4e^{2x}(\ln(1+e^{-x}) - e^{-x}).$$

Incidentally, the general solution is

$$y = 4e^x(c_1 + \ln(1+e^{-x})) + 4e^{2x}(c_2 + \ln(1+e^{-x})).$$

Let's apply variation of parameters to find a general solution to $xy'' - y' - 4x^3y = 8x^5$ given that $\{e^{x^2}, e^{-x^2}\}$ is a fundamental set of solutions to the complementary equation. The Wronskian is $W = -4x$.

Then $u' = \dfrac{fW_1}{W} = -\dfrac{fy_2}{W} = -\dfrac{8x^4e^{-x^2}}{-4x} = 2x^3e^{-x^2}$.

Using integrating by parts, $u = -(x^2+1)e^{-x^2}$ (choosing the constant of integration to be zero).

Then $v' = \dfrac{fW_1}{W} = \dfrac{fy_1}{W} = \dfrac{8x^4e^{x^2}}{-4x} = -2x^3e^{x^2}$. Again integrate by parts to get $v = (1-x^2)e^{x^2}$ (choosing $C = 0$).

So a particular solution is $y_p = uy_1 + vy_2 = -(x^2+1) + (1-x^2) = -2x^2$, and the general solution is

$$y = -2x^2 + c_1e^{x^2} + c_2e^{-x^2}.$$

To see how to generalize variation of parameters to higher orders let's look at the equations for third order differential equation $y''' + p(t)\,y'' + q(t)\,y' + r(t)\,y = f(t)$. As before we first find the complementary solution, $y_h = c_1y_1 + c_2y_2 + c_3y_3$. The Wronskian is

$$W = \det\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{pmatrix}$$

To find a particular solution we assume $y_p = uy_1 + vy_2 + wy_3$.
Then

$$u' = \frac{fW_1}{W}, \quad v' = \frac{fW_2}{W}, \quad w' = \frac{fW_3}{W},$$

where

$$W_1 = \det\begin{pmatrix} 0 & y_2 & y_3 \\ 0 & y_2{}' & y_3{}' \\ 1 & y_2{}'' & y_3{}'' \end{pmatrix}, \quad W_2 = \det\begin{pmatrix} y_1 & 0 & y_3 \\ y_1{}' & 0 & y_3{}' \\ y_1{}'' & 1 & y_3{}'' \end{pmatrix}, \quad W_3 = \det\begin{pmatrix} y_1 & y_2 & 0 \\ y_1{}' & y_2{}' & 0 \\ y_1{}'' & y_2{}'' & 1 \end{pmatrix}$$

Applications of Second Order o.d.e. to Spring-Mass Systems

Consider an object of mass m suspended from a spring (of negligible mass). We say that the spring-mass system is at *equilibrium* when the object is at rest and the sum of the forces acting on it is 0. The corresponding position is called the *equilibrium position.*

Let y be the displacement of the mass from the equilibrium where $y > 0$ is upward.

The following forces act on the object:

(1) Gravity: $F_g = -mg$ where g is the gravitational constant.

(2) Spring Force: $F_s = k\Delta L$ where k is the spring constant (Hooke's Law) and ΔL is the change in length (stretching is positive) from the spring equilibrium position with no mass attached.

(3) Damping Force: $F_d = -cy'$ where c is the damping constant (due to medium resistance – often negligible). *Simple harmonic motion* occurs when $c = 0$.

There could also be an another force F, independent of the motion, not accounted for above.

By Newton's second law we can write an o.d.e. to model the spring-mass system described above:

$$my'' = F - mg - cy' + k\Delta L.$$

Let's relate ΔL and y. Imagine the spring without a mass attached. When a mass m is attached, the spring stretches by some amount ℓ and at equilibrium, the sum of forces on the spring is 0, so $F_g + F_s = 0$. Hence $-mg + k\ell = 0$, or $\ell = mg/k$.

If at this point, a displacement of y occurs then we would have $\Delta L = \ell - y = mg/k - y$.

So $my'' = F - mg - cy' + k\Delta L$ becomes $my'' = F - cy' - ky$ or equivalently $my'' + cy' + ky = F$.

If $F = 0$ we say the motion is *free.* Otherwise, it is said to be *forced.*

Consider the free undamped system modeled by $y'' + \dfrac{k}{m}y = 0$. We know that a general solution is

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t), \text{ where } \omega = \sqrt{k/m}.$$

Example: Suppose an object is hung from a spring whose natural length is 98 cm and attains an equilibrium position of 1 m. Suppose the object is displaced downward 5 cm, but given an initial upward velocity of 11 cm/s. Find the displacement for time $t > 0$.

Since $g/\ell = k/m$, we have that $\omega = \sqrt{k/m} = \sqrt{g/\ell} = \sqrt{9.8/0.02} \approx 22$.

So $y \approx c_1 \cos(22t) + c_2 \sin(22t)$. Imposing $y(0) = -0.05$ yields $c_1 = -0.05$. Imposing $y'(0) = 0.11$ yields $22c_2 = 0.11$, so $c_2 = 0.005$.

Hence $y \approx -0.05 \cos(22t) + 0.005 \sin(22t)$.

Let's go back to a general solution: $y = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, where $\omega = \sqrt{k/m}$.

Let $A = \sqrt{c_1^2 + c_2^2}$. Then $c_1 = A\cos(\phi)$ and $c_2 = A\sin(\phi)$ for some angle ϕ in $(-\pi, \pi]$.

Using the identity, $\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)$ we get

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t) = A\cos(\omega t - \phi).$$

The A is the *amplitude* (motion goes from $-A$ to A). $T = 2\pi/\omega$ is the *period* (time it takes to go through a full cycle, such as from A back to A). ϕ is the *phase angle.* ω is the natural *angular frequency*, and $f = \omega/(2\pi) = 1/T$ is the *frequency.*

In the example above, the amplitude is $A \approx 0.0502$ m. The phase angle is $\phi \approx 0.1$ rad. The natural frequency is $\omega \approx 22$ rad/s (a frequency of $f \approx 3.5$ Hz). The period is $T \approx 0.28$ s.

Let's consider a curious forced undamped system (such a system is considered to be in *resonance*):

Suppose that the external force is $F = F_0 \cos(\omega t)$ where $\omega = \sqrt{k/m}$ (F_0 is a constant).

Then the system is modeled by $y'' + \omega^2 y = \dfrac{F_0}{m}\cos(\omega t)$. We already know that $\{\cos(\omega t), \sin \omega t\}$ is a fundamental set of solutions to the complementary homogeneous equation (for the free system).

Let's find a particular solution. We look for a solution of the form $y = t(a\cos(\omega t) + b\sin(\omega t))$ (won't work without the "t" – why?)

Then $y' = (a\cos(\omega t) + b\sin(\omega t)) + \omega t(-a\sin(\omega t) + b\cos(\omega t))$ and

$$y'' = 2\omega(-a\sin(\omega t) + b\cos(\omega t)) - \omega^2 t(a\cos(\omega t) + b\sin(\omega t)).$$

By substitution into $y'' + \omega^2 y = \dfrac{F_0}{m}\cos(\omega t)$ we get

$$2\omega(-a\sin(\omega t) + b\cos(\omega t)) = \frac{F_0}{m}\cos(\omega t).$$

This requires $a = 0$ and $b = \dfrac{F_0}{2m\omega} = \dfrac{F_0}{\sqrt{4mk}}$.

So $y = \dfrac{F_0}{\sqrt{4mk}} t \sin(\omega t)$ is a particular solution and a general solution is

$$y = \frac{F_0}{\sqrt{4mk}} t \sin(\omega t) + c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Then the amplitude goes toward ∞ as $t \to \infty$. This means that the system will eventually fail.

Now let's consider a free damped system: $my'' + cy' + ky = 0$ where $c \neq 0$.

The characteristic polynomial is $mr^2 + cr + k$. The roots of this polynomial are

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Case 1: Underdamped ($c < \sqrt{4mk}$)

Then we get two complex conjugate roots $\alpha \pm \omega i$ where $\alpha = -c/(2m)$ and $\omega = \dfrac{\sqrt{4mk - c^2}}{2m}$.

Hence a general solution is $y = e^{\alpha t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$ and since $\alpha < 0$ we get that $y \to 0$ as $t \to \infty$ (but with oscillations).

Case 2: Overdamped ($c > \sqrt{4mk}$)

Then we get two distinct real and negative roots r_1, r_2, and a general solution of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ which goes towards 0 as $t \to \infty$.

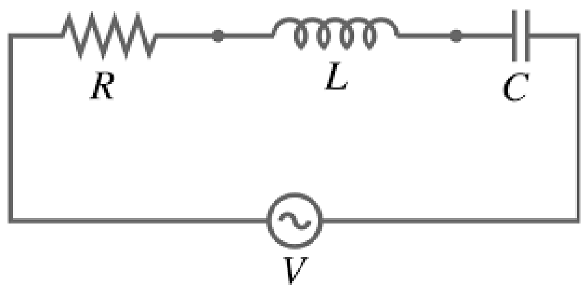
In this case no oscillation occurs as (ELFY) y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$.

Case 3: Critically damped ($c = \sqrt{4mk}$)

Then we get one repeated real and negative root $r = -c/(2m)$, and a general solution of $y = (c_1 + tc_2)e^{rt}$ which goes towards 0 as $t \to \infty$. As in the overdamped case, no oscillation occurs (ELFY) and y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$.

RLC circuits are like springs:

An RLC circuit is one that includes a voltage source of voltage $V(t)$ (SI units of volts) as a function of time, a resistor of resistance $R > 0$ (SI units ohms), a coiled wire inductor of inductance $L > 0$ (SI units henries), and a plate capacitor of capacitance $C > 0$ (SI units farads).



Let $Q(t)$ be the charge (SI units of coulombs) on the capacitor at time t. Then $I(t) = \frac{dQ}{dt}$ is the electrical current (SI units of amps) in the circuit. We have the following basic laws for the change in voltage across each circuit component:

- Resistor: $V = IR$
- Inductor: $V = L\frac{dI}{dt}$
- Capacitor: $V = \frac{Q}{C}$

Kirchhoffs law that says that the voltage change across any two points has to be independent of the path used to travel between the two points. Hence

$$L\frac{dI}{dt} + IR + \frac{Q(t)}{C} = V(t).$$

By differentiating both sides and using $I = \frac{dQ}{dt}$ we get

$$LI'' + RI' + \frac{1}{C}I = V'(t).$$

Note: One could also get a second order linear constant coefficient equation in terms of $Q(t)$ without differentiating.

Example: For an alternating current voltage source $V(t) = E\sin(\omega t)$ the differential equation becomes

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = \omega E\cos(\omega t).$$

Let's find the general solution of this differential equation. The complementary equation has a characteristic polynomial of $Lx^2 + Rx + 1/C = 0$ which has roots at

$$x = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} = r_1, r_2.$$

Then the general solution to the complementary equation is either $I = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ if

$R > 2\sqrt{L/C}$ (underdamped) or $I = (c_1 + c_2 t)e^{-\frac{R}{2L}t}$ if $R = 2\sqrt{L/C}$ (critically damped)

or $I = e^{-\frac{R}{2L}t}\left[c_1\cos\left(\frac{\sqrt{4L/C - R^2}}{2L}t\right) + c_2\sin\left(\frac{\sqrt{4L/C - R^2}}{2L}t\right)\right]$ if $R < 2\sqrt{L/C}$ (overdamped).

In each case the complementary solution decays towards 0 as $t \to \infty$.

The method of undetermined coefficients, where we guess a particular solution of the form $I_P = A\cos(\omega t) + B\sin(\omega t)$, yields (the reader is left to check the details)

$$A = -\frac{\omega E(L\omega^2 - 1/C)}{R^2\omega^2 + (L\omega^2 - 1/C)^2} \text{ and } B = \frac{R\omega^2 E}{R^2\omega^2 + (L\omega^2 - 1/C)^2}.$$

Putting the solutions together and imposing initial conditions allows us to solve for the current given any RLC circuit with any voltage source and initial conditions.

For example, suppose we have an RLC circuit where the resistance is $R = 600$ ohms, the capacitance is $C = 0.000001$ farads, the inductance is $L = 0.25$ henries, and the voltage source is $V(t) = 120\sin(120\pi t)$ volts.

With these values $R < 2\sqrt{L/C}$, $A \approx 0.0445$ and $B \approx 0.0104$. So the general solution is

$$I(t) \approx 0.0445\cos(120\pi t) + 0.0104\sin(120\pi t) + e^{-1200t}\left[c_1\cos(1600t) + c_2\sin(1600t)\right].$$

Then given initial conditions we can find c_1 and c_2 for a solution of the above form, called a **transient solution**, since the part of the solution involving c_1 and c_2 decays away quickly as $t \to \infty$. In the long run the current approaches the **steady state** solution of $I(t) \approx 0.0445\cos(120\pi t) + 0.0104\sin(120\pi t)$.

Laplace Transforms

Let $f(t)$ be defined for $t \geq 0$ and s represent a real number. Then the Laplace transform of f is

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The functions f and F are called a transform pair.

Notation: $\mathcal{L}(f) = F$ or $f \leftrightarrow F$.

Note: Technically the improper integral must be turned in a limit, but we will be lazy and never formally do this. We will allow "∞" to appear as though it were a real number (but with the understanding that we are taking a limit).

Let's calculate some basic Laplace transforms:

(i) $f(t) = 1$ (a constant function). Then $F(0)$ diverges (to ∞) and for $s \neq 0$,

$$F(s) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ \infty & \text{if } s < 0 \end{cases}. \text{ So we write } \mathcal{L}(1) = \frac{1}{s},\ s > 0.$$

(ii) $f(t) = t^2$. Then $F(0)$ diverges (to ∞) and for $s \neq 0$,

$$F(s) = \int_0^\infty t^2 e^{-st}\, dt = -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\, dt = -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\left[-\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt\right] =$$

$$= -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\left[-\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^\infty\right)\right] = \begin{cases} \frac{2}{s^3} & \text{if } s > 0 \\ \infty & \text{if } s < 0 \end{cases}.$$

So we write $\mathcal{L}(t^2) = \frac{2}{s^3}$, $s > 0$.

(iii) $f(t) = e^{at}$. $F(a)$ diverges to ∞.

$$F(s) = \int_0^\infty e^{-st}e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \infty & \text{if } s < a \end{cases}.$$

So we write $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $s > a$.

(iv) $f(t) = \begin{cases} 2-t & \text{if } 0 \le t < 1 \\ t & \text{if } t \ge 1 \end{cases}$

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 (2-t)e^{-st}\,dt + \int_1^\infty te^{-st}\,dt$$

$$= \left[-\frac{2-t}{s}e^{-st}\Big|_0^1 - \frac{1}{s}\int_0^1 e^{-st}\,dt \right] + \left[-\frac{t}{s}e^{-st}\Big|_1^\infty + \frac{1}{s}\int_1^\infty e^{-st}\,dt \right]$$

$$= \left[\frac{2}{s} - \frac{1}{s}e^{-s} + \frac{1}{s^2}(e^{-s}-1) \right] + \left[\frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-s} \right] = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s}.$$

As an appendix (last page of these notes) there a table of Laplace transforms.

Proposition (Linearity of Laplace transforms): If $\mathcal{L}(f)$ and $\mathcal{L}(g)$ both exist for $s > s_0$ then for any constants c_1, c_2 we have

$$\mathcal{L}(c_1 f(t) + c_2 g(t)) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g) \text{ for } s > s_0.$$

The proof is left as an (ELFY). It's very straightforward.

Let's find the Laplace transform of $\sinh(at) = \dfrac{e^{at} - e^{-at}}{2}$ where $a > 0$.

We know that $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ for $s > a$ and $\mathcal{L}(e^{-at}) = \dfrac{1}{s+a}$ for $s > -a$.

By linearity, $\mathcal{L}(\sinh(t)) = \dfrac{1}{2}\left(\dfrac{1}{s-a} - \dfrac{1}{s+a}\right) = \dfrac{a}{s^2-a^2}$ for $s > a$.

Suppose $\mathcal{L}(f) = F$. Let's determine $\mathcal{L}(e^{at}f(t))$ where a is a constant.

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st}f(t)\,dt$$

and thus $\mathcal{L}(e^{at}f(t)) = \displaystyle\int_0^\infty e^{-st}(e^{at}f(t))\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$

This property is referred to as the *first shifting property* of the Laplace transform.

Given that $\mathcal{L}(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2}$ (see appendix) we use the first shifting property to find $\mathcal{L}(e^{at}\cos(\omega t))$:

$$\mathcal{L}(e^{at}\cos(\omega t)) = \frac{s-a}{(s-a)^2+\omega^2}.$$

A function f is of real <u>exponential order s_0</u> if there are constants M and t_0 such that

$$|f(t)| \leq Me^{s_0 t} \text{ for all } t \geq t_0.$$

Often we say a function is of exponential order without a reference to a particular s_0.

To save time we state the following existence theorem, but do not give a proof.

Proposition: Suppose f is of exponential order s_0. Then $\mathcal{L}(f)$ is defined for $s > s_0$.

The inverse transform: Suppose $F = \mathcal{L}(f)$. Then we call f the inverse Laplace transform of F. Notation: $f = \mathcal{L}^{-1}(F)$.

While there is a formula for computing $\mathcal{L}^{-1}(F)$ we will not use it as it requires the theory of functions of a complex variable (complex analysis). We can however, make use the table (see appendix) to find inverse Laplace transforms.

Suppose we need $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4}\right)$. By looking through the table of Laplace transforms we find

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}.$$

So $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos(2t)$.

Note: Linearity also holds for the inverse transform (we omit the proof).

Let's find $\mathcal{L}^{-1}\left(\dfrac{2s+3}{s^2+4s+13}\right)$.

$$\mathcal{L}^{-1}\left(\frac{2s+3}{s^2+4s+13}\right) = \mathcal{L}^{-1}\left(\frac{2(s+2)-1}{(s+2)^2+9}\right) = 2\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+9}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right)$$

$$= 2e^{-2t}\cos(3t) - \frac{1}{3}e^{-2t}\sin(3t) = e^{-2t}\left(2\cos(3t) - \frac{1}{3}\sin(3t)\right).$$

In the last example, the denominator is irreducible over the reals. Our method changes when the denominator factors nicely:

Let's find $\mathcal{L}^{-1}\left(\dfrac{2s-1}{s^2-5s+6}\right)$.

Using a *P.F.D.* we get $\dfrac{2s-1}{s^2-5s+6} = \dfrac{2s-1}{(s-2)(s-3)} = \dfrac{5}{s-3} - \dfrac{3}{s-2}$.

Then $\mathcal{L}^{-1}\left(\dfrac{2s-1}{s^2-5s+6}\right) = 5\mathcal{L}^{-1}\left(\dfrac{1}{s-3}\right) - 3\mathcal{L}^{-1}\left(\dfrac{1}{s-2}\right) = 5e^{3t} - 3e^{2t}$.

Let's find $\mathcal{L}^{-1}\left(\dfrac{s}{s^2-8s+16}\right)$.

We can setup a *P.F.D.* as follows:

$$\frac{s}{s^2-8s+16} = \frac{s}{(s-4)^2} = \frac{A}{s-4} + \frac{4}{(s-4)^2}.$$

Clearing fractions and equating common-degree coefficients, we get $A = 1$.

So $\mathcal{L}^{-1}\left(\dfrac{s}{s^2-8s+16}\right) = \mathcal{L}^{-1}\left(\dfrac{1}{s-4}\right) + 4\mathcal{L}^{-1}\left(\dfrac{1}{(s-4)^2}\right) = e^{4t} + 4te^{4t} = e^{4t}(1+4t)$.

Proposition: Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then $\mathcal{L}(f)$ and $\mathcal{L}(f')$ are defined for $s > s_0$ and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

Proof: We will only prove it in the case where f' is continuous on $[0,\infty)$. You can see the proof in the other case in the textbook.

Assume f' is continuous on $[0,\infty)$. By integration by parts,

$$\mathcal{L}(f') = \int_0^T e^{-st} f'(t)\,dt = e^{-st}f(t)\big|_0^T + s\int_0^T e^{-st}f(t)\,dt.$$

Since f is of exponential order s_0, for $s > s_0$ we have $\lim_{T\to\infty} e^{-sT}f(T) = 0$ and $\mathcal{L}(f) = \int_0^\infty e^{-st}f(t)\,dt$ converges. Hence $\mathcal{L}(f')$ converges for $s > s_0$ and

$$\mathcal{L}(f') = -f(0) + s\mathcal{L}(f) \quad \square$$

Let's solve a really easy IVP using a Laplace transform!

Consider the basic first order IVP $y' = ay$, $y(0) = y_0$. We already know the solution is $y(t) = y_0 e^{at}$. Let's see if we can find the same result using our new tool!

Let $Y(s) = \mathcal{L}(y)$ be the Laplace transform of the solution $y(t)$ (pretend it is unknown).

Let's transform both sides of the o.d.e. $y' = ay$.

$\mathcal{L}(y') = \mathcal{L}(ay) \to s\mathcal{L}(y) - y(0) = a\mathcal{L}(y) \to (s-a)\mathcal{L}(y) = y_0 \to Y(s) = \mathcal{L}(y) = \dfrac{y_0}{s-a}$.

Then $y(t) = \mathcal{L}^{-1}\left(\dfrac{y_0}{s-a}\right) = y_0 e^{at}$. Neat!

What about a second order IVP? Well, we can use that

$$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s(s\mathcal{L}(y) - y(0)) - y'(0) = s^2\mathcal{L}(y) - y'(0) - sy(0).$$

Let's derive a simplified form of the Laplace transform of the solution of the IVP

$$ay'' + by' + cy = f(t),\ y(0) = k_0,\ y'(0) = k_1.$$

We begin with $a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = \mathcal{L}(f(t))$.

Then by substitution, this becomes $a(s^2\mathcal{L}(y) - k_1 - sk_0) + b(s\mathcal{L}(y) - k_0) + c\mathcal{L}(y) = \mathcal{L}(f(t))$.

Letting $Y(s) = \mathcal{L}(y)$ we get $(as^2 + bs + c)Y(s) = \mathcal{L}(f(t)) + ak_1 + ak_0 s + bk_0$ and hence

$$Y(s) = \frac{\mathcal{L}(f(t)) + a(k_1 + k_0 s) + bk_0}{as^2 + bs + c}.$$

Let's solve the IVP $y'' + 4y' - 5y = 6e^t$, $y(0) = 1$, $y'(0) = 0$ using Laplace transforms.

We would have that $Y(s) = \dfrac{\frac{6}{s-1} + (0 + 1s) + 4}{s^2 + 4s - 5}$ is the Laplace transform of the solution.

Simplifying this yields $Y(s) = \dfrac{6 + (s+4)(s-1)}{(s-1)^2(s+5)} = \dfrac{s^2 + 3s + 2}{(s-1)^2(s+5)}$.

We can begin our P.F.D. by having $Y(s) = \dfrac{s^2 + 3s + 2}{(s-1)^2(s+5)} = \dfrac{A}{s-1} + \dfrac{1}{(s-1)^2} + \dfrac{1/3}{s+5}$.

By clearing fractions,
$s^2+3s+2 = A(s-1)(s+5)+(s+5)+(1/3)(s-1)^2 = As^2+4As-5A+s+5+(1/3)s^2-(2/3)s+(1/3)$.

So $A = 2/3$ and $Y(s) = \dfrac{2/3}{s-1} + \dfrac{1}{(s-1)^2} + \dfrac{1/3}{s+5}$.

Then $y(t) = (2/3)\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + (1/3)\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = \frac{2}{3}e^t + te^t + \frac{1}{3}e^{-5t}$.

The <u>unit step function</u> $\mathcal{U}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ is a basic piecewise continuous function.

Consider a piecewise continuous function $f(t) = \begin{cases} f_0(t) & \text{if } 0 \leq t < t_1 \\ f_1(t) & \text{if } t \geq t_1 \end{cases}$.

We can rewrite $f(t)$ in a convenient form using the unit step function appropriately:

$$f(t) = f_0(t) + \mathcal{U}(t - t_1)[f_1(t) - f_0(t)].$$

Let's consider that $\mathcal{L}(\mathcal{U}(t-r)g(t)) = \int_0^\infty e^{-st}\mathcal{U}(t-r)g(t)\,dt = \int_r^\infty e^{-st}g(t)\,dt$.

With the substitution $u = t - r$, $du = dt$ we get

$$\int_r^\infty e^{-st}g(t)\,dt = \int_0^\infty e^{-s(u+r)}g(u+r)\,du = e^{-sr}\int_0^\infty e^{-st}g(t+r)\,dt.$$

(Last step since it doesn't matter what the integration variable is...)

Hence we have justified that:

$$\mathcal{L}(\mathcal{U}(t-r)g(t)) = e^{-sr}\mathcal{L}(g(t+r)).$$

Let's use this fact to find the Laplace transform of $f(t) = \begin{cases} 2-t & \text{if } 0 \leq t < 1 \\ t & \text{if } t \geq 1 \end{cases}$.

First we rewrite it as $f(t) = 2 - t + \mathcal{U}(t-1)[2t-2] = 2 - t + 2\mathcal{U}(t-1)[t-1]$.

Then $\mathcal{L}(f(t)) = \frac{2}{s} - \frac{1}{s^2} + 2\mathcal{L}(\mathcal{U}(t-1)[t-1]) = \frac{2}{s} - \frac{1}{s^2} + 2e^{-s}\mathcal{L}(t) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s}$.

Wasn't that nicer than how we did it before (example (iv) on page 9)?

Consider a piecewise continuous function $f(t) = \begin{cases} f_0(t) & \text{if } 0 \le t < t_1 \\ f_1(t) & \text{if } t_1 \le t < t_2 \\ f_2(t) & \text{if } t \ge t_2 \end{cases}$.

We can rewrite $f(t)$ in a convenient form using the unit step function appropriately:

$$f(t) = f_0(t) + \mathcal{U}(t - t_1)[f_1(t) - f_0(t)] + \mathcal{U}(t - t_2)[f_2(t) - f_1(t)].$$

Then we can find the Laplace transform of $f(t)$, using the above and

$$\mathcal{L}(\mathcal{U}(t - r)g(t)) = e^{-sr}\mathcal{L}(g(t + r)).$$

Of course this generalizes to any number of "pieces," but we will focus on functions that are either 2 or 3 "pieces."

Since $\mathcal{L}(\mathcal{U}(t - r)g(t)) = e^{-sr}\mathcal{L}(g(t + r))$ we get

$$\mathcal{L}(\mathcal{U}(t - r)g(t - r)) = e^{-sr}\mathcal{L}(g(t)),$$

which is called the *second shifting property.*

Use linearity and the second shifting property to find $\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right)$.

First we use linearity and then apply the second shifting property:

$$\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(e^{-s}\frac{1}{s^2}\right) = t + \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(t)\right) = t + \mathcal{U}(t - 1)[t - 1].$$

Of course we can rewrite this in the more familiar form $\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2t - 1 & \text{if } t \ge 1 \end{cases}$.

Let's solve the IVP $y'' + y = \begin{cases} t & \text{if } 0 \le t < 2\pi \\ -2t & \text{if } t \ge 2\pi \end{cases}, \qquad y(0) = 1,\ y'(0) = 2.$

Let's first find $\mathcal{L}(f)$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 2\pi \\ -2t & \text{if } t \ge 2\pi \end{cases}.$

We can rewrite this function as $f(t) = t - \mathcal{U}(t - 2\pi)[3t]$.

The $\mathcal{L}(f) = \mathcal{L}(t) - 3\mathcal{L}\left(\mathcal{U}(t-2\pi)[t]\right) = \frac{1}{s^2} - 3e^{-2\pi s}\mathcal{L}(t + 2\pi) = \frac{1}{s^2} - 3e^{-2\pi s}\left(\frac{1}{s^2} + \frac{2\pi}{s}\right)$.

Recall: $Y(s) = \dfrac{\mathcal{L}(f(t)) + a(k_1 + k_0 s) + bk_0}{as^2 + bs + c}$.

So

$$Y(s) = \frac{\frac{1}{s^2} - 3e^{-2\pi s}\left(\frac{1}{s^2} + \frac{2\pi}{s}\right) + (2+s)}{s^2+1} = \frac{1 + (s+2)(s^2)}{s^2(s^2+1)} - 3e^{-2\pi s}\left(\frac{1 + 2\pi s}{s^2(s^2+1)}\right).$$

So $Y(s) = Y_1(s) - 3Y_2(s)$ where $Y_1(s) = \dfrac{s^3 + 2s^2 + 1}{s^2(s^2+1)}$ and $Y_2(s) = e^{-2\pi s}\left(\dfrac{1 + 2\pi s}{s^2(s^2+1)}\right)$.

Let's find $\mathcal{L}^{-1}(Y_1(s)) = \mathcal{L}^{-1}\left(\dfrac{s^3 + 2s^2 + 1}{s^2(s^2+1)}\right)$.

We can begin a P.F.D. of $Y_1(s) = \dfrac{s^3 + 2s^2 + 1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{1}{s^2} + \dfrac{Bs + C}{s^2+1}$.

Clearing fractions yields $s^3 + 2s^2 + 1 = As(s^2+1) + s^2 + 1 + (Bs + C)s^2 = As^3 + As + s^2 + 1 + Bs^3 + Cs^2$.

Thus $A = 0$ and $B = C = 1$. So $Y_1(s) = \dfrac{1}{s^2} + \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$. Hence $\mathcal{L}^{-1}(Y_1(s)) = t + \cos(t) + \sin(t)$.

Now let's find $\mathcal{L}^{-1}(Y_2(s)) = \mathcal{L}^{-1}\left(e^{-2\pi s}\left(\dfrac{1 + 2\pi s}{s^2(s^2+1)}\right)\right)$. We actually begin with

$$\mathcal{L}^{-1}\left(\frac{1 + 2\pi s}{s^2(s^2+1)}\right).$$

We can begin with the P.F.D. $\dfrac{1 + 2\pi s}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{1}{s^2} + \dfrac{Bs + C}{s^2+1}$.

Clearing fractions yields $1 + 2\pi s = As(s^2+1) + s^2 + 1 + (Bs + C)s^2 = As^3 + As + s^2 + 1 + Bs^3 + Cs^2$.

Thus $A = 2\pi$, $B = -2\pi$, and $C = -1$. So $\dfrac{1 + 2\pi s}{s^2(s^2+1)} = \dfrac{2\pi}{s} + \dfrac{1}{s^2} - \dfrac{2\pi s + 1}{s^2+1}$.

Then $\mathcal{L}^{-1}\left(\dfrac{1+2\pi s}{s^2(s^2+1)}\right) = 2\pi + t - 2\pi\cos(t) - \sin(t)$.

So by the second shifting property,
$\mathcal{L}^{-1}\left(e^{-2\pi s}\dfrac{1+2\pi s}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(e^{-2\pi s}\mathcal{L}\left(2\pi + t - 2\pi\cos(t) - \sin(t)\right)\right) = \mathcal{U}(t-2\pi)[t-2\pi\cos(t-2\pi)-\sin(t-2\pi)]$.

Therefore the solution is $y(t) = \mathcal{L}^{-1}(Y(s)) = t+\cos(t)+\sin(t) - 3\mathcal{U}(t-2\pi)[t-2\pi\cos(t)-\sin(t)]$.

In a more familiar form,

$$y(t) = \begin{cases} t+\cos(t)+\sin(t) & \text{if } 0 \le t < 2\pi \\ -2t + (1+6\pi)\cos(t) + 4\sin(t) & \text{if } t \ge 2\pi \end{cases}.$$

Consider of a sequence of functions $f_1, f_2, f_3...$ where $f_n(t) = 0$ for t outside $[-1/n, 1/n]$ and $f_n(t) = n(1-n|t|)$ for t inside $[-1/n, 1/n]$.

The reader is left to check that for any positive integer n, $\displaystyle\int_I f_n \, dt = 1$ for any interval I containing $[-1/n, 1/n]$.

The <u>dirac delta function</u> δ is the limit $\delta = \lim\limits_{n\to\infty} f_n$. Here are the consequences:

- $\delta(t) = 0$ if $t \neq 0$.
- $\displaystyle\int_I \delta \, dt = 0$ for any closed interval I not containing 0.
- $\displaystyle\int_I \delta \, dt = 1$ for any closed interval I containing 0.
- Since $f_n(0) = n$ we have that $\delta(0)$ does not exist as a finite number.

Here is one way to "think" of δ:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}, \text{ such that } \int_I \delta(t)\, dt = 1 \text{ on any interval } I \text{ containing } 0.$$

For our purposes, it suffices to use the following result:

If $t_0 > 0$ then the solution to the IVP $ay'' + by' + cy = \delta(t - t_0)$, $y(0) = 0$, $y'(0) = 0$ is defined to be $y = \mathfrak{U}(t - t_0)w(t - t_0)$ where $w = \mathfrak{L}^{-1}\left(\dfrac{1}{as^2 + bs + c}\right)$.

The solution the IVP $ay'' + by' + cy = f(t) + \alpha\delta(t - t_0)$, $y(0) = k_0$, $y'(0) = k_1$ is

$$y = \hat{y} + \alpha\mathfrak{U}(t - t_0)w(t - t_0),$$

where $w = \mathfrak{L}^{-1}\left(\dfrac{1}{as^2 + bs + c}\right)$ and $\hat{y}$ is the solution to $ay'' + by' + cy = f(t)$, $y(0) = k_0$, $y'(0) = k_1$.

Justification: Let's find the Laplace Transform of $\delta(t - t_0)$.

For any real number s, $f(t) = e^{-st}$ is a continuous function on $[t_0 - \epsilon, t_0 + \epsilon]$ for any positive ϵ and by the Extreme Value Theorem attains extreme values of $e^{-s(t_0-\epsilon)}$ and $e^{-s(t_0+\epsilon)}$ (one is a max. and one is a min. – which is which depends on the value of s). Then

$\mathfrak{L}(\delta(t - t_0)) = \displaystyle\int_0^\infty e^{-st}\delta(t - t_0)\,dt = \lim_{\epsilon\to 0^+}\int_{t_0-\epsilon}^{t_0+\epsilon} e^{-st}\delta(t - t_0)\,dt$ and the integral is between $e^{-s(t_0-\epsilon)}$ and $e^{-s(t_0+\epsilon)}$.

Hence $\mathfrak{L}(\delta(t - t_0)) = e^{-st_0}$ by the Squeeze Theorem.

Then by taking the Laplace transform of both sides of $ay'' + by' + cy = \delta(t - t_0)$ we get $(as^2 + bs + c)Y(s) = e^{-st_0}$, so $Y(s) = \dfrac{e^{-st_0}}{as^2 + bs + c}$.

So by the second shifting property,

$$y = \mathfrak{U}(t - t_0)w(t - t_0)$$

where $w = \mathfrak{L}^{-1}\left(\dfrac{1}{as^2 + bs + c}\right)$.

Example: Let's solve $y'' - 7y' + 6y = 2e^{-7t} + 2\delta(t-1)$, $y(0) = 1$, $y'(0) = -1$.

(ELFY) Use Laplace transforms to show that

$$\hat{y} = -\frac{24}{65}e^{6t} + \frac{27}{20}e^{t} + \frac{1}{52}e^{-7t},$$

is a solution to $y'' - 7y' + 6y = 2e^{-7t}$, $y(0) = 1$, $y'(0) = -1$.

Next we find that $w = \mathfrak{L}^{-1}\left(\frac{1}{(s-6)(s-1)}\right) = \frac{1}{5}\left(e^{6t} - e^{t}\right)$.

Thus the solution is $y = \hat{y} + 2\mathcal{U}(t-1)\left[w(t-1)\right] = -\frac{24}{65}e^{6t} + \frac{27}{20}e^{t} + \frac{1}{52}e^{-7t} + 2\mathcal{U}(t-1)\left[\frac{1}{5}\left(e^{6(t-1)} - e^{t-1}\right)\right]$.

In a more familiar form,

$$y(t) = \begin{cases} -\frac{24}{65}e^{6t} + \frac{27}{20}e^{t} + \frac{1}{52}e^{-7t} & \text{if } 0 \le t < 1 \\ \left(-\frac{24}{65} + \frac{2}{5e^{6}}\right)e^{6t} + \left(\frac{27}{20} - \frac{2}{5e}\right)e^{t} + \frac{1}{52}e^{-7t} & \text{if } t \ge 1 \end{cases}.$$

Example: Let's solve $y'' + y = 1 + 2\delta(t-\pi) + 3\delta(t-2\pi)$, $y(0) = 1$, $y'(0) = -2$.

(ELFY) Use Laplace transforms to show that

$$\hat{y} = 1 - 2\sin(t),$$

is a solution to $y'' + y = 1$, $y(0) = 1$, $y'(0) = -2$.

Next we find that $w = \mathfrak{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$.

Thus the solution is

$y = \hat{y} + 2\mathcal{U}(t-\pi)\left[w(t-\pi)\right] + 3\mathcal{U}(t-2\pi)\left[w(t-2\pi)\right] = 1 - 2\sin(t) + 2\mathcal{U}(t-\pi)\left[\sin(t-\pi)\right] + 3\mathcal{U}(t-2\pi)\left[\sin(t-2\pi)\right].$

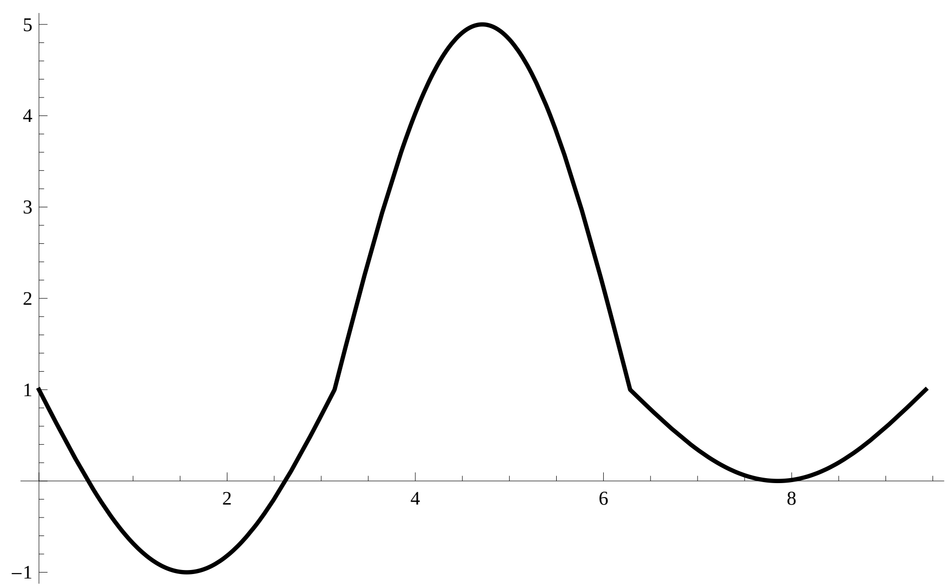
Using the basic identities $\sin(t-\pi) = -\sin(t)$ and $\sin(t-2\pi) = \sin(t)$, this becomes

$$y = 1 - 2\sin(t) - 2\mathcal{U}(t-\pi)\left[\sin(t)\right] + 3\mathcal{U}(t-2\pi)\left[\sin(t)\right].$$

Or in a more familiar form,

$$y(t) = \begin{cases} 1 - 2\sin(t) & \text{if } 0 \le t < \pi \\ 1 - 4\sin(t) & \text{if } \pi \le t < 2\pi \\ 1 - \sin(t) & \text{if } t \ge 2\pi \end{cases}.$$

Here is the graph of the solution:



Convolutions

Let $f(t), g(t)$ be functions. The <u>convolution</u> of f and g is the function $(f*g)(t) = \int_0^t f(r)g(t-r)\,dr$.

Let's compute the convolution of $f(t) = e^t$ and $g(t) = e^{-t}$:

$$(f*g)(t) = \int_0^t e^r e^{-t+r}\,dr = \int_0^t e^{-t}e^{2r}\,dr = e^{-t}\,\frac{1}{2}e^{2r}\Big|_0^t = \frac{e^{-t}}{2}\left(e^{2t}-1\right) = \frac{e^t - e^{-t}}{2} = \sinh{(t)}.$$

The following theorem is called convolution theorem (the proof is covered on an activity):

Proposition 4.4 : If $f(t), g(t)$ are functions such that $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$ then $\mathcal{L}(f*g) = FG$.

Let's find a formula for the solution of the IVP $y'' + 4y' - 5y = f(t)$, $y(0) = k_0$, $y'(0) = k_1$.

By taking the Laplace transform of both sides we get $(s^2+4s-5)Y(s) = F(s) + (k_1 + k_0 s) + 4k_0$, where $Y(s) = \mathcal{L}(y(t))$ and $F(s) = \mathcal{L}(f(t))$.

Then $Y(s) = F(s)G(s) + \dfrac{k_1 + 4k_0 + k_0 s}{(s+5)(s-1)} = F(s)G(s) + \dfrac{k_1+5k_0}{6(s-1)} - \dfrac{k_1 - k_0}{6(s+5)}$, where $G(s) = (s^2+4s-5)^{-1}$.

Then by taking the inverse Laplace transform we get

$$y(t) = \frac{k_1+5k_0}{6}e^t + \frac{k_0-k_1}{6}e^{-5t} + \int_0^t f(r)g(t-r)\,dr, \text{ where } \mathcal{L}(g) = G.$$

In particular, $g(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s^2+4s-5}\right) = \mathcal{L}^{-1}\left(\dfrac{1/6}{s-1} - \dfrac{1/6}{s+5}\right) = \dfrac{1}{6}\left(e^t - e^{-5t}\right)$.

So $y(t) = \dfrac{k_1+5k_0}{6}e^t + \dfrac{k_0-k_1}{6}e^{-5t} + \displaystyle\int_0^t \frac{f(r)}{6}\left(e^{t-r} - e^{-5t+5r}\right)\,dr$

Let's apply this formula to $f(t) = 6e^t$, $k_0 = 1$, $k_1 = 0$:

$$y(t) = \frac{5}{6}e^t + \frac{1}{6}e^{-5t} + \int_0^t \frac{6e^r}{6}\left(e^{t-r} - e^{-5t+5r}\right)\,dr = \frac{5}{6}e^t + \frac{1}{6}e^{-5t} + \int_0^t e^t - e^{-5t+6r}\,dr =$$

$$= \frac{5}{6}e^t + \frac{1}{6}e^{-5t} + \left(re^t - \frac{1}{6}e^{-5t+6r}\Big|_{r=0}^{r=t}\right) = \frac{5}{6}e^t + \frac{1}{6}e^{-5t} + te^t - \frac{1}{6}e^t + \frac{1}{6}e^{-5t} = \frac{2}{3}e^t + te^t + \frac{1}{3}e^{-5t}.$$

Compare to the solution we got earlier (top of page 13)!

Let's find a formula for the solution of the IVP $y'' + 2y' + 2 = f(t)$, $y(0) = k_0$, $y'(0) = k_1$.

Taking the Laplace transform of both sides yields $((s+1)^2 + 1)Y(s) = F(s) + (k_1 + sk_0) + 2k_0$, where $Y(s) = \mathcal{L}(y(t))$ and $F(s) = \mathcal{L}(f(t))$.

Thus $Y(s) = F(s)G(s) + \dfrac{k_0(s+1)}{(s+1)^2+1} + \dfrac{k_1+k_0}{(s+1)^2+1}$, where $G(s) = [(s+1)^2+1]^{-1}$.

Then by taking the inverse Laplace transform (using the table of Laplace transforms) we get

$$y(t) = k_0 e^{-t}\cos(t) + (k_0+k_1)e^{-t}\sin(t) + \int_0^t f(r)g(t-r)\,dr, \text{ where } \mathcal{L}(g) = G.$$

In particular, $g(t) = \mathcal{L}^{-1}\left(\dfrac{1}{(s+1)^2+1}\right) = e^{-t}\sin(t)$.

So $y(t) = k_0 e^{-t}\cos(t) + (k_0+k_1)e^{-t}\sin(t) + \displaystyle\int_0^t f(r)e^{-t+r}\sin(t-r)\,dr$.

One can also use this theory to evaluate convolution integrals. For example, let's evaluate $\displaystyle\int_0^t e^{-r}\sin(t-r)\,dr$.

This is the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$. Then $F(s) = \dfrac{1}{s+1}$ and $G(s) = \dfrac{1}{s^2+1}$ are the Laplace transforms respectively. All we need to do is find the inverse Laplace transform of $F(s)G(s) = \dfrac{1}{(s+1)(s^2+1)}$.

We begin with the P.F.D. $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{1/2}{s+1} + \dfrac{As+B}{s^2+1}$. From this we get $A = -1/2$ and $B = 1/2$.

So $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{1/2}{s+1} - (1/2)\dfrac{s-1}{s^2+1} = \dfrac{1}{2}\left[\dfrac{1}{s+1} + \dfrac{1}{s^2+1} - \dfrac{s}{s^2+1}\right]$ and the inverse Laplace transform, and therefore the integral, is

$$\frac{1}{2}\left[e^{-t} + \sin(t) - \cos(t)\right].$$

An integral equation is one of the form

$$\phi(t) + \int_{t_0}^{t} f(x, \phi(x))\, dx = g(t),\ t_0 \geq 0$$

where $\phi(t)$ is an unknown function.

An integro-differential equation is one of the form

$$\phi'(t) + \int_{t_0}^{t} f(x, \phi(x))\, dx = g(t, \phi(t)),\ t_0 \geq 0$$

where $\phi(t)$ is an unknown function.

When the integral part of either of the above is a convolution we can solve for ϕ by taking the Laplace transform of both sides, using partial fractions, and then taking the Inverse Laplace transform.

Example: Consider the equation $\phi'(t) + \displaystyle\int_0^t \phi(x)(t-x)\ dx = t$. Let's find a solution such that $\phi(0) = 1$.

Notice that $\phi(t) * t = \displaystyle\int_0^t \phi(x)(t-x)\, dx$. By taking the Laplace transform (where $\Phi(s) = \mathcal{L}(\phi(t))$) of both sides we get $s\Phi(s) - 1 + \dfrac{1}{s^2} \cdot \Phi(s) = \dfrac{1}{s^2}$.

Therefore, $\Phi(s) = \dfrac{\frac{1}{s^2}+1}{s+\frac{1}{s^2}} = \dfrac{s^2+1}{s^3+1} = \dfrac{s^2+1}{(s+1)(s^2-s+1)} = \dfrac{2}{3} \cdot \dfrac{1}{s+1} + \dfrac{1}{3} \cdot \dfrac{s+1}{s^2-s+1}$.

Since $\dfrac{s+1}{s^2-s+1} = \dfrac{s-1/2}{(s-1/2)^2+3/4} + \dfrac{3/2}{(s-1/2)^2+3/4} = \dfrac{s-1/2}{(s-1/2)^2+(\sqrt{3}/2)^2} + \dfrac{\sqrt{3}\cdot\sqrt{3}/2}{(s-1/2)^2+(\sqrt{3}/2)^2}$
it follows that $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2-s+1}\right) = e^{t/2}\cos(t\sqrt{3}/2) + \sqrt{3}e^{t/2}\sin(t\sqrt{3}/2)$.

So the solution is $\phi(t) = \dfrac{2}{3}e^{-t} + \dfrac{1}{3}e^{t/2}\cos(t\sqrt{3}/2) + \dfrac{\sqrt{3}}{3}e^{t/2}\sin(t\sqrt{3}/2)$

Euler's Method

Often the first order non-linear IVP $y' = f(x, y)$, $y(x_0) = y_0$ cannot be solved analytically for an implicit or explicit solution.

There are many numerical methods for approximating solutions at various values of x.

We will look at just one method:

Suppose we are interested in finding the values of y at equally spaced values $x_i = x_0 + i\Delta x$ for $i = 0, 2, ..., n$ where $\Delta x = \dfrac{x_n - x_0}{n}$.

Denote by y_i the approximate solution at x_i (for $i = 1, 2, .., n$). Then the error at the ith step is $e_i = y(x_i) - y_i$ (for $i = 0, 1, 2, .., n$). Clearly $e_0 = 0$ as we already know $y(x_0) = y_0$.

The following method is one type called **Euler's method** (also called the "forward Euler"):

Euler's method rests on having the tangent line to the graph of the solution curve at x_i approximate the curve on $[x_i, x_{i+1}]$.

The slope at (x_0, y_0) is $f(x_0, y_0)$ so the tangent line on $[x_0, x_1]$ is

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

So we get the approximate solution $y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + \Delta x f(x_0, y_0)$.

Then we can repeat to get the approximate solution $y_2 = y_1 + \Delta x f(x_1, y_1)$.

We continue in this way at the $(i + 1)^{\text{st}}$ step with $y_{i+1} = y_i + \Delta x f(x_i, y_i)$ for $i = 0, 1, ..., n - 1$.

Let's begin with an example for which we can solve it analytically:

Use Euler's method to find approximate solutions to the IVP $y' = \dfrac{y+1}{x^2+1}$, $y(0) = 1$ at the points $x = 0.1, 0.2, 0.3$ using $\Delta x = 0.1$.

$y_1 = y_0 + \Delta x f(x_0, y_0) = 1 + 0.1(2) = \dfrac{6}{5} = 1.2.$

$y_2 = y_1 + \Delta x f(x_1, y_1) = 1.2 + 0.1\left(\frac{2.2}{1.01}\right) = \dfrac{716}{505} \approx 1.417822.$

$y_3 = y_2 + \Delta x f(x_2, y_2) = \frac{716}{505} + 0.1\left(\frac{1221/505}{1.04}\right) = \dfrac{43337}{26260} \approx 1.650305.$

Now let's solve analytically the IVP $y' = \dfrac{y+1}{x^2+1}$, $y(0) = 1$ and determine how large the errors are for the approximate values y_1, y_2, y_3 found using Euler's method (on the last page):

The equation is separable:

$$\frac{1}{y+1}y' = \frac{1}{x^2+1}.$$

Integrating both sides and solving for y (relabeling the constant along the way) gives

$$\ln|y+1| = \tan^{-1}(x) + c \to y + 1 = ce^{\tan^{-1}(x)} \to y = -1 + ce^{\tan^{-1}(x)}.$$

By imposing the initial condition $y(0) = 1$ we get that $c = 2$.

So the solution to the IVP is

$$y = -1 + 2e^{\tan^{-1}(x)}.$$

We have that $y_1 = 1.2$. By plugging in $x = 0.1$ above,

$$y(0.1) = -1 + 2e^{\tan^{-1}(0.1)} \approx 1.209610.$$

So the approximation y_1 has an error of $e_1 \approx 0.0961$ (about an 8 percent underestimate).

We have that $y_2 \approx 1.417822$. By plugging in $x = 0.2$ above,

$$y(0.2) = -1 + 2e^{\tan^{-1}(0.2)} \approx 1.436452.$$

So the approximation y_2 has an error of $e_2 \approx 0.01863$ (about an 1.3 percent underestimate).

We have that $y_3 \approx 1.650305$. By plugging in $x = 0.3$ above,

$$y(0.3) = -1 + 2e^{\tan^{-1}(0.3)} \approx 1.676752.$$

So the approximation y_3 has an error of $e_3 \approx 0.02645$ (about an 1.6 percent underestimate).

Now let's apply Euler's method to the following linear non-homogeneous first order IVP to find approximate solutions at $x = 0.2, 0.4, 0.6, 0.8, 1$ with $\Delta x = 0.2$.

$$y' = 2xy + 1, \;\; y(0) = 1.$$

$$y_1 = y_0 + \Delta x f(x_0, y_0) = 1 + 0.2(1) = \frac{6}{5} = 1.2.$$

$$y_2 = y_1 + \Delta x f(x_1, y_1) = 1.2 + 0.2(2(0.2)(1.2) + 1) = \frac{187}{125} = 1.496.$$

$$y_3 = y_2 + \Delta x f(x_2, y_2) = 1.496 + 0.2(2(0.4)(1.496) + 1) = \frac{6,048}{3,125} = 1.93536.$$

$$y_4 = y_3 + \Delta x f(x_3, y_3) = 1.93536 + 0.2(2(0.6)(1.93536) + 1) = \frac{203,113}{78,125} = 2.5998464.$$

$$y_5 = y_4 + \Delta x f(x_4, y_4) = 2.5998464 + 0.2(2(0.8)(2.5998464) + 1) = \frac{7,093,354}{1,953,125} \approx 3.63179725.$$

What if we try to solve this analytically? The equation $y' - 2xy = 1$ has an integrating factor of $e^{\int -2x\,dx} = e^{-x^2}$.

Then

$$e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2} \to (e^{-x^2}y)' = e^{-x^2}.$$

But here, with the methods developed so far, we get stuck as the integral of e^{-x^2} cannot be solved analytically.

APPENDIX: Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
(1) 1	$\dfrac{1}{s}$
(2) $e^{at}f(t)$	$F(s-a)$
(3) $\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
(4) $f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
(5) $\delta(t)$	1
(6) $\delta(t-t_0)$	e^{-st_0}
(7) $t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$
(8) $f'(t)$	$sF(s) - f(0)$
(9) $f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
(10) $\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
(11) $t^n \ (n = 0, 1, \dots)$	$\dfrac{n!}{s^{n+1}}$
(12) $t^x \ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$
(13) $\sin kt$	$\dfrac{k}{s^2+k^2}$
(14) $\cos kt$	$\dfrac{s}{s^2+k^2}$
(15) e^{at}	$\dfrac{1}{s-a}$
(16) $\sinh kt$	$\dfrac{k}{s^2-k^2}$
(17) $\cosh kt$	$\dfrac{s}{s^2-k^2}$
(18) $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
(19) $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
(20) te^{at}	$\dfrac{1}{(s-a)^2}$
(21) $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
(22) $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
(23) $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
(24) $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
(25) $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
(26) $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
(27) $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
(28) $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
(29) $t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$
(30) $\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$
(31) $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
(32) $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
(33) $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$