Variation of Parameters

We explore how variation of parameters works for finding a particular solution to a linear second order o.d.e. given we have a fundamental set of solutions to the complementary homogeneous o.d.e. or can easily find one. Since we have a non-trivial solution the complementary equation, we could use reduction of order, but generally variation of parameters is easier...

We will explore this method by just working through an example:

Consider the linear second order o.d.e. \( y'' - 3y' + 2y = \frac{4}{1 + e^{-x}} \). Clearly we have that \( \{e^x, e^{2x}\} \) is a fundamental set of solutions to \( y'' - 3y' + 2y = 0 \).

Let’s look for a particular solution of the form \( y = ue^x + ve^{2x} \).

Then \( y' = u'e^x + ue^x + v'e^{2x} + 2ve^{2x} \).

To make things easier, we may assume that \( u, v \) satisfy \( u'e^x + v'e^{2x} = 0 \).

So \( y' = ue^x + 2ve^{2x} \) and \( y'' = u'e^x + 2v'e^{2x} + 4ve^{2x} \).

By substitution,

\[
(u'e^x + 2v'e^{2x} + 4ve^{2x}) - 3(ue^x + 2ve^{2x}) + 2(ue^x + ve^{2x}) = \frac{4}{1 + e^{-x}}, \text{ so }
\]

\[
u'e^x + 2v'e^{2x} + 4ve^{2x} - 3ue^x - 6ve^{2x} + 2ue^x + 2ve^{2x} = \frac{4}{1 + e^{-x}}, \text{ so }
\]

\[
u'e^x + u(e^x - 3e^x + 2e^{2x}) + 2v'e^{2x} + 2v(2e^{2x} - 3e^{2x} + e^{2x}) = \frac{4}{1 + e^{-x}}, \text{ so }
\]

\[
u'e^x + 2v'e^{2x} = \frac{4}{1 + e^{-x}}, \text{ so }
\]

\[
u'e^x + v'e^{2x} = 0,
\]

\[
u'e^x + 2v'e^{2x} = \frac{4}{1 + e^{-x}}.
\]

Subtracting these equations yields, \( v'e^{2x} = \frac{4}{1 + e^{-x}} \).

Hence \( v' = \frac{4e^{-2x}}{1 + e^{-x}} \).
By integrating both sides (letting the constant of integration be 0)

\[ v = \int \frac{4e^{-2x}}{1 + e^{-x}} \, dx = \int \frac{-4t}{1 + t} \, dt = -4 \int 1 - \frac{1}{1 + t} \, dt = -4(t - \ln(1 + t)) = 4(\ln(1 + e^{-x}) - e^{-x}). \]

Then \( 0 = u'e^x + v'e^{2x} = u'e^x + \left( \frac{4e^{-2x}}{1 + e^{-x}} \right) e^{2x} = u'e^x + \left( \frac{4}{1 + e^{-x}} \right) \) implies

\[ u' = \frac{-4e^{-x}}{1 + e^{-x}} \rightarrow u = 4 \ln(1 + e^{-x}) \text{ choosing zero for the constant}. \]

So a particular solution to \( y'' - 3y' + 2y = \frac{4}{1 + e^{-x}} \) is

\[ y = ue^x + ve^{2x} = 4e^x \ln(1 + e^{-x}) + 4e^{2x}(\ln(1 + e^{-x}) - e^{-x}). \]

Incidentally, a general solution is

\[ y = 4e^x(c_1 + \ln(1 + e^{-x})) + 4e^{2x}(c_2 + \ln(1 + e^{-x})). \]

Let’s apply variation of parameters to find a general solution to \( xy'' - y' - 4x^3y = 8x^5 \) given that \( \{e^{x^2}, e^{-x^2}\} \) is a fundamental set of solutions to the complementary equation.

Let \( y_1 = e^{x^2} \) and \( y_2 = e^{-x^2} \). Set \( y_p = u_1y_1 + v_1y_2 \) to be a particular solution. Then \( y' = u_1y'_1 + v_1y'_2 \) assuming \( u'y_1 + v'y_2 = 0 \).

Then \( y'' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \). When substituting this into the equation \( xy'' - y' - 4x^3y = 8x^5 \) and simplifying we get \( x(u'y_1' + v'y_2') = 8x^5 \).

So \( u'y_1 + v'y_2 = 0 \) and \( u'y_1' + v'y_2' = 8x^4 \). Multiplying the first equation by \( y_2' \), then the second by \( -y_2 \), and then adding the resulting equations yields

\[ u'(y_1y_2' - y_1'y_2) = -8x^4y_2. \]

By simplifying with \( y_1 = e^{x^2} \) and \( y_2 = e^{-x^2} \), and solving for \( u' \), this becomes

\[ u' = -8x^4e^{-x^2}/(-2x - 2x) = 2x^3e^{-x^2}. \]

Using integrating by parts, \( u = -(x^2 + 1)e^{-x^2} \) (choosing the constant of integration to be zero).

Then \( v' = -u'y_1/y_2 = -2x^3e^{-x^2} \). Again integrate by parts to get \( v = (1 - x^2)e^{x^2} \) (choosing \( C = 0 \)).

So a particular solution is \( y_p = uy_1 + vy_2 = -(x^2 + 1) + (1 - x^2) = -2x^2 \), and a general solution is

\[ y = -2x^2 + c_1e^{x^2} + c_2e^{-x^2}. \]
Applications of Second Order o.d.e. to Spring-Mass Systems

Consider an object of mass $m$ suspended from a spring (of negligible mass). We say that the spring-mass system is at equilibrium when the object is at rest and the sum of the forces acting on it is 0. The corresponding position is called the equilibrium position.

Let $y$ be the displacement of the mass from the equilibrium where $y > 0$ is upward.

The following forces act on the object:

1. Gravity: $F_g = -mg$ where $g$ is the gravitational constant.
2. Spring Force: $F_s = k\Delta L$ where $k$ is the spring constant (Hooke’s Law) and $\Delta L$ is the change in length (stretching is positive) from the spring equilibrium position with no mass attached.
3. Damping Force: $F_d = -cy'$ where $c$ is the damping constant (due to medium resistance – often negligible). Simple harmonic motion occurs when $c = 0$.

There could also be another force $F$, independent of the motion, not accounted for above.

By Newton’s second law we can write an o.d.e. to model the spring-mass system described above:

$$my'' = F - mg - cy' + k\Delta L.$$  

Let’s relate $\Delta L$ and $y$. Imagine the spring without a mass attached. When a mass $m$ is attached, the spring stretches by some amount $\ell$ and at equilibrium, the sum of forces on the spring is 0, so $F_g + F_s = 0$. Hence $-mg + k\ell = 0$, or $\ell = mg/k$.

If at this point, a displacement of $y$ occurs then we would have $\Delta L = \ell - y = mg/k - y$.

So $my'' = F - mg - cy' + k\Delta L$ becomes $my'' = F - cy' - ky$ or equivalently $my'' + cy' + ky = F$.

If $F = 0$ we say the motion is free. Otherwise, it is said to be forced.

Consider the free undamped system modeled by $y'' + \frac{k}{m}y = 0$. We know that a general solution is

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t),$$

where $\omega = \sqrt{k/m}$.

Example: Suppose an object is hung from a spring whose natural length is 98 cm and attains an equilibrium position of 1 m. Suppose the object is displaced downward 5 cm, but given an initial upward velocity of 11 cm/s. Find the displacement for time $t > 0$. 
Since \( \frac{g}{\ell} = \frac{k}{m} \), we have that \( \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\ell}} = \sqrt{9.8/0.02} \approx 22 \).

So \( y \approx c_1 \cos (22t) + c_2 \sin (22t) \). Imposing \( y(0) = -0.05 \) yields \( c_1 = -0.05 \). Imposing \( y'(0) = 0.11 \) yields \( 22c_2 = 0.11 \), so \( c_2 = 0.005 \).

Hence \( y \approx -0.05 \cos (22t) + 0.005 \sin (22t) \).

Let’s go back to a general solution: \( y = c_1 \cos (\omega t) + c_2 \sin (\omega t) \), where \( \omega = \sqrt{\frac{k}{m}} \).

Let \( A = \sqrt{c_1^2 + c_2^2} \). Then \( c_1 = A \cos (\phi) \) and \( c_2 = A \sin (\phi) \) for some angle \( \phi \) in \((-\pi, \pi]\).

Using the identity, \( \cos (\omega t - \phi) = \cos (\omega t) \cos (\phi) + \sin (\omega t) \sin (\phi) \) we get \( y = c_1 \cos (\omega t) + c_2 \sin (\omega t) = A \cos (\phi) \cos (\omega t) + A \sin (\phi) \sin (\omega t) = A \cos (\omega t - \phi) \).

The \( A \) is the \textit{amplitude} (motion goes from \(-A\) to \(A\)). \( T = \frac{2\pi}{\omega} \) is the \textit{period} (time it takes to go through a full cycle, such as from \(A\) back to \(A\)). \( \phi \) is the \textit{phase angle}. \( \omega \) is the natural \textit{angular frequency}, and \( f = \omega/(2\pi) = 1/T \) is the \textit{frequency}.

In the example above, the amplitude is \( A \approx 0.0502 \) m. The phase angle is \( \phi \approx 0.1 \) rad. The natural frequency is \( \omega \approx 22 \) rad/s (a frequency of \( f \approx 3.5 \) Hz). The period is \( T \approx 0.28 \) s.

Let’s consider a curious forced undamped system (such a system is considered to be in \textit{resonance}): Suppose that the external force is \( F = F_0 \cos (\omega t) \) where \( \omega = \sqrt{\frac{k}{m}} \) (\( F_0 \) is a constant).

Then the system is modeled by \( y'' + \omega^2 y = \frac{F_0}{m} \cos (\omega t) \). We already know that \( \{ \cos (\omega t), \sin (\omega t) \} \) is a fundamental set of solutions to the complementary homogeneous equation (for the free system).

Let’s find a particular solution. We look for a solution of the form \( y = t(a \cos (\omega t) + b \sin (\omega t)) \) (won’t work without the “\( t \)” – why?)

Then \( y' = (a \cos (\omega t) + b \sin (\omega t)) + \omega t(-a \sin (\omega t) + b \cos (\omega t)) \) and

\[
y'' = 2\omega(-a \sin (\omega t) + b \cos (\omega t)) - \omega^2 t(a \cos (\omega t) + b \sin (\omega t)).
\]

By substitution into \( y'' + \omega^2 y = \frac{F_0}{m} \cos (\omega t) \) we get

\[
2\omega(-a \sin (\omega t) + b \cos (\omega t)) = \frac{F_0}{m} \cos (\omega t).
\]
This requires \( a = 0 \) and \( b = \frac{F_0}{2m\omega} = \frac{F_0}{\sqrt{4mk}} \).

So \( y = \frac{F_0}{\sqrt{4mk}} t \sin(\omega t) \) is a particular solution and a general solution is

\[
y = \frac{F_0}{\sqrt{4mk}} t \sin(\omega t) + c_1 \cos(\omega t) + c_2 \sin(\omega t).
\]

Then the amplitude goes toward \( \infty \) as \( t \to \infty \). This means that the system will fail (like the Tacoma Narrows Bridge).

Now let’s consider a free damped system: \( my'' + cy' + ky = 0 \) where \( c \neq 0 \).

The characteristic polynomial is \( mr^2 + cr + k \). The roots of this polynomial are

\[
r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.
\]

Case 1: Underdamped \( (c < \sqrt{4mk}) \)

Then we get two complex conjugate roots \( \alpha \pm \omega i \) where \( \alpha = -c/(2m) \) and \( \omega = \frac{\sqrt{4mk - c^2}}{2m} \).

Hence a general solution is \( y = e^{\alpha t} (c_1 \cos(\omega t) + c_2 \sin(\omega t)) \) and since \( \alpha < 0 \) we get that \( y \to 0 \) as \( t \to \infty \) (but with oscillations).

Case 2: Overdamped \( (c > \sqrt{4mk}) \)

Then we get two distinct real and negative roots \( r_1, r_2 \), and a general solution of

\[
y = c_1 e^{r_1 t} + c_2 e^{r_2 t}
\]

which goes towards 0 as \( t \to \infty \).

In this case no oscillation occurs as (ELFY) \( y \) can’t equal zero for more than one value of \( t \) unless \( c_1 = c_2 = 0 \).

Case 3: Critically damped \( (c = \sqrt{4mk}) \)

Then we get one repeated real and negative root \( r = -c/(2m) \), and a general solution of

\[
y = (c_1 + tc_2)e^{rt}
\]

which goes towards 0 as \( t \to \infty \). As in the overdamped case, no oscillation occurs (ELFY) and \( y \) can’t equal zero for more than one value of \( t \) unless \( c_1 = c_2 = 0 \).
Laplace Transforms

Let \( f(t) \) be defined for \( t \geq 0 \) and \( s \) represent a real number. Then the Laplace transform of \( f \) is

\[
F(s) = \int_0^\infty e^{-st} f(t) \, dt.
\]

The functions \( f \) and \( F \) are called a transform pair.

Notation: \( \mathcal{L}(f) = F \) or \( f \leftrightarrow F \).

Note: Technically the improper integral must be turned in a limit, but we will be lazy and never formally do this. We will allow “\( \infty \)” to appear as though it were a real number (but with the understanding that we are taking a limit).

Let’s calculate some basic Laplace transforms:

(i) \( f(t) = 1 \) (a constant function). Then \( F(0) \) diverges (to \( \infty \)) and for \( s \neq 0 \),

\[
F(s) = \int_0^\infty e^{-st} \, dt = -\frac{1}{s} e^{-st} \bigg|_0^\infty = \begin{cases} 
\frac{1}{s} & \text{if } s > 0 \\
\infty & \text{if } s < 0 
\end{cases}.
\]

So we write \( \mathcal{L}(1) = \frac{1}{s}, \, s > 0 \).

(ii) \( f(t) = t^2 \). Then \( F(0) \) diverges (to \( \infty \)) and for \( s \neq 0 \),

\[
F(s) = \int_0^\infty t^2 e^{-st} \, dt = -\frac{t^2}{s} e^{-st} \bigg|_0^\infty + \frac{2}{s^3} \int_0^\infty t e^{-st} \, dt = -\frac{t^2}{s} e^{-st} \bigg|_0^\infty + \frac{2}{s} \left[ -\frac{t}{s} e^{-st} \bigg|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \, dt \right] =
\]

\[
= -\frac{t^2}{s} e^{-st} \bigg|_0^\infty + \frac{2}{s} \left[ -\frac{t}{s} e^{-st} \bigg|_0^\infty + \frac{1}{s} \left( -\frac{1}{s} e^{-st} \bigg|_0^\infty \right) \right] = \begin{cases} 
\frac{2}{s^3} & \text{if } s > 0 \\
\infty & \text{if } s < 0 
\end{cases}.
\]

So we write \( \mathcal{L}(t^2) = \frac{2}{s^3}, \, s > 0 \).

(iii) \( f(t) = e^{at} \). \( F(a) \) diverges to \( \infty \).

\[
F(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt = \frac{1}{a-s} e^{(a-s)t} \bigg|_0^\infty = \begin{cases} 
\frac{1}{s-a} & \text{if } s > a \\
\infty & \text{if } s < a
\end{cases}.
\]

So we write \( \mathcal{L}(e^{at}) = \frac{1}{s-a}, \, s > a \).
(iv) \( f(t) = \begin{cases} 
2 - t & \text{if } 0 \leq t < 1 \\
t & \text{if } t \geq 1
\end{cases} \)

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt = \int_0^1 (2 - t)e^{-st} \, dt + \int_1^\infty te^{-st} \, dt
\]

\[
= \left[ -\frac{2 - t}{s}e^{-st} \right]_0^1 - \frac{1}{s} \int_0^1 e^{-st} \, dt + \left[ -\frac{t}{s}e^{-st} \right]_1^\infty + \frac{1}{s} \int_1^\infty e^{-st} \, dt
\]

\[
= \left[ \frac{2}{s} - \frac{1}{s}e^{-s} + \frac{1}{s^2}(e^{-s} - 1) \right] + \left[ \frac{1}{s}e^{-s} + \frac{1}{s^2}e^{-s} \right] = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s}.
\]

As an appendix (last page of these notes) there a table of Laplace transforms.

**Proposition 4.1 (Linearity of Laplace transforms):** If \( \mathcal{L}(f) \) and \( \mathcal{L}(g) \) both exist for \( s > s_0 \) then for any constants \( c_1, c_2 \) we have

\[
\mathcal{L}(c_1f(t) + c_2g(t)) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g) \text{ for } s > s_0.
\]

The proof is left as an (ELFY). It’s very straightforward.

Let’s find the Laplace transform of \( \sinh(at) = \frac{e^{at} - e^{-at}}{2} \) where \( a > 0 \).

We know that \( \mathcal{L}(e^{at}) = \frac{1}{s - a} \) for \( s > a \) and \( \mathcal{L}(e^{-at}) = \frac{1}{s + a} \) for \( s > -a \).

By linearity, \( \mathcal{L}(\sinh(t)) = \frac{1}{2} \left( \frac{1}{s - a} - \frac{1}{s + a} \right) = \frac{a}{s^2 - a^2} \) for \( s > a \).

Suppose \( \mathcal{L}(f) = F \). Let’s determine \( \mathcal{L}(e^{at}f(t)) \) where \( a \) is a constant.

\[
F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) \, dt
\]

and thus \( \mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}(e^{at}f(t)) \, dt = \int_0^\infty e^{-(s-at)}f(t) \, dt = F(s - a) \).

This property is referred to as the *first shifting property* of the Laplace transform.
Given that \( \mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2} \) (see appendix) we use the first shifting property to find \( \mathcal{L}(e^{at} \cos(\omega t)) \):

\[
\mathcal{L}(e^{at} \cos(\omega t)) = \frac{s - a}{(s - a)^2 + \omega^2}.
\]

A function \( f \) is of real exponential order \( s_0 \) if there are constants \( M \) and \( t_0 \) such that

\[
|f(t)| \leq Me^{s_0 t} \quad \text{for all} \quad t \geq t_0.
\]

Often we say a function is of exponential order without a reference to a particular \( s_0 \).

To save time we state the following existence theorem, but do not give a proof.

**Proposition 4.2**: Suppose \( f \) is of exponential order \( s_0 \). Then \( \mathcal{L}(f) \) is defined for \( s > s_0 \).

The inverse transform: Suppose \( F = \mathcal{L}(f) \). Then we call \( f \) the inverse Laplace transform of \( F \). Notation: \( f = \mathcal{L}^{-1}(F) \).

While there is a formula for computing \( \mathcal{L}^{-1}(F) \) we will not use it as it requires the theory of functions of a complex variable (complex analysis). We can however, make use the table (see appendix) to find inverse Laplace transforms.

Suppose we need \( \mathcal{L}^{-1} \left( \frac{s}{s^2 + 4} \right) \). By looking through the table of Laplace transforms we find

\[
\mathcal{L}(\cos(kt)) = \frac{s}{s^2 + k^2}.
\]

So \( \mathcal{L}^{-1} \left( \frac{s}{s^2 + 4} \right) = \cos(2t) \).

Note: Linearity also holds for the inverse transform (we omit the proof).

Let’s find \( \mathcal{L}^{-1} \left( \frac{2s + 3}{s^2 + 4s + 13} \right) \).

\[
\mathcal{L}^{-1} \left( \frac{2s + 1}{s^2 + 4s + 13} \right) = \mathcal{L}^{-1} \left( \frac{2(s + 2) - 1}{(s + 2)^2 + 9} \right) = 2\mathcal{L}^{-1} \left( \frac{s + 2}{(s + 2)^2 + 9} \right) - \frac{1}{3} \mathcal{L}^{-1} \left( \frac{3}{(s + 2)^2 + 9} \right)
\]

\[
= 2e^{-2t} \cos(3t) - \frac{1}{3} e^{-2t} \sin(3t) = e^{-2t} \left( 2 \cos(3t) - \frac{1}{3} \sin(3t) \right).
\]
In the last example, the denominator is irreducible over the reals. Our method changes when the denominator factors nicely:

Let’s find \( \mathcal{L}^{-1} \left( \frac{2s - 1}{s^2 - 5s + 6} \right) \).

Using a \( P.F.D. \) we get
\[
\frac{2s - 1}{s^2 - 5s + 6} = \frac{2s - 1}{(s - 2)(s - 3)} = \frac{5}{s - 3} - \frac{3}{s - 2}.
\]

Then \( \mathcal{L}^{-1} \left( \frac{2s - 1}{s^2 - 5s + 6} \right) = 5\mathcal{L}^{-1} \left( \frac{1}{s - 3} \right) - 3\mathcal{L}^{-1} \left( \frac{1}{s - 2} \right) = 5e^{3t} - 3e^{2t} \).

Let’s find \( \mathcal{L}^{-1} \left( \frac{s}{s^2 - 8s + 16} \right) \).

We can setup a \( P.F.D. \) as follows:
\[
\frac{s}{s^2 - 8s + 16} = \frac{s}{(s - 4)^2} = \frac{A}{s - 4} + \frac{4}{(s - 4)^2}.
\]

Clearing fractions and equating common-degree coefficients, we get \( A = 1 \).

So \( \mathcal{L}^{-1} \left( \frac{s}{s^2 - 8s + 16} \right) = \mathcal{L}^{-1} \left( \frac{1}{s - 4} \right) + 4\mathcal{L}^{-1} \left( \frac{1}{(s - 4)^2} \right) = e^{4t} + 4te^{4t} = e^{4t}(1 + 4t) \).

**Proposition 4.3** : Suppose \( f \) is continuous on \([0, \infty)\) and of exponential order \( s_0 \), and \( f' \) is piecewise continuous on \([0, \infty)\). Then \( \mathcal{L}(f) \) and \( \mathcal{L}(f') \) are defined for \( s > s_0 \) and
\[
\mathcal{L}(f') = s\mathcal{L}(f) - f(0).
\]

**Proof**: We will only prove it in the case where \( f' \) is continuous on \([0, \infty)\). You can see the proof in the other case in the textbook.

Assume \( f' \) is continuous on \([0, \infty)\). By integration by parts,
\[
\mathcal{L}(f') = \int_0^T e^{-st} f'(t) \, dt = e^{-st} f(t)|_0^T + s \int_0^T e^{-st} f(t) \, dt.
\]

Since \( f \) is of exponential order \( s_0 \), for \( s > s_0 \) we have \( \lim_{T \to \infty} e^{-sT} f(T) = 0 \) and \( \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) \, dt \) converges. Hence \( \mathcal{L}(f') \) converges for \( s > s_0 \) and
\[
\mathcal{L}(f') = -f(0) + s\mathcal{L}(f) \]
Let’s solve a really easy IVP using a Laplace transform!

Consider the basic first order IVP \( y' = ay, \ y(0) = y_0. \) We already know the solution is \( y(t) = y_0e^{at}. \)

Let’s see if we can find the same result using our new tool!

Let \( Y(s) = \mathcal{L}(y) \) be the Laplace transform of the solution \( y(t) \) (pretend it is unknown).

Let’s transform both sides of the o.d.e. \( y' = ay. \)

\[
\mathcal{L}(y') = \mathcal{L}(ay) \rightarrow s\mathcal{L}(y) - y(0) = a\mathcal{L}(y) \rightarrow (s - a)\mathcal{L}(y) = y_0 \rightarrow Y(s) = \mathcal{L}(y) = \frac{y_0}{s - a}.
\]

Then \( y(t) = \mathcal{L}^{-1}\left(\frac{y_0}{s - a}\right) = y_0e^{at}. \) Neat!

What about a second order IVP? Well, we can use that \( \mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s(s\mathcal{L}(y) - y(0)) - y'(0) = s^2\mathcal{L}(y) - y'(0) - sy(0). \)

Let’s derive a simplified form of the Laplace transform of the solution of the IVP

\[
ay'' + by' + cy = f(t), \ y(0) = k_0, \ y'(0) = k_1.
\]

We begin with \( a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = \mathcal{L}(f(t)). \)

Then by substitution, this becomes \( a(s^2\mathcal{L}(y) - k_1 - sk_0) + b(s\mathcal{L}(y) - k_0) + c\mathcal{L}(y) = \mathcal{L}(f(t)). \)

Letting \( Y(s) = \mathcal{L}(y) \) we get \( (as^2 + bs + c)Y(s) = \mathcal{L}(f(t)) + ak_1 + ak_0s + bk_0 \) and hence

\[
Y(s) = \frac{\mathcal{L}(f(t)) + ak_1 + ak_0s + bk_0}{as^2 + bs + c}.
\]

Let’s solve the IVP \( y'' + 4y' - 5y = 6e^t, \ y(0) = 1, \ y'(0) = 0 \) using Laplace transforms.

We would have that \( Y(s) = \frac{6 + (0 + 1s) + 4}{s^2 + 4s - 5} \) is the Laplace transform of the solution.

Simplifying this yields \( Y(s) = \frac{6 + (s + 4)(s - 1)}{(s - 1)^2(s + 5)} = \frac{s^2 + 3s + 2}{(s - 1)^2(s + 5)}. \)

We can begin our P.F.D. by having \( Y(s) = \frac{s^2 + 3s + 2}{(s - 1)^2(s + 5)} = \frac{A}{s - 1} + \frac{1}{(s - 1)^2} + \frac{1/3}{s + 5}. \)

By clearing fractions,

\[
s^2 + 3s + 2 = A(s - 1)(s + 5) + (s + 5) + (1/3)(s - 1)^2 = As^2 + 4As - 5A + s + 5 + (1/3)s^2 - (2/3)s + (1/3).
\]

So \( A = 2/3 \) and \( Y(s) = \frac{2/3}{s - 1} + \frac{1}{(s - 1)^2} + \frac{1/3}{s + 5}. \)
Then \( y(t) = \left(\frac{2}{3}\right)\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \left(\frac{1}{3}\right)\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = \frac{2}{3}e^t + te^t + \frac{1}{3}e^{-5t}. \)

The unit step function \( U(t) = \begin{cases} 
0 & \text{if } t < 0 \\
1 & \text{if } t \geq 0 
\end{cases} \) is a basic piecewise continuous function.

Consider a piecewise continuous function \( f(t) = \begin{cases} 
f_0(t) & \text{if } 0 \leq t < t_1 \\
f_1(t) & \text{if } t \geq t_1 
\end{cases}. \)

We can rewrite \( f(t) \) in a convenient form using the unit step function appropriately:

\[ f(t) = f_0(t) + U(t-t_1)[f_1(t) - f_0(t)]. \]

The proof of the following fact is in the textbook:

\[ \mathcal{L}(U(t-r)g(t)) = e^{-sr}\mathcal{L}(g(t+r)). \]

Let’s use this fact to find the Laplace transform of \( f(t) = \begin{cases} 
2-t & \text{if } 0 \leq t < 1 \\
t & \text{if } t \geq 1 
\end{cases}. \)

First we rewrite it as \( f(t) = 2-t + U(t-1)[2t-2] = 2-t + 2U(t-1)[t-1]. \)

Then \( \mathcal{L}(f(t)) = \frac{2}{s} - \frac{1}{s^2} + 2\mathcal{L}(U(t-1)[t-1]) = \frac{2}{s} - \frac{1}{s^2} + 2e^{-s}\mathcal{L}(t) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s}. \)

Wasn’t that nicer than how we did it before (example (iv) on page 11)?
Consider a piecewise continuous function $f(t) = \begin{cases} f_0(t) & \text{if } 0 \leq t < t_1 \\ f_1(t) & \text{if } t_1 \leq t < t_2 \\ f_2(t) & \text{if } t \geq t_2 \end{cases}$.

We can rewrite $f(t)$ in a convenient form using the unit step function appropriately:

$$f(t) = f_0(t) + \mathcal{U}(t-t_1)[f_1(t) - f_0(t)] + \mathcal{U}(t-t_2)[f_2(t) - f_1(t)].$$

Then we can find the Laplace transform of $f(t)$, using the above and $\mathcal{L}(\mathcal{U}(t-r)g(t)) = e^{-sr}\mathcal{L}(g(t+r))$.

Of course this generalizes to any number of “pieces,” but we will focus on functions that are either 2 or 3 “pieces.”

Since $\mathcal{L}(\mathcal{U}(t-r)g(t)) = e^{-sr}\mathcal{L}(g(t+r))$ we get

$$\mathcal{L}(\mathcal{U}(t-r)g(t-r)) = e^{-sr}\mathcal{L}(g(t)),$$

which is called the second shifting property.

Use linearity and the second shifting property to find $\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right)$.

First we use linearity and then apply the second shifting property:

$$\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(e^{-s}\frac{1}{s^2}\right) = t + \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(t)\right) = t + \mathcal{U}(t-1)[t-1].$$

Of course we can rewrite this in the more familiar form $\mathcal{L}^{-1}\left(\frac{1 + e^{-s}}{s^2}\right) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2t - 1 & \text{if } t \geq 1 \end{cases}$. 
Let’s solve the IVP \( y'' + y = \begin{cases} t & \text{if } 0 \leq t < 2\pi, \\ -2t & \text{if } t \geq 2\pi \end{cases} \), \( y(0) = 1 \), \( y'(0) = 2 \).

Let’s first find \( L(f) \) where \( f(t) = \begin{cases} t & \text{if } 0 \leq t < 2\pi, \\ -2t & \text{if } t \geq 2\pi \end{cases} \).

We can rewrite this function as \( f(t) = t - \lfloor t - 2\pi \rfloor [3t] \).

The \( L(f) = L(t) - 3L(\lfloor t - 2\pi \rfloor [t]) = \frac{1}{s^2} - 3e^{-2\pi s}L(t + 2\pi) = \frac{1}{s^2} - 3e^{-2\pi s} \left( \frac{1}{s^2} + \frac{2}{s} \right) \).

Recall: \( Y(s) = \frac{L(f(t))}{as^2 + bs + c} \).

So \( Y(s) = \frac{1}{s^2} - 3e^{-2\pi s} \left( \frac{1}{s^2} + \frac{2}{s} \right) + (2 + s) = \frac{1 + (s + 2)(s^2)}{s^2(s^2 + 1)} - 3e^{-2\pi s} \left( \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) \).

So \( Y(s) = Y_1(s) - 3Y_2(s) \) where \( Y_1(s) = \frac{s^3 + 2s^2 + 1}{s^2(s^2 + 1)} \) and \( Y_2(s) = e^{-2\pi s} \left( \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) \).

Let’s find \( L^{-1}(Y_1(s)) = L^{-1} \left( \frac{s^3 + 2s^2 + 1}{s^2(s^2 + 1)} \right) \).

We can begin a P.F.D. of \( Y_1(s) = \frac{s^3 + 2s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{1}{s^2} + \frac{Bs + C}{s^2 + 1} \).

Clearing fractions yields \( s^3 + 2s^2 + 1 = As(s^2 + 1) + s^2 + 1 + (Bs + C)s^2 = As^3 + As + s^2 + 1 + Bs^3 + Cs^2 \).

Thus \( A = 0 \) and \( B = C = 1 \). So \( Y_1(s) = \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} \). Hence \( L^{-1}(Y_1(s)) = t + \cos(t) + \sin(t) \).

Now let’s find \( L^{-1}(Y_2(s)) = L^{-1} \left( e^{-2\pi s} \left( \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) \right) \). We actually begin with \( L^{-1} \left( \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) \).

We can begin with the P.F.D. \( \frac{1 + 2\pi s}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{1}{s^2} + \frac{Bs + C}{s^2 + 1} \).

Clearing fractions yields \( 1 + 2\pi s = As(s^2 + 1) + s^2 + 1 + (Bs + C)s^2 = As^3 + As + s^2 + 1 + Bs^3 + Cs^2 \).

Thus \( A = 2\pi, B = -2\pi, \) and \( C = -1 \). So \( \frac{1 + 2\pi s}{s^2(s^2 + 1)} = \frac{2\pi}{s} + \frac{1}{s^2} - \frac{2\pi s + 1}{s^2 + 1} \).
Then \( \mathcal{L}^{-1}\left( \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) = 2\pi + t - 2\pi \cos(t) - \sin(t) \).

So by the second shifting property,
\[
\mathcal{L}^{-1}\left( e^{-2\pi s} \frac{1 + 2\pi s}{s^2(s^2 + 1)} \right) = \mathcal{L}^{-1}\left( e^{-2\pi s} \mathcal{L}(2\pi + t - 2\pi \cos(t) - \sin(t)) \right) = \mathcal{U}(t-2\pi)(t-2\pi \cos(t) - \sin(t)).
\]

Therefore the solution is \( y(t) = \mathcal{L}^{-1}(Y(s)) = t + \cos(t) + \sin(t) - 3\mathcal{U}(t-2\pi)(t-2\pi \cos(t) - \sin(t)) \).

In a more familiar form,
\[
y(t) = \begin{cases} 
  t + \cos(t) + \sin(t) & \text{if } 0 \leq t < 2\pi \\
  -2t + (1 + 6\pi) \cos(t) + 4\sin(t) & \text{if } t \geq 2\pi 
\end{cases}.
\]

The rigorous definition of the dirac delta function is beyond the scope of our study, but we will just use the heuristic definition:
\[
\delta(t) = \begin{cases} 
  \infty & \text{if } t = 0 \\
  0 & \text{if } t \neq 0
\end{cases}, \text{ such that } \int_I \delta(t) \, dt = 1 \text{ on any interval } I \text{ containing } 0.
\]

For our purposes, it suffices to use the following result:

If \( t_0 > 0 \) then the solution to the IVP \( ay'' + by' + cy = \delta(t - t_0) \), \( y(0) = 0 \), \( y'(0) = 0 \) is defined to be \( y = \mathcal{U}(t - t_0)w(t - t_0) \) where \( w = \mathcal{L}^{-1}\left( \frac{1}{as^2 + bs + c} \right) \).

The solution the IVP \( ay'' + by' + cy = f(t) + \alpha \delta(t - t_0) \), \( y(0) = k_0 \), \( y'(0) = k_1 \) is
\[
y = \hat{y} + \alpha \mathcal{U}(t - t_0)w(t - t_0),
\]
where \( w = \mathcal{L}^{-1}\left( \frac{1}{as^2 + bs + c} \right) \) and \( \hat{y} \) is the solution to \( ay'' + by' + cy = f(t) \), \( y(0) = k_0 \), \( y'(0) = k_1 \).
Example: Let’s solve $y'' - 7y' + 6y = 2e^{-7t} + 2\delta(t - 1)$, $y(0) = 1$, $y'(0) = -1$.

(ELFY) Use Laplace transforms to show that 

$$\hat{y} = \frac{-24}{65}e^{6t} + \frac{27}{20}e^t + \frac{1}{52}e^{-7t},$$

is a solution to $y'' - 7y' + 6y = 2e^{-7t}$, $y(0) = 1$, $y'(0) = -1$.

Next we find that $w = \mathcal{L}^{-1}\left(\frac{1}{(s - 6)(s - 1)}\right) = \frac{1}{5}(e^{6t} - e^t)$.

Thus the solution is 

$$y = \hat{y} + 2\mathcal{U}(t-1)[w(t-1)] = -\frac{24}{65}e^{6t} + \frac{27}{20}e^t + \frac{1}{52}e^{-7t} + 2\mathcal{U}(t-1)\left[\frac{1}{5}(e^{6(t-1)} - e^{t-1})\right].$$

In a more familiar form,

$$y(t) = \begin{cases} 
-\frac{24}{65}e^{6t} + \frac{27}{20}e^t + \frac{1}{52}e^{-7t} & \text{if } 0 \leq t < 1 \\
\left(-\frac{24}{65} + \frac{2}{5e^6}\right)e^{6t} + \left(\frac{27}{20} - \frac{2}{5e}\right)e^t + \frac{1}{52}e^{-7t} & \text{if } t \geq 1
\end{cases}.$$ 

Example: Let’s solve $y'' + y = 1 + 2\delta(t - \pi) + 3\delta(t - 2\pi)$, $y(0) = 1$, $y'(0) = -2$.

(ELFY) Use Laplace transforms to show that 

$$\hat{y} = 1 - 2\sin(t),$$

is a solution to $y'' + y = 1$, $y(0) = 1$, $y'(0) = -2$.

Next we find that $w = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) = \sin(t)$.

Thus the solution is 

$$y = \hat{y} + 2\mathcal{U}(t-\pi)[w(t-\pi)] + 3\mathcal{U}(t-2\pi)[w(t-2\pi)] = 1 - 2\sin(t) + 2\mathcal{U}(t-\pi)\sin(t-\pi) + 3\mathcal{U}(t-2\pi)\sin(t-2\pi).$$

Using the basic identities $\sin(t - \pi) = -\sin(t)$ and $\sin(t - 2\pi) = \sin(t)$, this becomes 

$$y = 1 - 2\sin(t) - 2\mathcal{U}(t-\pi)[\sin(t)] + 3\mathcal{U}(t-2\pi)[\sin(t)].$$

Or in a more familiar form,

$$y(t) = \begin{cases} 
1 - 2\sin(t) & \text{if } 0 \leq t < \pi \\
1 - 4\sin(t) & \text{if } \pi \leq t < 2\pi \\
1 - \sin(t) & \text{if } t \geq 2\pi
\end{cases}.$$
Here is the graph of the solution:
Convolutions

Let \( f(t), g(t) \) be functions. The convolution of \( f \) and \( g \) is the function \((f * g)(t) = \int_0^t f(r)g(t-r) \, dr\).

Let’s compute the convolution of \( f(t) = e^t \) and \( g(t) = e^{-t} \):

\[
(f * g)(t) = \int_0^t e^r e^{-t+r} \, dr = \int_0^t e^{-t} e^{2r} \, dr = e^{-t} \frac{1}{2} e^{2r}\bigg|_0^t = \frac{e^{-t}}{2} (e^t - 1) = \frac{e^t - e^{-t}}{2} = \sinh(t).
\]

The following theorem is called convolution theorem (the proof is beyond the scope of our course):

**Proposition 4.4**: If \( f(t), g(t) \) are functions such that \( \mathcal{L}(f) = F \) and \( \mathcal{L}(g) = G \) then \( \mathcal{L}(f * g) = FG \).

Let’s find a formula for the solution of the IVP \( y'' + 4y' - 5y = f(t), \ y(0) = k_0, \ y'(0) = k_1 \).

By taking the Laplace transform of both sides we get \((s^2 + 4s - 5)Y(s) = F(s) + (k_1 + k_0s) + 4k_0, \) where \( Y(s) = \mathcal{L}(y(t)) \) and \( F(s) = \mathcal{L}(f(t)) \).

Then \( Y(s) = F(s)G(s) + \frac{k_1 + 4k_0 + k_0s}{(s+5)(s-1)} = F(s)G(s) + \frac{k_1 + 5k_0}{6(s-1)} - \frac{k_1 - k_0}{6(s+5)}, \) where \( G(s) = (s^2 + 4s - 5)^{-1}. \)

Then by taking the inverse Laplace transform we get

\[
y(t) = \frac{k_1 + 5k_0}{6} e^t + \frac{k_0 - k_1}{6} e^{-5t} + \int_0^t f(r)g(t-r) \, dr, \text{ where } \mathcal{L}(g) = G.
\]

In particular, \( g(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 4s - 5}\right) = \mathcal{L}^{-1}\left(\frac{1/6}{s-1} - \frac{1/6}{s+5}\right) = \frac{1}{6} (e^t - e^{-5t}). \)

So \( y(t) = \frac{k_1 + 5k_0}{6} e^t + \frac{k_0 - k_1}{6} e^{-5t} + \int_0^t \frac{1}{6} f(r)\left(e^{t-r} - e^{-5(t+5r)}\right) \, dr \)

Let’s apply this formula to \( f(t) = 6e^t, \ k_0 = 1, \ k_1 = 0: \)

\[
y(t) = \frac{5}{6} e^t + \frac{1}{6} e^{-5t} + \int_0^t \frac{6e^r}{6} \left(e^{t-r} - e^{-5(t+5r)}\right) \, dr = \frac{5}{6} e^t + \frac{1}{6} e^{-5t} + \int_0^t e^t - e^{-5t+6r} \, dr = \]

\[
= \frac{5}{6} e^t + \frac{1}{6} e^{-5t} + \left(re^t - \frac{1}{6} e^{-5t+6r}\bigg|_{r=0}^{r=t}\right) = \frac{5}{6} e^t + \frac{1}{6} e^{-5t} + \frac{1}{6} e^t - \frac{1}{6} e^t + \frac{1}{6} e^{-5t} = \frac{2}{3} e^t + te^t + \frac{1}{3} e^{-5t}.
\]

Compare to the solution we got earlier (top of page 15)!
Let’s find a formula for the solution of the IVP \( y'' + 2y' + 2y = f(t), \ y(0) = k_0, \ y'(0) = k_1. \)

Taking the Laplace transform of both sides yields \(((s + 1)^2 + 1)Y(s) = F(s) + (k_1 + sk_0) + 2k_0, \)

where \( Y(s) = \mathcal{L}(y(t)) \) and \( F(s) = \mathcal{L}(f(t)) \).

Thus \( Y(s) = F(s)G(s) + \frac{k_0(s + 1)}{(s + 1)^2 + 1} + \frac{k_1 + k_0}{(s + 1)^2 + 1} \), where \( G(s) = [(s + 1)^2 + 1]^{-1} \).

Then by taking the inverse Laplace transform (using the table of Laplace transforms) we get

\[
y(t) = k_0e^{-t} \cos(t) + (k_0 + k_1)e^{-t} \sin(t) + \int_0^t f(r)g(t - r) \, dr, \quad \text{where } \mathcal{L}(g) = G.
\]

In particular, \( g(t) = \mathcal{L}^{-1}\left(\frac{1}{(s + 1)^2 + 1}\right) = e^{-t} \sin(t). \)

So \( y(t) = k_0e^{-t} \cos(t) + (k_0 + k_1)e^{-t} \sin(t) + \int_0^t f(r)e^{-t+r} \sin(t - r) \, dr. \)

One can also use this theory to evaluate convolution integrals. For example, let’s evaluate \( \int_0^t e^{-r} \sin(t - r) \, dr. \)

This is the convolution of \( f(t) = e^{-t} \) and \( g(t) = \sin(t) \). Then \( F(s) = \frac{1}{s+1} \) and \( G(s) = \frac{1}{s^2 + 1} \) are the Laplace transforms respectively. All we need to do is find the inverse Laplace transform of \( F(s)G(s) = \frac{1}{(s + 1)(s^2 + 1)}. \)

We begin with the P.F.D. \( \frac{1}{(s + 1)(s^2 + 1)} = \frac{1/2}{s + 1} + \frac{As + B}{s^2 + 1} \). From this we get \( A = -1/2 \) and \( B = 1/2. \)

So \( \frac{1}{(s + 1)(s^2 + 1)} = \frac{1/2}{s + 1} - \frac{(1/2)(s - 1)}{s^2 + 1} = \frac{1}{2} \left( \frac{1}{s + 1} + \frac{1}{s^2 + 1} - \frac{s}{s^2 + 1} \right) \) and the inverse Laplace transform, and therefore the integral, is

\[
\frac{1}{2} \left[ e^{-t} + \sin(t) - \cos(t) \right].
\]
Euler’s Method

Often the first order non-linear IVP \( y' = f(x, y), \ y(x_0) = y_0 \) cannot be solved analytically for an implicit or explicit solution.

There are numerical methods for approximating solutions at various values of \( x \). Suppose we are interested in finding the values of \( y \) at equally spaced values \( x_i = x_0 + i\Delta x \) for \( i = 0, 2, ..., n \) where \( \Delta x = \frac{x_n - x_0}{n} \).

Denote by \( y_i \) the approximate solution at \( x_i \) (for \( i = 1, 2, ..., n \)). Then the error at the \( i \)th step is \( e_i = y(x_i) - y_i \) (for \( i = 0, 1, 2, ..., n \)). Clearly \( e_0 = 0 \) as we already know \( y(x_0) = y_0 \).

The following method is called Euler’s method:

Euler’s method rests on having the tangent line to the graph of the solution curve at \( x_i \) approximate the curve on \([x_i, x_{i+1}]\).

The slope at \((x_0, y_0)\) is \( f(x_0, y_0) \) so the tangent line on \([x_0, x_1]\) is

\[
y = y_0 + f(x_0, y_0)(x - x_0).
\]

So we get the approximate solution \( y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + \Delta x f(x_0, y_0) \).

Then we can repeat to get the approximate solution \( y_2 = y_1 + \Delta x f(x_1, y_1) \).

We continue in this way at the \((i + 1)\)st step with \( y_{i+1} = y_i + \Delta x f(x_i, y_i) \) for \( i = 0, 1, ..., n - 1 \).

Let’s begin with an example for which we can solve it analytically:

Use Euler’s method to find approximate solutions to the IVP \( y' = \frac{y + 1}{x^2 + 1}, \ y(0) = 1 \) at the points \( x = 0.1, 0.2, 0.3 \) using \( \Delta x = 0.1 \).

\[
y_1 = y_0 + \Delta x f(x_0, y_0) = 1 + 0.1(2) = \frac{6}{5} = 1.2.
\]

\[
y_2 = y_1 + \Delta x f(x_1, y_1) = 1.2 + 0.1 \left( \frac{1221/505}{1.04} \right) = \frac{43337}{26260} \approx 1.650305.
\]
Now let’s solve analytically the IVP $y' = \frac{y + 1}{x^2 + 1}$, $y(0) = 1$ and determine how large the errors are for the approximate values $y_1, y_2, y_3$ found using Euler’s method (on the last page):

The equation is separable:

$$\frac{1}{y + 1} y' = \frac{1}{x^2 + 1}.$$ 

Integrating both sides and solving for $y$ (relabeling the constant along the way) gives

$$\ln |y + 1| = \tan^{-1}(x) + c \rightarrow y + 1 = ce^{\tan^{-1}(x)} \rightarrow y = -1 + ce^{\tan^{-1}(x)}.$$ 

By imposing the initial condition $y(0) = 1$ we get that $c = 2$.

So the solution to the IVP is

$$y = -1 + 2e^{\tan^{-1}(x)}.$$ 

We have that $y_1 = 1.2$. By plugging in $x = 0.1$ above,

$$y(0.1) = -1 + 2e^{\tan^{-1}(0.1)} \approx 1.209610.$$ 

So the approximation $y_1$ has an error of $e_1 \approx 0.00961$ (about an 0.8 percent underestimate).

We have that $y_2 \approx 1.417822$. By plugging in $x = 0.2$ above,

$$y(0.2) = -1 + 2e^{\tan^{-1}(0.2)} \approx 1.436452.$$ 

So the approximation $y_2$ has an error of $e_2 \approx 0.01863$ (about an 1.3 percent underestimate).

We have that $y_3 \approx 1.650305$. By plugging in $x = 0.3$ above,

$$y(0.3) = -1 + 2e^{\tan^{-1}(0.3)} \approx 1.676752.$$ 

So the approximation $y_3$ has an error of $e_3 \approx 0.02645$ (about an 1.6 percent underestimate).
Now let’s apply Euler’s method to the following linear non-homogeneous first order IVP to find approximate solutions at $x = 0.2, 0.4, 0.6, 0.8, 1$ with $\Delta x = 0.2$.

$$y' = 2xy + 1, \quad y(0) = 1.$$  

\[ y_1 = y_0 + \Delta x f(x_0, y_0) = 1 + 0.2(1) = \frac{6}{5} = 1.2. \]

\[ y_2 = y_1 + \Delta x f(x_1, y_1) = 1.2 + 0.2(2(0.2)(1.2) + 1) = \frac{187}{125} = 1.496. \]

\[ y_3 = y_2 + \Delta x f(x_2, y_2) = 1.496 + 0.2(2(0.4)(1.496) + 1) = \frac{6,048}{3,125} = 1.93536. \]

\[ y_4 = y_3 + \Delta x f(x_3, y_3) = 1.93536 + 0.2(2(0.6)(1.93536) + 1) = \frac{203,113}{78,125} = 2.5998464. \]

\[ y_5 = y_4 + \Delta x f(x_4, y_4) = 2.5998464 + 0.2(2(0.8)(2.5998464) + 1) = \frac{7,093,354}{1,953,125} \approx 3.63179725. \]

What if we try to solve this analytically? The equation $y' - 2xy = 1$ has an integrating factor of $e^{\int -2x \, dx} = e^{-x^2}$.

Then

$$e^{-x^2} y' - 2xe^{-x^2} y = e^{-x^2} \rightarrow (e^{-x^2} y)' = e^{-x^2}.$$  

But here, with the methods developed so far, we get stuck as the integral of $e^{-x^2}$ cannot be solved analytically.
**APPENDIX: Table of Laplace Transforms**

<table>
<thead>
<tr>
<th>f(t)</th>
<th>L[f(t)] = F(s)</th>
<th>f(t)</th>
<th>L[f(t)] = F(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 1</td>
<td>( \frac{1}{s} )</td>
<td>(19) ( ae^{at} - be^{bt} )</td>
<td>( \frac{s}{a-b} )</td>
</tr>
<tr>
<td>(2) ( e^{at} f(t) )</td>
<td>( F(s-a) )</td>
<td>(20) ( te^{at} )</td>
<td>( \frac{1}{(s-a)^2} )</td>
</tr>
<tr>
<td>(3) ( \mathcal{U}(t-a) )</td>
<td>( \frac{e^{-as}}{s} )</td>
<td>(21) ( t^n e^{at} )</td>
<td>( \frac{n!}{(s-a)^{n+1}} )</td>
</tr>
<tr>
<td>(4) ( f(t-a)\mathcal{U}(t-a) )</td>
<td>( e^{-as} F(s) )</td>
<td>(22) ( e^{at} \sin kt )</td>
<td>( \frac{k}{(s-a)^2 + k^2} )</td>
</tr>
<tr>
<td>(5) ( \delta(t) )</td>
<td>1</td>
<td>(23) ( e^{at} \cos kt )</td>
<td>( \frac{s-a}{(s-a)^2 + k^2} )</td>
</tr>
<tr>
<td>(6) ( \delta(t-t_0) )</td>
<td>( e^{-st_0} )</td>
<td>(24) ( e^{at} \sinh kt )</td>
<td>( \frac{k}{(s-a)^2 - k^2} )</td>
</tr>
<tr>
<td>(7) ( t^n f(t) )</td>
<td>( (-1)^n \frac{d^n F(s)}{ds^n} )</td>
<td>(25) ( e^{at} \cosh kt )</td>
<td>( \frac{s-a}{(s-a)^2 - k^2} )</td>
</tr>
<tr>
<td>(8) ( f'(t) )</td>
<td>( sF(s) - f(0) )</td>
<td>(26) ( t \sin kt )</td>
<td>( \frac{2ks}{(s^2 + k^2)^2} )</td>
</tr>
<tr>
<td>(9) ( f^n(t) )</td>
<td>( s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0) )</td>
<td>(27) t ( \cos kt )</td>
<td>( \frac{s^2 - k^2}{(s^2 + k^2)^2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(28) t ( \sinh kt )</td>
<td>( \frac{2ks}{(s^2 - k^2)^2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(29) t ( \cosh kt )</td>
<td>( \frac{s^2 - k^2}{(s^2 - k^2)^2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(30) ( \frac{\sin at}{t} )</td>
<td>( \arctan \frac{a}{s} )</td>
</tr>
<tr>
<td>(10) ( \int_0^t f(x)g(t-x)dx ) ( F(s)G(s) )</td>
<td></td>
<td>(31) ( \frac{1}{\sqrt{\pi t}} e^{-a^2/4t} )</td>
<td>( \frac{e^{-a\sqrt{s}}}{\sqrt{s}} )</td>
</tr>
<tr>
<td>(11) ( t^n ) ( n = 0,1, \ldots ) ( \frac{n!}{s^{n+1}} )</td>
<td></td>
<td>(32) ( \frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t} )</td>
<td>( \frac{e^{-a\sqrt{s}}}{s} )</td>
</tr>
<tr>
<td>(12) ( t^x ) ( x \geq -1 \in \mathbb{R} ) ( \frac{\Gamma(x+1)}{s^{x+1}} )</td>
<td></td>
<td>(33) ( \text{erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
<td>( \frac{e^{-a\sqrt{s}}}{s} )</td>
</tr>
<tr>
<td>(13) ( \sin kt )</td>
<td>( \frac{k}{s^2 + k^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(14) ( \cos kt )</td>
<td>( \frac{s}{s^2 + k^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(15) ( e^{at} )</td>
<td>( \frac{1}{s-a} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(16) ( \sinh kt )</td>
<td>( \frac{k}{s^2 - k^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(17) ( \cosh kt )</td>
<td>( \frac{s}{s^2 - k^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(18) ( \frac{e^{at} - e^{bt}}{a-b} )</td>
<td>( \frac{1}{(s-a)(s-b)} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>