(1) True or False?

(a) If the series \( \sum_{n=k}^{\infty} a_n \) converges, then the sequence \( a_n \) converges to 0.

\[ \text{TRUE} \]

(b) If \( a_n \) converges to 0 then \( \sum_{n=k}^{\infty} a_n \) converges.

\[ \text{FALSE} \]

(c) If \( a_n \) is positive, decreasing, and \( a_n \to 0 \) then \( \sum_{n=k}^{\infty} (-1)^n a_n \) converges.

\[ \text{TRUE} \]

(d) For all real numbers \( x \) in the interval \( (-1, 1) \), \( \sum_{k=j}^{\infty} x^k = \frac{x^j}{1-x} \).

\[ \text{TRUE} \]

(e) \[ \sum_{k=1}^{\infty} 2^{-k} = 1. \]

\[ \text{TRUE} \]

(f) The alternating harmonic series is absolutely convergent.

\[ \text{FALSE} \]

(g) The radius of convergence remains the same when a power series is integrated or differentiated.

\[ \text{TRUE} \]
Suppose that \( s_n = \frac{4n}{2n + 1} \) is the \( n \)th partial sum of a series \( \sum_{k=1}^{\infty} a_k \).

(a) Find the sum of the series \( \sum_{k=1}^{\infty} a_k \).

The sum of the series is the limit of the partial sums as \( n \rightarrow \infty \), which is 2.

(b) Find a formula for \( a_n \).

\[
a_n = s_n - s_{n-1} = \frac{4n}{2n + 1} - \frac{4(n-1)}{2(n-1) + 1} = \cdots = \frac{4}{4n^2 - 1}.
\]

(3) The Koch Snowflake problem:

(a) Let \( p_i \) denote the perimeter at the \( i \)th step where \( p_0 = 3s \) is the perimeter of the original triangle. Find a formula for \( p_i \). What does this tell us about the perimeter of the Koch Snowflake?

\[
p_i = \left(\frac{4}{3}\right)p_{i-1} \text{ for } i = 1, 2, 3, \ldots \text{ hence } p_i = \left(\frac{4}{3}\right)^i p_0 = 3s\left(\frac{4}{3}\right)^i. \text{ Since } p_i \rightarrow \infty \text{ as } i \rightarrow \infty, \text{ the Koch Snowflake has infinite perimeter.}
\]

(b) Let \( A_i \) denote the increased area at the \( i \)th step (from the \((i-1)\) step) where \( A_0 = \frac{\sqrt{3}}{4}s^2 \) is the area of the original triangle. Find a formula for \( A_i \) and use it to find the area of the Koch Snowflake.

\[
A_i = \left(\frac{1}{3}\right)(4/9)^{i-1} \text{ for } i = 1, 2, 3, \ldots \text{ so the area of the Koch Snowflake is...}
\]

\[
A = A_0 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)(4/9)^{i-1}A_0 = A_0 \left[1 + \frac{1}{3} \cdot \frac{1}{1 - \frac{4}{9}}\right] = A_0 \cdot \frac{8}{5} = \frac{2\sqrt{3}}{5}s^2.
\]
(4) Find the sum of each series:

(a) \[ \sum_{n=0}^{\infty} 5^{-n} 2^{n-1} \cdot \frac{5}{6} \]

(b) \[ \sum_{n=0}^{\infty} \frac{12^n}{13^n n!} \cdot e^{12/13} \]

(c) \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} \cdot \frac{5}{6} \]

(d) \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \cdot 1 \]

(5) Use the integral test to determine if the series is convergent or divergent:

\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]

Since \( \int_2^{\infty} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \ln \ln (\ln x) \bigg|_2^t = \infty \), this series is divergent.
(6) Use a partial sum of the series below to approximate the sum with an error of no more than 0.001:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n+1}{4n^4}.
\]

First solve for the first positive integer \( n \) such that \( \frac{1}{4(n+1)^4} \leq \frac{1}{1000} \).

This is equivalent to \((n + 1)^4 \geq 250\) which requires that \( n \geq 3 \).

So \( s_3 = \frac{1}{4} - \frac{1}{64} + \frac{1}{324} \) approximates the sum to within 0.001.

(7) Find the radius and interval of convergence for the following power series:

(a) \[
\sum_{n=0}^{\infty} \frac{(2x)^n}{(2n)!}.
\]

By the ratio test the radius is \( R = \infty \) and so the interval is \( I = (-\infty, \infty) \).

(b) \[
\sum_{n=1}^{\infty} \frac{(2x - 4)^n}{n}.
\]

By the ratio test the radius is \( R = \frac{1}{2} \), and by testing endpoints, the interval is \( I = [1.5, 2.5) \).

(c) \[
\sum_{n=1}^{\infty} \left( \frac{x - 5}{3} \right)^{2n}.
\]

By the root test the radius is \( R = 3 \), and by testing endpoints, the interval is \( I = (2, 8) \).

[Note: One could also see this directly as this series is a geometric series.]
(8) Derive a power series representation of \( f(x) = x \tan^{-1}(x) \). Find the radius and interval of convergence.

\[
f(x) = x \tan^{-1}(x) = x \int_0^x \frac{1}{1 + t^2} dt = x \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} x^{2n}.
\]

The radius of convergence is \( R = 1 \) and the interval is \( I = [-1, 1] \).

(9) Determine if the following series converges or diverges. Justify your answer.

\[
\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}.
\]

It is a \( p \)-series with \( p > 1 \), so it converges.

(10) For what values of a complex variable \( z \) does the following series converge? That is, find the circle of convergence.

\[
\sum_{n=1}^{\infty} n(z - i)^n.
\]

The radius of convergence is \( R = 1 \). So the circle of convergence is \( \{ z : |z - i| < 1 \} \). For all \( z \) such that \( |z - i| = 1 \) the series clearly diverges.
(11) Test each series for convergence or divergence.

(a) \[ \sum_{n=1}^{\infty} \frac{n^n}{(2n)!}. \]

*This series converges.*

(b) \[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right). \]

*This series diverges.*

(c) \[ \sum_{n=1}^{\infty} \left( e^{1/n^2} - 1 \right). \]

*This series converges.*
(12) Let \( f(x) = \sqrt{1 + 2x^2} \). Compute the fourth degree Taylor polynomial of \( f(x) \) (centered at the origin).

\[
T_3(x) = 1 + x^2 - \frac{1}{2}x^4.
\]

(13) Determine if each series given below is absolutely convergent, conditionally convergent, or divergent:

(a) \[
\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}.
\]

This series is absolutely convergent. [Note: The sum of this series is \((e^{-2} - 1)\).]

(b) \[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5n - 4}.
\]

This series is conditionally convergent.

(c) \[
\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n.
\]

This series is divergent.
(14) Use the Taylor series of \( f(x) = \cos(4x) \) (centered at \( x = 0 \)) to evaluate:

\[
\lim_{x \to 0} \frac{1 - \cos(4x)}{(3x)^2}.
\]

\( 8/9 \)

(15) Find the Taylor series of each function and then the radius of convergence and the interval of convergence:

(a) \( f(x) = \frac{1}{(1-x)^2} \) about \( x = 0 \).

\[
f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.
\]

The radius of convergence is \( R = 1 \) and the interval is \( I = (-1,1) \).

(b) \( f(x) = (x-1) \ln x \) about \( x = 1 \).

Begin with

\[
\frac{d}{dx} \ln x = \frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.
\]

Then \( \ln x = C + \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \) since \( \ln 1 = C = 0 \).

The radius of convergence is \( R = 1 \) and the interval is \( I = (0,2] \).

(c) \( f(x) = xe^{-x^2} \) about \( x = 0 \).

\[
f(x) = xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}.
\]

The radius is \( R = \infty \) and the interval is \( I = (-\infty,\infty) \).
(16) Use the Taylor series for \( f(x) = \tan^{-1} x \) centered about \( x = 0 \) to find the sum of following series:

\[
1 - 1/3 + 1/5 - 1/7 + \cdots
\]

\[
\frac{\pi}{4}
\]

(17) Use the Taylor series for \( f(x) = \sin (x^2) \) centered about \( x = 0 \) to estimate the following integral to within 0.005:

\[
\int_0^1 \sin(x^2)dx.
\]

\[
\int_0^1 \sin(x^2)dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{4n+2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n + 3)(2n+1)!}.
\]

To estimate this as it is an alternating series find the first non-negative integer such that...

\[
\frac{1}{(4(n + 1) + 3)(2(n + 1) + 1)!} \leq \frac{1}{200}
\]

We get \( n = 1 \).

The estimate is \( \frac{1}{3} - \frac{1}{42} = \frac{13}{42} \).