(1) True or False?

(a) All non-zero square matrices are invertible.

(b) All square matrices with a non-zero determinant are invertible.

(c) For all $m \times n$ matrices $A$, the equation $Ax = 0$ is consistent.

(d) If $u, v$ are linearly dependent and non-zero, then there exists $c$ such that $u = cv$.

(e) For all $n \times n$ matrices $A, B, C$, if $AB = AC$ and $A \neq 0$ then $B = C$.

(f) If the columns of $m \times n$ matrix $A$ add to the zero vector then the columns are linearly dependent.

(g) If the columns are linearly dependent then the columns of $m \times n$ matrix $A$ add to the zero vector.

(h) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(0, 0) \neq (0, 0)$.

(2) Find $h, k$ such that the system below has (a) no solutions, (b) a unique solution, or (c) infinitely many solutions.

\[
\begin{align*}
2x_1 - x_2 &= h, \\
10x_1 - kx_2 &= 2.
\end{align*}
\]

(3) Solve the system. Describe the solution set geometrically:

\[
\begin{align*}
2x - 2y + 5z &= -3, \\
-x + 2y - z &= 5, \\
3x + 12z &= 6.
\end{align*}
\]
(4) Consider the linear system:

\[
\begin{align*}
    x_1 - 3x_2 - 4x_3 &= b_1, \\
    -3x_1 + 3x_2 + 6x_3 &= b_2, \\
    5x_1 - 3x_2 - 8x_3 &= b_3.
\end{align*}
\]

(a) Show this system is inconsistent for some \((b_1, b_2, b_3)\).

(b) Find the solution set to the homogeneous system where \(b_1 = b_2 = b_3 = 0\).

(c) What three vectors in \(\mathbb{R}^3\) are shown to be linearly dependent based on the solution set to the homogenous system?

(5) Let \(A\) be a \(3 \times 3\) matrix such that it’s columns (in order) are the vectors \(a_1, a_2, a_3\). Describe the \((i, j)\)-entry of \(A^TA\) as a dot product.

(6) Let \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^4\) be the linear transformation given by

\[T(x, y, z) = (-3x + 2y + z, x - 4y + z, y - z, 2x - y + z)\].

(a) Find \(T(5, 4, 7)\).

(b) Find the standard matrix of \(T\).

(7) Consider the vector equation:

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
- \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}
+ \begin{bmatrix}
    c \\
    1 \\
    1
\end{bmatrix}
= \begin{bmatrix}
    6 \\
    2 \\
    4
\end{bmatrix}.
\]

(a) Rewrite this equation in a matrix form \(Ax = b\).

(b) For what value(s) (if any) of \(c\) will there be a unique solution?

(c) For what value(s) of \(c\) (if any) will there infinitely-many solutions?

(d) For what value(s) of \(c\) (if any) will there no solutions?
(8) Determine if each list of vectors below are linearly independent, and if not, find an explicit dependence relation:

(a) \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -12 \\ 25 \end{pmatrix}.

(b) \begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 3 \\ 5 \end{pmatrix}.

(c) \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.

(d) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ -9 \end{pmatrix}.

(9) Determine if matrix is invertible. Note: I am not asking you to calculate the inverse of the matrix (when it exists).

(a) \[ A = \begin{pmatrix} 5 & -10 \\ 7 & -14 \end{pmatrix} \].

(b) \[ A = \begin{pmatrix} -1 & -3 & 1 \\ 2 & 0 & -1 \\ 3 & 5 & -4 \end{pmatrix} \].

(c) \[ A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & -3 \end{pmatrix} \].
(10) Find all eigenvalues of the matrices below, and for each eigenvalue, find some linearly independent eigenvectors (as many as possible) corresponding to that eigenvalue. In each case, if possible, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

(a) 
$$A = \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}.$$  

(b) 
$$A = \begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix}.$$  

(c) 
$$A = \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$  

(d) 
$$A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}.$$  

(11) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects all vectors in the plane $x + y = 0$, followed by tripling the length.

(a) Find the standard matrix of $T$. Call it $A$. Hint: In the $xy$-plane $T$ reflects along the line $y = -x$.

(b) Use a geometric argument to find all the eigenvalues of $A$. For each eigenvalue, find linearly independent eigenvectors (as many as possible) corresponding to that eigenvalue.

(c) Confirm this by calculating the characteristic polynomial of $A$ and factoring it.

(d) Does there exists an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$?