(1) True of False?

(a) All non-zero square matrices are invertible.
   \textit{FALSE}

(b) All square matrices with a non-zero determinant are invertible.
   \textit{TRUE}

(c) For all $m \times n$ matrices $A$, the equation $Ax = 0$ is consistent.
   \textit{TRUE}

(d) If $u, v$ are linearly dependent and non-zero, then there exists $c$ such that $u = cv$.
   \textit{TRUE}

(e) For all $n \times n$ matrices $A, B, C$, if $AB = AC$ and $A \neq 0$ then $B = C$.
   \textit{FALSE}

(f) If the columns of $m \times n$ matrix $A$ add to the zero vector then the columns are linearly dependent.
   \textit{TRUE}

(g) If the columns are linearly dependent then the columns of $m \times n$ matrix $A$ add to the zero vector.
   \textit{FALSE}

(h) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(0, 0) \neq (0, 0)$.
   \textit{FALSE}
(2) Find $h, k$ such that the system below has (a) no solutions, (b) a unique solution, or (c) infinitely many solutions.

\[
\begin{align*}
2x_1 - x_2 &= h, \\
10x_1 - kx_2 &= 2.
\end{align*}
\]

(a) $k = 5$ and $h \neq \frac{2}{5}$.
(b) $k \neq 5$ ($h$ can be any real number).
(c) $k = 5$ and $h = \frac{2}{5}$.

(3) Solve the system. Describe the solution set geometrically:

\[
\begin{align*}
2x - 2y + 5z &= -3, \\
-x + 2y - z &= 5, \\
3x + 12z &= 6.
\end{align*}
\]

The solution set is \{(2 - 4t, \frac{7}{2} - \frac{3}{2}t, t) | t \in \mathbb{R}\}. Note: Answers can look different.

Here I choose $z = t$ as a free variable.

Geometrically, this solution set is a line

\[
\mathbf{r}(t) = \begin{pmatrix}
2 \\
3.5 \\
0
\end{pmatrix}
+ t
\begin{pmatrix}
-4 \\
-1.5 \\
1
\end{pmatrix}
\text{ in } \mathbb{R}^3.
\]
(4) Consider the linear system:

\[ \begin{align*}
    x_1 - 3x_2 - 4x_3 &= b_1, \\
    -3x_1 + 3x_2 + 6x_3 &= b_2, \\
    5x_1 - 3x_2 - 8x_3 &= b_3.
\end{align*} \]

(a) Show this system is not consistent for all \((b_1, b_2, b_3)\).

\[
\begin{pmatrix}
    1 & -3 & -4 & b_1 \\
    -3 & 3 & 6 & b_2 \\
    5 & -3 & -8 & b_3
\end{pmatrix}
\sim
\begin{pmatrix}
    1 & -3 & -4 & b_1 \\
    0 & -6 & -6 & 3b_1 + b_2 \\
    0 & 12 & 12 & -5b_1 + b_3
\end{pmatrix}
\sim
\begin{pmatrix}
    1 & -3 & -4 & b_1 \\
    0 & -6 & -6 & 3b_1 + b_2 \\
    0 & 0 & 0 & b_1 + 2b_2 + b_3
\end{pmatrix}
\]

In particular, the system is not consistent whenever \(b_1 + 2b_2 + b_3 \neq 0\). For an explicit triple, this system is not consistent for \((b_1, b_2, b_3) = (1, 0, 0)\).

(b) Find the solution set to the homogeneous system where \(b_1 = b_2 = b_3 = 0\).

Working from above

\[
\begin{pmatrix}
    1 & -3 & -4 & 0 \\
    0 & -6 & -6 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
    1 & 0 & -1 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

The solution set is \(\{(t, -t, t)|t \in \mathbb{R}\}\). Note: Answers can look different.

Here I choose \(z = t\) as a free variable.

(c) What three vectors in \(\mathbb{R}^3\) are shown to be linearly dependent based on the solution set to the homogenous system?

\[
\begin{pmatrix}
    1 \\
    -3 \\
    5
\end{pmatrix}, \begin{pmatrix}
    -3 \\
    3 \\
    -3
\end{pmatrix}, \begin{pmatrix}
    -4 \\
    6 \\
    -8
\end{pmatrix}.
\]
(5) Let $A$ be a $3 \times 3$ matrix such that its columns (in order) are the vectors $a_1, a_2, a_3$. Describe the $(i, j)$-entry of $A^T A$ as a dot product.

The $(i, j)$-entry of $A^T A$ is $a_i \cdot a_j$.

(6) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the linear transformation given by

$$T(x, y, z) = (-3x + 2y + z, x - 4y + z, y - z, 2x - y + z).$$

(a) Find $T(5, 4, 7)$.

$T(5, 4, 7) = (0, -4, -3, 13)$.

(b) Find the standard matrix of $T$.

$$A = \begin{pmatrix}
-3 & 2 & 1 \\
1 & -4 & 1 \\
0 & 1 & -1 \\
2 & -1 & 1 
\end{pmatrix}.$$
(7) Consider the vector equation:

\[
\begin{align*}
x \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - y \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + z \begin{pmatrix} c \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}.
\end{align*}
\]

(a) Rewrite this equation in a matrix form $Ax = b$.

\[
\begin{pmatrix} 1 & 0 & c \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}.
\]

(b) For what value(s) (if any) of $c$ will there be a unique solution?

For all values of $c$ except $c = 1$ there is a unique solution.

(c) For what value(s) of $c$ (if any) will there infinitely-many solutions?

For $c = 1$ there are infinitely-many solutions. Can you describe them?

(d) For what value(s) of $c$ (if any) will there no solutions?

There are no values of $c$ such that this equation has no solution.
(8) Determine if each list of vectors below are linearly independent, and if not, find an explicit dependence relation:

(a) \[
\begin{pmatrix}
2 \\
-4
\end{pmatrix},
\begin{pmatrix}
-12 \\
25
\end{pmatrix}.
\]

These vectors are independent as the determinant of the matrix with these columns is not zero.

(b) \[
\begin{pmatrix}
5 \\
2 \\
1 \\
-3
\end{pmatrix},
\begin{pmatrix}
-1 \\
2 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
6 \\
3 \\
5
\end{pmatrix}.
\]

These vectors are independent (the only solution corresponding homogenous system is the trivial one).

(c) \[
\begin{pmatrix}
5 \\
3 \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix},
\begin{pmatrix}
3 \\
2 \\
1
\end{pmatrix}.
\]

These vectors are dependent since the determinant of the matrix with these columns is zero. By solving the homogenous system we can produce an explicit non-trivial dependence relation (answers can vary):

\[
\begin{pmatrix}
5 \\
3 \\
1
\end{pmatrix} + \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} - 2 \begin{pmatrix}
3 \\
2 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]
(d) 
\[ \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ -9 \end{pmatrix}. \]

These vectors are dependent (by inspection) since they are scalar multiples of each other. We can produce an explicit non-trivial dependence relation (answers can vary) setting one vector as a multiple of the other and manipulating it, or by just by solving the corresponding homogenous system:

\[ 3 \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} + 2 \begin{pmatrix} -6 \\ 3 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

(9) Determine if matrix is invertible. Note: I am not asking you to calculate the inverse of the matrix (when it exists).

(a) 
\[ A = \begin{pmatrix} 5 & -10 \\ 7 & -14 \end{pmatrix}. \]

This matrix is not invertible since the determinant of the matrix is zero.

(b) 
\[ A = \begin{pmatrix} -1 & -3 & 1 \\ 2 & 0 & -1 \\ 3 & 5 & -4 \end{pmatrix}. \]

This matrix is invertible since the determinant of the matrix is not zero.

(c) 
\[ A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & -3 \end{pmatrix}. \]

This matrix is invertible since the determinant of the matrix is not zero.
(10) Find all eigenvalues of the matrices below, and for each eigenvalue, find some linearly independent eigenvectors (as many as possible) corresponding to that eigenvalue. In each case, if possible, find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

(a) 
\[
A = \begin{pmatrix}
1 & 3 \\
5 & -1
\end{pmatrix}.
\]

The eigenvalues are \( \pm 4 \).

All eigenvectors corresponding to \( 4 \) are multiples of \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

All eigenvectors corresponding to \( -4 \) are multiples of \( \begin{pmatrix} 3 \\ -5 \end{pmatrix} \).

We get \( A = PDP^{-1} \) by setting 
\[
P = \begin{pmatrix}
1 & 3 \\
1 & -5
\end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}
4 & 0 \\
0 & -4
\end{pmatrix}.
\]

(b) 
\[
A = \begin{pmatrix}
-2 & 4 \\
3 & -6
\end{pmatrix}.
\]

The eigenvalues are \( 0, -8 \).

All eigenvectors corresponding to \( 0 \) are multiples of \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).

All eigenvectors corresponding to \( -8 \) are multiples of \( \begin{pmatrix} 2 \\ -3 \end{pmatrix} \).

We get \( A = PDP^{-1} \) by setting 
\[
P = \begin{pmatrix}
2 & 2 \\
1 & -3
\end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}
0 & 0 \\
0 & -8
\end{pmatrix}.
\]
(c) 
\[ A = \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \].

The eigenvalues are \( 3 \pm i\sqrt{3} \).

All eigenvectors corresponding to \( 3 - i\sqrt{3} \) are multiples of \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \).

All eigenvectors corresponding to \( 3 + i\sqrt{3} \) are multiples of \( \begin{pmatrix} i \\ 1 \end{pmatrix} \).

We get \( A = PDP^{-1} \) by setting
\[ P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 - i\sqrt{3} & 0 \\ 0 & 3 + i\sqrt{3} \end{pmatrix} \].

(d) 
\[ A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix} \].

The eigenvalues are 1, 2.

All eigenvectors corresponding to 1 are multiples of \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

All eigenvectors corresponding to 2 are multiples of \( \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \).

It is not possible to find an invertible \( P \) and a diagonal \( D \) such that \( A = PDP^{-1} \).
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects all vectors in the plane $x + y = 0$, followed by tripling the length.

(a) Find the standard matrix of $T$. Call it $A$. Hint: In the $xy$-plane $T$ reflects along the line $y = -x$.

The standard matrix is

$$
A = \begin{pmatrix}
0 & -3 & 0 \\
-3 & 0 & 0 \\
0 & 0 & 3
\end{pmatrix}.
$$

(b) Use a geometric argument to find all the eigenvalues of $A$. For each eigenvalue, find linearly independent eigenvectors (as many as possible) corresponding to that eigenvalue.

Vectors perpendicular to the plane (one dimensional) are sent to 3 times their opposite, so $-3$ is an eigenvalue. Vectors in the plane (two-dimensional) are sent to 3 times themselves. So 3 is the only other eigenvalue (by counting dimensions).

All eigenvectors corresponding to $-3$ are multiples of

$$
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}.
$$

All eigenvectors corresponding to 3 are linear combinations of

$$
\begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
$$

(c) Confirm this by calculating the characteristic polynomial of $A$ and factoring it.

The characteristic polynomial is $f(\lambda) = (3 - \lambda)(\lambda^2 - 9)$ which subsequently factors as $f(\lambda) = -(\lambda + 3)(\lambda - 3)^2$ confirming the geometric arguments that gave us the eigenvalues.

(d) Does there exists an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$?

Yes. Can you produce such a $P$ and $D$?