We will not go through chapter 3 (as it was covered in 341), but I expect you to know how to calculate determinants via cofactor expansions and via row reduction. I also expect you know the general theory of determinants (if you don’t remember, read chapter 3 as a refresher).

**Polynomials**

Some background material on Polynomials...

Here we use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. Really, $\mathbb{F}$ represents any set with the same fundamental properties as $\mathbb{R}$ and $\mathbb{C}$ (which are examples of what are called fields).

A function $f : \mathbb{F} \to \mathbb{F}$ is a polynomial over $\mathbb{F}$ if there exist coefficients $a_0, a_1, ..., a_n \in \mathbb{F}$ such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \text{ for all } x \in \mathbb{F}.$$  

Formally, the degree of $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ (assuming $a_n \neq 0$) is $n$ unless $f(x) = 0$, which has degree $-\infty$; that special case is important, for instance, it makes the rule $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$ work for all polynomials $f(x), g(x)$.

As we have already seen, $\mathbb{P}_n$, the polynomials of degree at most $n$, form a vector space.

A number $\lambda \in \mathbb{F}$ is called a root of $f(x) \in \mathbb{P}_n$ if $f(\lambda) = 0$.

The following proposition is sometimes refereed to as the “Root-Factor Correspondence:”

**Proposition:** Suppose $f(x) \in \mathbb{P}_n$, $\deg(f(x)) \geq 1$. $\lambda \in \mathbb{F}$ is a root of $f(x)$ iff there exists $g(x) \in \mathbb{P}_{n-1}$ such that $f(x) = (x-\lambda)g(x)$.

**Proof:** The reverse direction is trivial. Assume $f(x) \in \mathbb{P}_n$, $\deg(f(x)) \geq 1$, and $\lambda \in \mathbb{F}$ is a root of $f(x)$.

Let $a_0, a_1, ..., a_m \in \mathbb{F}$ such that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, $a_m \neq 0$.

Then $f(\lambda) = 0 = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_m \lambda^m$. 


By subtracting equations, \( f(x) = a_1(x - \lambda) + a_2(x^2 - \lambda^2) + \cdots + a_m(x^m - \lambda^m) \).

For \( k = 2, 3, \ldots, m \) we have the following basic result from algebra:

\[
(x^k - \lambda^k) = (x - \lambda)(x^{k-1} + x^{k-2}\lambda + \cdots + x\lambda^{k-2} + \lambda^{k-1}).
\]

Hence \( f(x) = (x - \lambda)g(x) \) for some \( g(x) \in \mathbb{P}_n \) with degree \( m - 1 \) \( \square \)

**Corollary:** If \( f(x) \in \mathbb{P}_n \) is not 0 then \( f(x) \) can have at most \( n \) roots in \( \mathbb{F} \).

(ELFY) Prove this using induction on the degree of \( f(x) \) (base case is degree 0) and the proposition we just proved.

**Corollary:** Suppose \( a_0, a_1, \ldots, a_n \in \mathbb{F} \). If \( a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \) for all \( x \in \mathbb{F} \), then \( a_0 = a_1 = \cdots = a_n = 0 \).

**Proof:** Let \( a_0, a_1, \ldots, a_n \in \mathbb{F} \). Let \( x \in \mathbb{F} \). Assume \( a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \). But then \( x \) is a root of \( f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \), so \( f(x) \in \mathbb{P}_n \) has infinitely many roots, which means the degree cannot be non-negative, which means \( \deg(f(x)) = -\infty \) and thus \( a_0 = a_1 = \cdots = a_n = 0 \) \( \square \)

**Division Algorithm:** Suppose \( f(x), d(x) \in \mathbb{P}_n \) with \( d(x) \neq 0 \). Then there exist \( q(x), r(x) \in \mathbb{P}_n \) such that \( f(x) = q(x)d(x) + r(x) \) and \( \deg(r(x)) < \deg(d(x)) \).

**Proof:** Choose \( q(x) \in \mathbb{P}_n \) such that \( f(x) - q(x)d(x) \in \mathbb{P}_n \) and has degree as small as possible. Let \( r(x) = f(x) - q(x)d(x) \).

Then \( f(x) = q(x)d(x) + r(x) \), so it only remains to show that \( \deg(r(x)) < \deg(d(x)) \). Suppose otherwise, that is, \( \deg(r(x)) \geq \deg(d(x)) \).

Let \( j = \deg(r(x)) - \deg(d(x)) \), which is non-negative and less than or equal to \( n \). Let \( c \) be the leading coefficient of \( r(x) \) divided by the leading coefficient of \( d(x) \). Then \( q(x) + cx^j \in \mathbb{P}_n \) and \( f(x) - (q(x) + cx^j)d(x) = r(x) - cx^j d(x) \) has degree less than \( r(x) \), which is contradicts how \( q(x) \) was chosen. This completes the proof \( \square \)
So far we have been using $\mathbb{F}$ for $\mathbb{R}$ and $\mathbb{C}$. That’s because everything we have done works in either case without changing the arguments (and also works for arbitrary fields).

Now we state a property that $\mathbb{C}$ has but $\mathbb{R}$ does not. The proof requires advanced tools (complex analysis) that are beyond the scope of this course. We use the notation $\mathbb{P}^n(\mathbb{C})$ to denote polynomials over $\mathbb{C}$ of degree at most $n$. It’s tradition to use the variable $z$ instead of $x$ when the polynomials are over $\mathbb{C}$. 

**Fundamental Theorem of Algebra**: Suppose $f(z) \in \mathbb{P}^n(\mathbb{C})$. If $f(z)$ is non-constant (degree at least 1) then $f(z)$ has a root in $\mathbb{C}$.

An important consequence (proof omitted) is that every non-constant polynomial $f(z) \in \mathbb{P}^n(\mathbb{C})$ has a unique factorization:

$$f(x) = c(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$ where $c, \lambda_1, ..., \lambda_n \in \mathbb{C}$.

Some comments about $\mathbb{C}$:

Let $z \in \mathbb{C}$. Then $z = a + bi$ where $a, b \in \mathbb{R}$. $a$ is the real part and $bi$ the imaginary part. Note that $a + bi = 0$ iff $a = b = 0$. In particular, since $1, i$ are linearly independent over $\mathbb{R}$, and they span $\mathbb{C}$, it follows that $\mathbb{C}$ is a two-dimensional real vector space...

The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$.

The absolute value (aka magnitude, size, length) of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Some nice properties (verifications are left to you):

1. $z\bar{z} = |z|^2$ for all $z \in \mathbb{C}$.
2. $z_1 + \bar{z}_2 = \bar{z}_1 + z_2$ for all $z_1, z_2 \in \mathbb{C}$.
3. $z_1 \bar{z}_2 = \bar{z}_1 z_2$ for all $z_1, z_2 \in \mathbb{C}$.
4. $\bar{z} = z$ for all $z \in \mathbb{C}$.
5. $|z_1 z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$.
6. $f(\bar{z}) = \overline{f(z)}$ for all $f(z) \in \mathbb{P}^n(\mathbb{C})$ and $z \in \mathbb{C}$.

In particular, (6) justifies that if $\lambda \in \mathbb{C}$ is a root of a polynomial over $\mathbb{R}$ then so is $\bar{\lambda}$. That holds because the conjugate of a real coefficient is just that real coefficient.
By the quadratic formula, we know explicitly that \( x^2 + bx + c = 0 \) has some real root(s) iff \( b^2 - 4ac \geq 0 \).

Thus when \( b^2 - 4ac < 0 \) the roots of \( x^2 + bx + c = 0 \) are complex conjugates \( \alpha + \beta i, \alpha - \beta i \) with \( \alpha, \beta \in \mathbb{R}, \beta \neq 0 \).

**Proposition:** Suppose \( f(x) \in \mathbb{P}_n(\mathbb{R}) \) is non-constant. Then \( f(x) \) factors into linear (degree 1) and quadratic (degree 2) factors in \( \mathbb{P}_n(\mathbb{R}) \).

**Proof:** Let \( f(x) \in \mathbb{P}_n(\mathbb{R}) \) be non-constant. If \( f(x) \) has degree 1 then \( f(x) = c(x - r) \) for \( c, r \in \mathbb{R}, c \neq 0 \).

If \( f(x) \) has degree 2, then either \( f(x) = c(x - r_1)(x - r_2) \) with \( c, r_1, r_2 \in \mathbb{R}, c \neq 0 \) or \( f(x) = a(x^2 + bx + c) \) with \( a, b, c \in \mathbb{R}, a \neq 0, b^2 - 4ac < 0 \).

These cases form the base cases for the following inductive step:

Let \( m \geq 2 \). Assume for all non-constant \( f(x) \in \mathbb{P}_m(\mathbb{R}) \) factors into linear and quadratic factors in \( \mathbb{P}_m(\mathbb{R}) \).

Let \( g(x) \in \mathbb{P}_{m+1}(\mathbb{R}) \) be of degree \( m + 1 \).

Suppose \( g(x) \) has a real root \( r \), then by the root-factor correspondence, \( g(x) = (x - r)f(x) \) and by the inductive hypothesis \( f(x) \) factors into linear and quadratic factors in \( \mathbb{P}_m(\mathbb{R}) \) since the degree is \( m \). Then so does \( g(x) \).

Suppose \( g(x) \) does not have a real root \( r \). Then by the fundamental theorem of algebra, \( g(x) \) has a non-real root \( \lambda \in \mathbb{C} \). Then \( (x - \lambda)(x - \bar{\lambda}) = x^2 + bx + c \), where (ELFY) \( b = -\lambda - \bar{\lambda} \in \mathbb{R}, c = \lambda \bar{\lambda} \in \mathbb{R} \) and \( b^2 - 4c < 0 \). Then \( x^2 + bx + c \) is a factor of \( g(x) \) by the root-factor correspondence. Thus \( g(x) = (x^2 + bx + c)f(x) \) and by the inductive hypothesis \( f(x) \) factors into linear and quadratic factors in \( \mathbb{P}_m(\mathbb{R}) \) since the degree is \( m - 1 \). Then so does \( g(x) \).