(1) Which of the following is a binary operation on $\mathbb{Z}^+ = \{n \in \mathbb{Z} | n > 0\}$? What about on $\mathbb{R}^+ = \{r \in \mathbb{R} | r > 0\}$? In each case, if it is a binary operation, determine (i) if there is an identity, and if so, exhibit it, and (ii) if it is commutative. Justify your answers.

(a) $*$ given by $a * b = a + b$.
(b) $*$ given by $a * b = a - b$.
(c) $*$ given by $a * b = ab$.
(d) $*$ given by $a * b = a/b$.
(e) $*$ given by $a * b = ab/2$.
(f) $*$ given by $a * b = a^{[b]}$, where $|x|$ denotes the greatest integer less than or equal to $x$ (aka the “floor function”).

(2) Let $H = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

(a) Show that $(H, +)$ is a commutative binary structure with identity and $(H, +)$ is isomorphic (as a binary structure) to $(\mathbb{C}, +)$

Hint: Define $\phi : H \to \mathbb{C}$ by $\phi \left( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) = a + bi$.

(b) Show that $(H, \cdot)$ is a commutative binary structure with identity and $(H, \cdot)$ is isomorphic (as a binary structure) to $(\mathbb{C}, \cdot)$

Hint: Use the same map $\phi$ as in part (a).

(3) Let $n \in \mathbb{N}$. Show that $(\mathbb{R}^+, *)$ where $x * y = \frac{xy}{n}$ is a group.

(4) Let $n \in \mathbb{N}$. Recall the following basic fact from linear algebra: If $A, B \in M_n$ then $\det(AB) = \det(A)\det(B)$.

Let $O_n = \{ A \in M_n | \det(A) = \pm 1 \}$. Show that $O_n$ is a group (under matrix multiplication).

(5) Let $S = \{ r \in \mathbb{R} | r \neq -1 \}$. Let $*$ be given by $x * y = x + y + xy$. Show that $*$ is a binary operation on $S$ and that $(S, *)$ forms an abelian group. Solve the equation $2 * x * 3 = 7$ for $x \in S$. 